Mathtrace – user guide

II. Linear algebra and geometry

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20. Geometry calculations

Perimeters and areas of figures, volumes and surfaces of solids are computed by a form like the other applications (Applications → Geometry): pick the figure and what you are looking for, and enter the dimensions with their units. Figures: square, rectangle, triangle (a side and its height, or three sides – Heron's formula), parallelogram, rhombus, trapezoid, regular polygon, circle, annulus, arc and sector, the Pythagorean theorem and similar triangles; solids: cube, cuboid, prism, cylinder, square pyramid, cone and sphere.

The working writes the formula, substitutes and computes; when an intermediate value is needed (a cone's slant height, a pyramid's slant height), it comes first. It works the other way too – the radius from the area, the edge from the volume, a leg from the hypotenuse. The result is exact with π and roots and approximate (25π cm² ≈ 78.54 cm²). Units are converted (cm to dm, dm³ to litres).

Examples

A circle's area – exact with π and approximate.

Input: `model(circle; find=S[cm2]; r=5[cm])`

Result

$$S = 25\pi\,\mathrm{cm}^2 \approx 78.54\,\mathrm{cm}^2$$

Steps

- A circle is the points at the distance r from the centre, a disc the circle with its inside. Circumference o = 2πr = πd, area S = πr². The number π ≈ 3.14159 is the circumference over the diameter – the result exact (with π) and approximate. Formulas
  $o = \pi d$
  $d = 2r$
  $o = 2\pi r$
  $S = \pi r^2$
- Find S; given: $r = 5\,\mathrm{cm}$
- Substituting: $S = \pi \cdot 5^2 = 25\pi\,\mathrm{cm}^2 \approx 78.54\,\mathrm{cm}^2$

The other way: the radius from the area.

Input: `model(circle; find=r[cm]; S=50[cm2])`

Result

$$r \approx 3.989\,\mathrm{cm}$$

Steps

- A circle is the points at the distance r from the centre, a disc the circle with its inside. Circumference o = 2πr = πd, area S = πr². The number π ≈ 3.14159 is the circumference over the diameter – the result exact (with π) and approximate. Formulas

  $o = \pi d$

  $d = 2r$

  $o = 2\pi r$

  $S = \pi r^2$

- Find r; given: $S = 50\,\mathrm{cm}^2$
- Expressing the sought quantity: $r = \sqrt{\frac{S}{\pi}}$
- Substituting: $r = \sqrt{\frac{50}{\pi}} \approx 3.989\,\mathrm{cm}$

A triangle from three sides – Heron's formula.

Input: `model(triangle; find=S[cm2]; a=3[cm]; b=4[cm]; c=5[cm]; input=sides)`

Result

$$S = 6\,\mathrm{cm}^2$$

Steps

- A triangle's area is half a side times its height: S = a·v_a/2. From the three sides by Heron's formula S = √(s(s − a)(s − b)(s − c)), s = (a + b + c)/2 being half the perimeter. The sides must meet the triangle inequality – any two together longer than the third. Formulas

  $o = a + b + c$

  $S = \frac{a v_a}{2}$

  $s = \frac{a + b + c}{2}$

  $S = \sqrt{s(s-a)(s-b)(s-c)}$

- Find S; given: $a = 3\,\mathrm{cm},\ b = 4\,\mathrm{cm},\ c = 5\,\mathrm{cm}$
- Substituting: $s = \frac{3 + 4 + 5}{2} = 6\,\mathrm{cm}$
- Substituting: $S = \sqrt{6 \cdot (6-3)(6-4)(6-5)} = 6\,\mathrm{cm}^2$

A leg from the hypotenuse.

Input: `model(pythagoras; find=b[cm]; a=5[cm]; c=13[cm])`

Result

$$b = 12\,\text{cm}$$

Steps

- In a right triangle the square on the hypotenuse equals the sum of the squares on the legs: c² = a² + b². The hypotenuse c lies opposite the right angle and is the longest side; a leg is b = √(c² − a²). Formulas
  $$c^2 = a^2 + b^2$$
- Find b; given: $a = 5\,\text{cm},\ c = 13\,\text{cm}$
- Expressing the sought quantity: $b = \sqrt{c^2 - a^2}$
- Substituting: $b = \sqrt{13^2 - 5^2} = 12\,\text{cm}$

A cone's surface – the slant height first.

Input: `model(cone; find=S[cm2]; r=3[cm]; v=4[cm])`

Result

$$S = 24\pi\,\text{cm}^2 \approx 75.4\,\text{cm}^2$$

Steps

- A cone has a circular base of radius r and the height v. Its volume is a third of the cylinder's: V = πr²v/3. The slant height s is the hypotenuse with the legs r and v; the lateral surface is a sector S_{pl} = πrs, surface S = πr² + πrs. Formulas
  $$V = \frac{\pi r^2 v}{3}$$
  $$s^2 = r^2 + v^2$$
  $$S_{pl} = \pi r s$$
  $$S = \pi r^2 + S_{pl}$$
- Find S; given: $r = 3\,\text{cm},\ v = 4\,\text{cm}$
- Expressing the sought quantity: $s = \sqrt{r^2 + v^2}$
- Substituting: $s = \sqrt{3^2 + 4^2} = 5\,\text{cm}$
- Substituting: $S_{pl} = \pi \cdot 3 \cdot 5 = 15\pi\,\text{cm}^2 \approx 47.12\,\text{cm}^2$
- Substituting: $S = \pi \cdot 3^2 + 15\pi = 24\pi\,\text{cm}^2 \approx 75.4\,\text{cm}^2$

A cube's volume in litres.

Input: `model(cube; find=V[l]; a=10[cm])`

Result

$$V = 1\,\mathrm{l}$$

Steps

- A cube has six equal square faces with the edge a.
  Volume V = a³, surface S = 6a², space diagonal u = a√3.
  The edge from the volume is its cube root: a = ∛V.
  Formulas
  $V = a^3$
  $S = 6a^2$
  $u = \sqrt{3}a$
- Find V; given: $a = 10\,\mathrm{cm}$
- Converting the units: $a = 10\,\mathrm{cm} = 1\,\mathrm{dm}$
- Substituting: $V = 1^3 = 1\,\mathrm{dm}^3$
- Converting the units: $V = 1\,\mathrm{dm}^3 = 1\,\mathrm{l}$

Known limitations

- A sketch of the figure with its dimensions is still missing.

21. Matrices

Write a matrix in square brackets as in Matlab: entries separated by spaces, rows by a semicolon or a new line. An entry may be a fraction, a root or a letter. The key with a grid on the f(x) tab opens a table to fill in cell by cell – or to paste into from the clipboard (cells copied from Excel or Google Sheets) or load from a CSV file.

For a single matrix a chip says what to compute: the determinant (2×2 by the formula, 3×3 by the rule of Sarrus, otherwise by expansion along a row or column or by row reduction to a triangular matrix – the method is a chip too), the inverse (2×2 by the formula, otherwise by Gauss–Jordan elimination with the identity beside it), the rank (row reduction to row echelon form) or the transpose.

Name several matrices with capital letters, each on its own line, and write what to compute on the last line: a sum, difference, multiple, product (row by column, with the sizes checked), power, transpose A^T, inverse A^-1, det(A), rank(A) – or a matrix equation AX = B or XA = B, which the app solves with the inverse. A system of equations given by a matrix (an augmented matrix, Ax = b) is covered in the chapter Systems of equations.

If a matrix contains a parameter (a letter), the app discusses its rank by the parameter's values just like a system – row reduction to echelon form, the critical values put in. For an inverse it first computes the determinant and the condition for which values of the parameter it isn't zero (the inverse exists). A 3×3 or bigger inverse is computed by Gauss–Jordan elimination or – by the chip – by the adjugate matrix (cofactors, A^{-1} = adj A / det A).

For a square matrix you can also compute the eigenvalues and eigenvectors: the characteristic polynomial det(A − λE) is written out like a determinant, factored, and its roots are the eigenvalues (a quadratic factor by the discriminant, complex ones too; a cubic without a rational root approximately). The eigenvectors are the solutions of (A − λE)v = o by row operations to the reduced row echelon form; for each eigenvalue the app gives its algebraic and geometric multiplicity. The Diagonalization chip also builds P and D ($A = PDP^{-1}$), or explains why the matrix can't be diagonalized. The singular value decomposition $A = U\Sigma V^T$ (SVD) and the pseudoinverse A^+ work for a matrix of any size: SVD through the eigenvalues of A^TA (AA^T for a wide matrix), the normalized eigenvectors and the other side computed from them; the pseudoinverse of a full-rank matrix by the formula $(A^TA)^{-1}A^T$, otherwise from the decomposition $A^+ = V\Sigma^+U^T$ – and if the singular values would be nested radicals, exactly from the rank factorization A = BC by $A^+ = C^T(CC^T)^{-1}(B^TB)^{-1}B^T$.

The app also works out the eigenvalues and the diagonalization of a matrix with one parameter (up to 3×3): it factors the characteristic polynomial with the parameter, writes the eigenvalues as its functions and finds the special values of the parameter – where two eigenvalues coincide (two factors have a common root, a quadratic factor a zero discriminant) or where the general computation of the

eigenvectors would divide by zero; it also tells for which values the eigenvalues are complex. The general case is solved with the parameter, the special values are put in and discussed separately. The Diagonalization chip then answers for which values of the parameter the matrix is diagonalizable.

The Jordan form of a square matrix (the chip Jordan form, written jordan(A)) comes from the ranks of the powers $(A - \lambda E)^k$: they give the number and the sizes of the Jordan blocks, the chains of generalized eigenvectors are built from the top vector down to an eigenvector and P, J are checked ($A = PJP^{-1}$). The Jordan form also gives the power A^n for a general n (the chip Power A^n, written A^n; a block's power by the binomial theorem, the result like $2^{n-1}(n + 2)$) or a big number (A^10), and the matrix exponential e^{At} and e^A (the chip Matrix exponential, written e^A, e^(At), exp(A), expm(A*t)) – useful for systems of differential equations $x' = Ax$; complex eigenvalues are rewritten as cosines and sines. The minimal polynomial (the chip Minimal polynomial, written minpoly(A)) is found by trying the candidates – the divisors of the characteristic polynomial – from the lowest degree; the app recalls the Cayley–Hamilton theorem p(A) = O and for an invertible matrix gets the inverse from m(A) = O.

How to enter it

`[1 2 3; 4 5 6; 7 8 10]`	Entries by spaces, rows by a semicolon (or on new lines inside the brackets).

`A = [1 2; 3 4]`
`B = [0 1; 1 0]`
`A*B - 2B^T`
	Named matrices a line each, the computation on the last line.

`det(A), A^-1, A^T, rank(A)`	Determinant, inverse and transpose, rank.
`eig(A), diagonalize(A), svd(A), pinv(A), A^+`	Eigenvalues, diagonalization, singular value decomposition, pseudoinverse.
`jordan(A), A^n, e^A, e^(At), minpoly(A)`	Jordan form, a general power, the matrix exponential, the minimal polynomial.

Examples

A 3×3 determinant by the rule of Sarrus.

Input: `[1 2 3; 4 5 6; 7 8 10]` Determinant

Result

$$\det A = -3$$

Steps

- Rule of Sarrus, the products along the main diagonals minus those along the other ones

$$\begin{aligned}\det A &= \begin{vmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 10\end{vmatrix}\\ &= 1\cdot 5\cdot 10 + 2\cdot 6\cdot 7 + 3\cdot 4\cdot 8\\ &\quad - 3\cdot 5\cdot 7 - 1\cdot 6\cdot 8 - 2\cdot 4\cdot 10\\ &= 50 + 84 + 96 - 105 - 48 - 80\\ &= -3\end{aligned}$$

Expansion along the column with the most zeros.

Input: `[1 0 2; 3 0 1; 4 5 6]` Expansion

Result

$$\det A = 25$$

Steps

- Expansion along column 2, each entry times its cofactor with the sign of its position

$$\begin{aligned}\det A &= \begin{vmatrix}1 & 0 & 2\\ 3 & 0 & 1\\ 4 & 5 & 6\end{vmatrix}\\ &= -5\cdot \begin{vmatrix}1 & 2\\ 3 & 1\end{vmatrix}\\ &= -5\cdot(-5)\\ &= 25\end{aligned}$$

The inverse by Gauss–Jordan elimination.

Input: `[1 2 3; 4 5 6; 7 8 10]` Inverse matrix

Result

$$A^{-1} = \begin{pmatrix} -\frac{2}{3} & -\frac{4}{3} & 1 \\ -\frac{2}{3} & \frac{11}{3} & -2 \\ 1 & -2 & 1 \end{pmatrix}$$

Steps

- Write the identity matrix next to the matrix. Row operations turn the left part into the identity
$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 10 & 0 & 0 & 1 \end{array}\right)$$
- Row operations $R_2 \to R_2 - 4R_1,\ R_3 \to R_3 - 7R_1$
$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & -6 & -11 & -7 & 0 & 1 \end{array}\right)$$
- Row operations $R_3 \to R_3 - 2R_2$
$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$$
- Row operations $R_2 \to R_2 + 6R_3,\ R_1 \to R_1 - 3R_3$
$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -2 & 6 & -3 \\ 0 & -3 & 0 & 2 & -11 & 6 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$$
- Row operations $R_1 \to 3R_1 + 2R_2$
$$\left(\begin{array}{ccc|ccc} 3 & 0 & 0 & -2 & -4 & 3 \\ 0 & -3 & 0 & 2 & -11 & 6 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$$
- Row operations $R_1 : 3,\ R_2 : (-3)$
$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{2}{3} & -\frac{4}{3} & 1 \\ 0 & 1 & 0 & -\frac{2}{3} & \frac{11}{3} & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$$
- The identity on the left, the inverse on the right
$$A^{-1} = \begin{pmatrix} -\frac{2}{3} & -\frac{4}{3} & 1 \\ -\frac{2}{3} & \frac{11}{3} & -2 \\ 1 & -2 & 1 \end{pmatrix}$$

The rank by row reduction.

Input: `[1 2 3; 4 5 6; 7 8 9]` Rank

Result

$$h(A) = 2$$

Steps

- Row operations
$$R_2 \to \frac{1}{3}(R_2 - 4R_1),\ R_3 \to \frac{1}{6}(R_3 - 7R_1)$$
$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{pmatrix}$$
- Row operations $R_3 \to R_3 - R_2$
$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$
- The matrix in row echelon form
$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$
- The rank is the number of non-zero rows of the row echelon form: $h(A) = 2$

The rank of a matrix with a parameter.

Input: `[a 1 1; 1 a 1; 1 1 a]` Rank

Result

$$\begin{cases} h(A) = 3, & a \neq -2,\ a \neq 1 \\ h(A) = 1, & a = 1 \\ h(A) = 2, & a = -2 \end{cases}$$

Steps

- The matrix contains the parameter a – any real number. The rank may depend on its value, so we go through the cases.
- Row operations $R_1 \leftrightarrow R_2$

$$\begin{pmatrix} 1 & a & 1 \\ a & 1 & 1 \\ 1 & 1 & a \end{pmatrix}$$

- Row operations $R_2 \to R_2 - aR_1,\ R_3 \to R_3 - R_1$

$$\begin{pmatrix} 1 & a & 1 \\ 0 & (1-a)(a+1) & 1-a \\ 0 & 1-a & a-1 \end{pmatrix}$$

- We go on with the entry $1 - a$, which can be zero – so we assume $a \neq 1$; the case $a = 1$ is dealt with separately.
- Row operations $R_2 \leftrightarrow R_3$

$$\begin{pmatrix} 1 & a & 1 \\ 0 & 1-a & a-1 \\ 0 & (1-a)(a+1) & 1-a \end{pmatrix}$$

- Row operations $R_3 \to \dfrac{1}{1-a}(R_3 - (a+1)R_2)$

$$\begin{pmatrix} 1 & a & 1 \\ 0 & 1-a & a-1 \\ 0 & 0 & a+2 \end{pmatrix}$$

- We go on with the entry $a + 2$, which can be zero – so we assume $a \neq -2$; the case $a = -2$ is dealt with separately.
- The matrix in row echelon form

$$\begin{pmatrix} 1 & a & 1 \\ 0 & 1-a & a-1 \\ 0 & 0 & a+2 \end{pmatrix}$$

- For $a \neq -2,\ a \neq 1$: $h(A) = 3$
- For $a = 1$: $h(A) = 1$ – the matrix with the value put in

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- For $a = -2$: $h(A) = 2$ – the matrix with the value put in

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

The inverse with a parameter by the adjugate matrix.

Input: `[1 0 a; 0 1 0; 1 0 1]` Adjugate matrix

Result

$$A^{-1} = \begin{pmatrix} -\frac{1}{a-1} & 0 & \frac{a}{a-1} \\ 0 & 1 & 0 \\ \frac{1}{a-1} & 0 & -\frac{1}{a-1} \end{pmatrix}$$

Conditions: $a \neq 1$

Steps

- Rule of Sarrus, the products along the main diagonals minus those along the other ones

$$\begin{aligned} \det A &= \begin{vmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} \\ &= 1 \cdot 1 \cdot 1 + 0 \cdot 0 \cdot 1 + a \cdot 0 \cdot 0 \\ &\quad - a \cdot 1 \cdot 1 - 1 \cdot 0 \cdot 0 - 0 \cdot 0 \cdot 1 \\ &= 1 + 0 + 0 - a - 0 - 0 \\ &= 1 - a \end{aligned}$$

- $\det A = 1 - a$ – the inverse exists exactly when the determinant isn't zero, i.e. for $a \neq 1$
- The cofactors of the entries (with the sign (−1)^(i+j))

$$A_{11} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$A_{12} = -\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = 0$$

$$A_{13} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

$$A_{21} = -\begin{vmatrix} 0 & a \\ 0 & 1 \end{vmatrix} = 0$$

$$A_{22} = \begin{vmatrix} 1 & a \\ 1 & 1 \end{vmatrix} = 1 - a$$

$$A_{23} = -\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 0$$

$$A_{31} = \begin{vmatrix} 0 & a \\ 1 & 0 \end{vmatrix} = -a$$

$$A_{32} = -\begin{vmatrix} 1 & a \\ 0 & 0 \end{vmatrix} = 0$$

$$A_{33} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

- The adjugate is the transpose of the matrix of cofactors; divide it by the determinant

$$A^{-1} = \frac{1}{1-a} \begin{pmatrix} 1 & 0 & -a \\ 0 & 1-a & 0 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{a-1} & 0 & \frac{a}{a-1} \\ 0 & 1 & 0 \\ \frac{1}{a-1} & 0 & -\frac{1}{a-} \end{pmatrix}$$

Product, transpose, multiple and difference of matrices.

Input: `A = [1 2; 3 4]`
`B = [0 1; 1 0]`
`A*B - 2B^T`

Result

$$AB - 2B^T = \begin{pmatrix} 2 & -1 \\ 2 & 3 \end{pmatrix}$$

Steps

- Multiply row by column (sizes $2 \times 2 \cdot 2 \times 2 \to 2 \times 2$)

$$AB = \begin{pmatrix} 1 \cdot 0 + 2 \cdot 1 & 1 \cdot 1 + 2 \cdot 0 \\ 3 \cdot 0 + 4 \cdot 1 & 3 \cdot 1 + 4 \cdot 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$$

- Transpose, the rows become columns

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ B^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- Multiply every entry by the number

$$2B^T = \begin{pmatrix} 2 \cdot 0 & 2 \cdot 1 \\ 2 \cdot 1 & 2 \cdot 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

- Add (subtract) the entries in the same positions

$$AB - 2B^T = \begin{pmatrix} 2 - 0 & 1 - 2 \\ 4 - 2 & 3 - 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 2 & 3 \end{pmatrix}$$

A matrix equation AX = B by the inverse.

Input: `A = [1 2; 3 4]`

`B = [5; 6]`

`AX = B`

Result

$$X = \begin{pmatrix} -4 \\ \frac{9}{2} \end{pmatrix}$$

Steps

- Multiply the equation by the inverse on the side of the unknown: $AX = B \;\Rightarrow\; X = A^{-1}B$
- A 2×2 determinant is the product on the main diagonal minus the product on the other one

  $$\det A = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \cdot 4 - 2 \cdot 3 = -2$$

- $\det A = -2 \neq 0$, the inverse exists.
- For a 2×2 matrix swap the entries on the main diagonal, change the signs of the other two and divide by the determinant

  $$A^{-1} = \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

- Multiply row by column (sizes $2 \times 2 \cdot 2 \times 1 \to 2 \times 1$)

  $$X = A^{-1}B = \begin{pmatrix} (-2) \cdot 5 + 1 \cdot 6 \\ \frac{3}{2} \cdot 5 + (-\frac{1}{2}) \cdot 6 \end{pmatrix} = \begin{pmatrix} -4 \\ \frac{9}{2} \end{pmatrix}$$

The eigenvalues and eigenvectors of a symmetric matrix.

Input: `[2 1; 1 2]` Eigenvalues

Result

$$\lambda_1 = 1, \mathbf{v}_1 = t\begin{pmatrix} 1 \\ -1 \end{pmatrix},\ t \neq 0$$

$$\lambda_2 = 3, \mathbf{v}_2 = t\begin{pmatrix} 1 \\ 1 \end{pmatrix},\ t \neq 0$$

Steps

- The eigenvalues are the roots of the characteristic equation det(A − λE) = 0 (E the identity matrix). We subtract λ on the diagonal

  $$A - \lambda E = \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix}$$

- A 2×2 determinant is the product on the main diagonal minus the product on the other one

  $$\det(A - \lambda E) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)\cdot(2-\lambda) - 1$$

- The characteristic polynomial, factored

  $$\det(A - \lambda E) = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3) = 0$$

- The eigenvalues (a multiple root has more names): $\lambda_1 = 1,\ \lambda_2 = 3$

- The eigenvectors for $\lambda_1 = 1$ are the non-zero solutions of the homogeneous system; the matrix in the reduced row echelon form

  $$(A - \lambda_1 E)\mathbf{v} = \mathbf{o}: \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

- The eigenvectors for $\lambda_1 = 1$ (algebraic multiplicity 1, geometric 1 − the number of free unknowns)

  $$\mathbf{v}_1 = t\begin{pmatrix} 1 \\ -1 \end{pmatrix},\ t \neq 0$$

- The eigenvectors for $\lambda_2 = 3$ are the non-zero solutions of the homogeneous system; the matrix in the reduced row echelon form

  $$(A - \lambda_2 E)\mathbf{v} = \mathbf{o}: \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

- The eigenvectors for $\lambda_2 = 3$ (algebraic multiplicity 1, geometric 1 − the number of free unknowns)

  $$\mathbf{v}_2 = t\begin{pmatrix} 1 \\ 1 \end{pmatrix},\ t \neq 0$$

Irrational eigenvalues by the discriminant.

Input: `[1 2; 3 4]` Eigenvalues

Result

$$\lambda_1 = \frac{5-\sqrt{33}}{2}, \mathbf{v}_1 = t\begin{pmatrix}3+\sqrt{33}\\-6\end{pmatrix},\ t$$

$$\lambda_2 = \frac{5+\sqrt{33}}{2}, \mathbf{v}_2 = t\begin{pmatrix}-3+\sqrt{33}\\6\end{pmatrix},$$

Steps

- The eigenvalues are the roots of the characteristic equation det(A − λE) = 0 (E the identity matrix). We subtract λ on the diagonal

$$A-\lambda E = \begin{pmatrix}1-\lambda & 2\\ 3 & 4-\lambda\end{pmatrix}$$

- A 2×2 determinant is the product on the main diagonal minus the product on the other one

$$\det(A-\lambda E) = \begin{vmatrix}1-\lambda & 2\\ 3 & 4-\lambda\end{vmatrix} = (1-\lambda)\cdot(4-\lambda) - 2$$

- The characteristic polynomial, factored:
$\det(A-\lambda E) = \lambda^2 - 5\lambda - 2 = 0$

- The roots of the quadratic factor by the discriminant

$$\lambda^2 - 5\lambda - 2 = 0:\ D = 33,\ \lambda_{1,2} = \frac{5\pm\sqrt{33}}{2} = \frac{5-\sqrt{3}}{2}$$

- The eigenvalues (a multiple root has more names):
$\lambda_1 = \frac{5-\sqrt{33}}{2},\ \lambda_2 = \frac{5+\sqrt{33}}{2}$

- The eigenvectors for $\lambda_1 = \frac{5-\sqrt{33}}{2}$ are the non-zero solutions of the homogeneous system; the matrix in the reduced row echelon form

$$(A-\lambda_1 E)\mathbf{v} = \mathbf{o}:\ \begin{pmatrix}\frac{-3+\sqrt{33}}{2} & 2\\ 3 & \frac{3+\sqrt{33}}{2}\end{pmatrix} \sim \begin{pmatrix}1 & \frac{3+\sqrt{33}}{6}\\ 0 & 0\end{pmatrix}$$

- The eigenvectors for $\lambda_1 = \frac{5-\sqrt{33}}{2}$ (algebraic multiplicity 1, geometric 1 − the number of free unknowns)

$$\mathbf{v}_1 = t\begin{pmatrix}3+\sqrt{33}\\-6\end{pmatrix},\ t\neq 0$$

- The eigenvectors for $\lambda_2 = \frac{5+\sqrt{33}}{2}$ are the non-zero solutions of the homogeneous system; the matrix in the reduced row echelon form

$$(A-\lambda_2 E)\mathbf{v} = \mathbf{o}:\ \begin{pmatrix}\frac{-3-\sqrt{33}}{2} & 2\\ 3 & \frac{3-\sqrt{33}}{2}\end{pmatrix} \sim \begin{pmatrix}1 & \frac{3-\sqrt{33}}{6}\\ 0 & 0\end{pmatrix}$$

- The eigenvectors for $\lambda_2 = \frac{5+\sqrt{33}}{2}$ (algebraic multiplicity 1, geometric 1 − the number of free unknowns)

$$\mathbf{v}_2 = t\begin{pmatrix}-3+\sqrt{33}\\6\end{pmatrix},\ t\neq 0$$

The diagonalization A = PDP⁻¹.

Input: `[2 0 0; 0 3 4; 0 4 9]` Diagonalization

Result

$$A = PDP^{-1},\ P = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \\ -1 & 0 & 2 \end{pmatrix},\ D$$

Steps

- The eigenvalues are the roots of the characteristic equation det(A − λE) = 0 (E the identity matrix). We subtract λ on the diagonal

$$A - \lambda E = \begin{pmatrix} 2-\lambda & 0 & 0 \\ 0 & 3-\lambda & 4 \\ 0 & 4 & 9-\lambda \end{pmatrix}$$

- Rule of Sarrus, the products along the main diagonals minus those along the other ones

$$\begin{aligned} \det(A-\lambda E) &= \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 3-\lambda & 4 \\ 0 & 4 & 9-\lambda \end{vmatrix} \\ &= (2-\lambda)\cdot(3-\lambda)\cdot(9-\lambda) + 0\cdot 4\cdot 0 + \\ &\quad - 0\cdot(3-\lambda)\cdot 0 - (2-\lambda)\cdot 4\cdot 4 - 0\cdot \\ &= \left(-\lambda^3 + 14\lambda^2 - 51\lambda + 54\right) + 0 + 0 - 0 \\ &= -\lambda^3 + 14\lambda^2 - 35\lambda + 22 \end{aligned}$$

- The characteristic polynomial, factored

$$\det(A-\lambda E) = -\lambda^3 + 14\lambda^2 - 35\lambda + 22 = -(\lambda-1)(\lambda$$

- The eigenvalues (a multiple root has more names):

$$\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 11$$

- The eigenvectors for $\lambda_1 = 1$ are the non-zero solutions of the homogeneous system; the matrix in the reduced row echelon form

$$(A-\lambda_1 E)\mathbf{v} = \mathbf{o}: \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

- The eigenvectors for $\lambda_1 = 1$ (algebraic multiplicity 1, geometric 1 – the number of free unknowns)

$$\mathbf{v}_1 = t\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix},\ t \neq 0$$

- The eigenvectors for $\lambda_2 = 2$ are the non-zero solutions of the homogeneous system; the matrix in the reduced row echelon form

$$(A-\lambda_2 E)\mathbf{v} = \mathbf{o}: \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 4 & 7 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

- The eigenvectors for $\lambda_2 = 2$ (algebraic multiplicity 1, geometric 1 – the number of free unknowns)

$$\mathbf{v}_2 = t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ t \neq 0$$

- The eigenvectors for $\lambda_3 = 11$ are the non-zero solutions of the homogeneous system; the matrix in the reduced row echelon form

$$(A-\lambda_3 E)\mathbf{v} = \mathbf{o}: \begin{pmatrix} -9 & 0 & 0 \\ 0 & -8 & 4 \\ 0 & 4 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

- The eigenvectors for $\lambda_3 = 11$ (algebraic multiplicity 1, geometric 1 – the number of free unknowns)

$$\mathbf{v}_3 = t\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix},\ t \neq 0$$

- The matrix has as many linearly independent eigenvectors as rows, so it is diagonalizable: P has the eigenvectors as its columns, D the eigenvalues on its diagonal in the same order, and $A = PDP^{-1}$

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \\ -1 & 0 & 2 \end{pmatrix},\ D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 11 \end{pmatrix},\ P^{-1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

- The matrix is symmetric: its eigenvalues are real and the eigenvectors of different eigenvalues are orthogonal. Normalized, they make P orthogonal, $P^{-1} = P^T$

A matrix that can't be diagonalized.

Input: `[1 1 0; 0 1 0; 0 0 2]` Diagonalization

Result

The matrix can't be diagonalized – it hasn't enough linearly independent eigenvectors.

Steps

- The eigenvalues are the roots of the characteristic equation det(A − λE) = 0 (E the identity matrix). We subtract λ on the diagonal

$$A - \lambda E = \begin{pmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix}$$

- Rule of Sarrus, the products along the main diagonals minus those along the other ones

$$\begin{aligned} \det(A-\lambda E) &= \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} \\ &= (1-\lambda)\cdot(1-\lambda)\cdot(2-\lambda) + 1\cdot 0\cdot 0 + \\ &\quad - 0\cdot(1-\lambda)\cdot 0 - (1-\lambda)\cdot 0\cdot 0 - 1\cdot \\ &= \left(-\lambda^3 + 4\lambda^2 - 5\lambda + 2\right) + 0 + 0 - 0 - \\ &= -\lambda^3 + 4\lambda^2 - 5\lambda + 2 \end{aligned}$$

- The characteristic polynomial, factored

$$\det(A-\lambda E) = -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = -(\lambda-1)^2(\lambda -$$

- The eigenvalues (a multiple root has more names):

$$\lambda_1 = \lambda_2 = 1,\ \lambda_3 = 2$$

- The eigenvectors for $\lambda_1 = 1$ are the non-zero solutions of the homogeneous system; the matrix in the reduced row echelon form

$$(A-\lambda_1 E)\mathbf{v} = \mathbf{o}: \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

- The eigenvectors for $\lambda_1 = 1$ (algebraic multiplicity 2, geometric 1 − the number of free unknowns)

$$\mathbf{v}_1 = t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ t \neq 0$$

- The eigenvectors for $\lambda_3 = 2$ are the non-zero solutions of the homogeneous system; the matrix in the reduced row echelon form

$$(A-\lambda_3 E)\mathbf{v} = \mathbf{o}: \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- The eigenvectors for $\lambda_3 = 2$ (algebraic multiplicity 1, geometric 1 − the number of free unknowns)

$$\mathbf{v}_3 = t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\ t \neq 0$$

- For $\lambda = 1$ the geometric multiplicity 1 is less than the algebraic one 2 − the matrix hasn't enough linearly independent eigenvectors, so it can't be diagonalized.

Eigenvalues of a matrix with a parameter – the special values and their cases.

Input: `[1 1 0; 0 a 0; 0 0 2]` Eigenvalues

Result

$$\begin{cases} \lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = a, & a \neq 1,\ a \\ \lambda_1 = \lambda_2 = 1,\ \lambda_3 = 2, & a = 1 \\ \lambda_1 = 1,\ \lambda_2 = \lambda_3 = 2, & a = 2 \end{cases}$$

Steps

- The eigenvalues are the roots of the characteristic equation det(A − λE) = 0 (E the identity matrix). We subtract λ on the diagonal

$$A - \lambda E = \begin{pmatrix} 1-\lambda & 1 & 0 \\ 0 & a-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix}$$

- Rule of Sarrus, the products along the main diagonals minus those along the other ones

$$\begin{aligned} \det(A - \lambda E) &= \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & a-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} \\ &= (1-\lambda)\cdot(a-\lambda)\cdot(2-\lambda) + 1\cdot 0\cdot 0 + \\ &\quad - 0\cdot(a-\lambda)\cdot 0 - (1-\lambda)\cdot 0\cdot 0 - 1\cdot \\ &= \left(-\lambda^3 + 3\lambda^2 + a\lambda^2 - 2\lambda - 3a\lambda + 2a\right) \cdot \\ &= -\lambda^3 + 3\lambda^2 + a\lambda^2 - 2\lambda - 3a\lambda + 2a \end{aligned}$$

- The characteristic polynomial, factored

$$\det(A - \lambda E) = -\lambda^3 + 3\lambda^2 + a\lambda^2 - 2\lambda - 3a\lambda + 2a =$$

- The eigenvalues (a multiple root has more names):
$\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = a$

- Special values of the parameter – two eigenvalues coincide there, or the eigenvectors' computation would divide by zero: $a = 1,\ a = 2$. They are cases of their own ().

$$a = 1:\ \lambda_1 = \lambda_2 = 1,\ \lambda_3 = 2, \quad a = 2:\ \lambda_1 = 1,\ \lambda_2 =$$

- The general case $a \neq 1,\ a \neq 2$: the eigenvalues are distinct, so the matrix has as many linearly independent eigenvectors as rows and is diagonalizable.

- The eigenvectors for $\lambda_1 = 1$ are the non-zero solutions of the homogeneous system; the matrix in the reduced row echelon form

$$(A - \lambda_1 E)\mathbf{v} = \mathbf{o}:\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & a-1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

- The eigenvectors for $\lambda_1 = 1$ (algebraic multiplicity 1, geometric 1 – the number of free unknowns)

$$\mathbf{v}_1 = t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ t \neq 0$$

- The eigenvectors for $\lambda_2 = 2$ are the non-zero solutions of the homogeneous system; the matrix in the reduced row echelon form

$$(A - \lambda_2 E)\mathbf{v} = \mathbf{o}:\ \begin{pmatrix} -1 & 1 & 0 \\ 0 & a-2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- The eigenvectors for $\lambda_2 = 2$ (algebraic multiplicity 1, geometric 1 – the number of free unknowns)

$$\mathbf{v}_2 = t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\ t \neq 0$$

- The eigenvectors for $\lambda_3 = a$ are the non-zero solutions of the homogeneous system; the matrix in the reduced row echelon form

$$(A - \lambda_3 E)\mathbf{v} = \mathbf{o}:\ \begin{pmatrix} 1-a & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2-a \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{1}{1-a} \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

- The eigenvectors for $\lambda_3 = a$ (algebraic multiplicity 1, geometric 1 – the number of free unknowns)

$$\mathbf{v}_3 = t\begin{pmatrix} 1 \\ a-1 \\ 0 \end{pmatrix},\ t \neq 0$$

- The case $a = 1$ – put into the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

- The characteristic polynomial, factored

$$\det(A - \lambda E) = -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = -(\lambda - 1)^2(\lambda -$$

- The eigenvalues (a multiple root has more names):
  $\lambda_1 = \lambda_2 = 1,\ \lambda_3 = 2$
- The eigenvectors for $\lambda_1 = 1$ are the non-zero solutions of the homogeneous system; the matrix in the reduced row echelon form
  $$(A - \lambda_1 E)\mathbf{v} = \mathbf{o}: \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
- The eigenvectors for $\lambda_1 = 1$ (algebraic multiplicity 2, geometric 1 – the number of free unknowns)
  $$\mathbf{v}_1 = t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ t \neq 0$$
- The eigenvectors for $\lambda_3 = 2$ are the non-zero solutions of the homogeneous system; the matrix in the reduced row echelon form
  $$(A - \lambda_3 E)\mathbf{v} = \mathbf{o}: \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
- The eigenvectors for $\lambda_3 = 2$ (algebraic multiplicity 1, geometric 1 – the number of free unknowns)
  $$\mathbf{v}_3 = t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\ t \neq 0$$
- For $\lambda = 1$ the geometric multiplicity 1 is less than the algebraic one 2 – the matrix hasn't enough linearly independent eigenvectors, so it can't be diagonalized.
- The case $a = 2$ – put into the matrix
  $$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$
- The characteristic polynomial, factored
  $$\det(A - \lambda E) = -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = -(\lambda - 1)(\lambda -$$
- The eigenvalues (a multiple root has more names):
  $\lambda_1 = 1,\ \lambda_2 = \lambda_3 = 2$
- The eigenvectors for $\lambda_1 = 1$ are the non-zero solutions of the homogeneous system; the matrix in the reduced row echelon form
  $$(A - \lambda_1 E)\mathbf{v} = \mathbf{o}: \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
- The eigenvectors for $\lambda_1 = 1$ (algebraic multiplicity 1, geometric 1 – the number of free unknowns)
  $$\mathbf{v}_1 = t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ t \neq 0$$
- The eigenvectors for $\lambda_2 = 2$ are the non-zero solutions of the homogeneous system; the matrix in the reduced row echelon form
  $$(A - \lambda_2 E)\mathbf{v} = \mathbf{o}: \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
- The eigenvectors for $\lambda_2 = 2$ (algebraic multiplicity 2, geometric 2 – the number of free unknowns)
  $$\mathbf{v}_2 = t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\ (t, s) \neq (0, 0)$$

For which values of the parameter the matrix is diagonalizable.

Input: `[0 1; a 0]` Diagonalization

Result

The matrix is diagonalizable for: $a \neq 0$

Steps

- The eigenvalues are the roots of the characteristic equation det(A − λE) = 0 (E the identity matrix). We subtract λ on the diagonal

$$A - \lambda E = \begin{pmatrix} -\lambda & 1 \\ a & -\lambda \end{pmatrix}$$

- A 2×2 determinant is the product on the main diagonal minus the product on the other one

$$\det(A - \lambda E) = \begin{vmatrix} -\lambda & 1 \\ a & -\lambda \end{vmatrix} = (-\lambda)\cdot(-\lambda) - 1 \cdot a = \lambda^2$$

- The characteristic polynomial, factored: $\det(A - \lambda E) = \lambda^2 - a = 0$
- The roots of the quadratic factor by the discriminant

$$\lambda^2 - a = 0:\ D = 4a,\ \lambda_{1,2} = \frac{\pm\sqrt{4a}}{2} = -\sqrt{a},\ \sqrt{a}$$

- The eigenvalues (a multiple root has more names): $\lambda_1 = -\sqrt{a},\ \lambda_2 = \sqrt{a}$
- Special values of the parameter – two eigenvalues coincide there, or the eigenvectors' computation would divide by zero: $a = 0$. They are cases of their own ($a = 0:\ \lambda_1 = \lambda_2 = 0$).
- For $a \in (-\infty,\ 0)$ the discriminant is negative and the eigenvalues are complex conjugates (diagonalizable over the complex numbers only).
- The general case $a \neq 0$: the eigenvalues are distinct, so the matrix has as many linearly independent eigenvectors as rows and is diagonalizable.
- The case $a = 0$ – put into the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

- The characteristic polynomial, factored: $\det(A - \lambda E) = \lambda^2 = 0$
- The eigenvalues (a multiple root has more names): $\lambda_1 = \lambda_2 = 0$
- The eigenvectors for $\lambda_1 = 0$ are the non-zero solutions of the homogeneous system; the matrix in the reduced row echelon form

$$(A - \lambda_1 E)\mathbf{v} = \mathbf{o}:\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

- The eigenvectors for $\lambda_1 = 0$ (algebraic multiplicity 2, geometric 1 – the number of free unknowns)

$$\mathbf{v}_1 = t\begin{pmatrix} 1 \\ 0 \end{pmatrix},\ t \neq 0$$

- For $\lambda = 0$ the geometric multiplicity 1 is less than the algebraic one 2 – the matrix hasn't enough linearly independent eigenvectors, so it can't be diagonalized.

The singular value decomposition of a 2×3 matrix.

Input: `[3 2 2; 2 3 -2]` Singular value decomposition (SVD)

Result

$$A = U\Sigma V^T,\ U = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix},\ \Sigma$$

Steps

- The singular value decomposition $A = U\Sigma V^T$: the matrix has more columns than rows, so we start with the smaller AA^T – its eigenvalues are the squares of the singular values, its eigenvectors make up U. We compute AA^T
- Multiply row by column (sizes $2 \times 3 \cdot 3 \times 2 \to 2 \times 2$)
$$AA^T = \begin{pmatrix} 3\cdot 3+2\cdot 2+2\cdot 2 & 3\cdot 2+2\cdot 3+2\cdot(- \\ 2\cdot 3+3\cdot 2+(-2)\cdot 2 & 2\cdot 2+3\cdot 3+(-2)\cdot \end{pmatrix}$$
- The eigenvalues are the roots of the characteristic equation det(A − λE) = 0 (E the identity matrix). We subtract λ on the diagonal
$$AA^T - \lambda E = \begin{pmatrix} 17-\lambda & 8 \\ 8 & 17-\lambda \end{pmatrix}$$
- A 2×2 determinant is the product on the main diagonal minus the product on the other one
$$\det\left(AA^T - \lambda E\right) = \begin{vmatrix} 17-\lambda & 8 \\ 8 & 17-\lambda \end{vmatrix} = (17-\lambda)\cdot(17$$
- The characteristic polynomial, factored
$$\det\left(AA^T - \lambda E\right) = \lambda^2 - 34\lambda + 225 = (\lambda - 9)(\lambda - 25)$$
- The eigenvalues (a multiple root has more names):
$\lambda_1 = 25,\ \lambda_2 = 9$
- The eigenvectors for $\lambda_1 = 25$ are the non-zero solutions of the homogeneous system; the matrix in the reduced row echelon form
$$\left(AA^T - \lambda_1 E\right)\mathbf{v} = \mathbf{o}:\ \begin{pmatrix} -8 & 8 \\ 8 & -8 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$
- The eigenvectors for $\lambda_1 = 25$ (algebraic multiplicity 1, geometric 1 – the number of free unknowns)
$$\mathbf{v}_1 = t\begin{pmatrix} 1 \\ 1 \end{pmatrix},\ t \neq 0$$
- The eigenvectors for $\lambda_2 = 9$ are the non-zero solutions of the homogeneous system; the matrix in the reduced row echelon form
$$\left(AA^T - \lambda_2 E\right)\mathbf{v} = \mathbf{o}:\ \begin{pmatrix} 8 & 8 \\ 8 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$
- The eigenvectors for $\lambda_2 = 9$ (algebraic multiplicity 1, geometric 1 – the number of free unknowns)
$$\mathbf{v}_2 = t\begin{pmatrix} 1 \\ -1 \end{pmatrix},\ t \neq 0$$
- The singular values are the square roots of the eigenvalues, in descending order:
$\sigma_1 = \sqrt{25} = 5,\ \sigma_2 = \sqrt{9} = 3$. Σ has the size of A
$$\Sigma = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$$
- The eigenvectors normalized (divided by their magnitude); a multiple eigenvalue's first made orthogonal by the Gram–Schmidt process. They are the columns of
$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix},\ \mathbf{u}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix}$$
$$U = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}$$
- The columns of V for the non-zero singular values, $\mathbf{v}_i = A^T\mathbf{u}_i/\sigma_i$
$$\mathbf{v}_1 = \frac{1}{\sigma_1}A^T\mathbf{u}_1 = \frac{1}{5}\begin{pmatrix} \frac{5\sqrt{2}}{2} \\ \frac{5\sqrt{2}}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix},\ \mathbf{v}_2 = \frac{1}{\sigma_2}A^T\mathbf{u}_2$$
- The missing columns of V complete an orthonormal basis – normalized solutions of $A\mathbf{x} = \mathbf{o}$, orthogonal to the columns before
$$\mathbf{v}_3 = \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$$
- The singular value decomposition
$$A = U\Sigma V^T = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{6} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{6} \\ 0 & \frac{2\sqrt{2}}{3} \end{pmatrix}$$

The pseudoinverse of a matrix with independent columns.

Input: `[1 0; 0 1; 1 1]` Pseudoinverse

Result

$$A^{+} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

Steps

- The matrix's columns are linearly independent (rank 2 = the number of columns), so $A^T A$ is invertible and the pseudoinverse is given by the formula:
- $A^{+} = \left(A^T A\right)^{-1} A^T$
- Multiply row by column (sizes $2 \times 3 \cdot 3 \times 2 \to 2 \times 2$)

$$A^T A = \begin{pmatrix} 1 \cdot 1 + 0 \cdot 0 + 1 \cdot 1 & 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 1 \\ 0 \cdot 1 + 1 \cdot 0 + 1 \cdot 1 & 0 \cdot 0 + 1 \cdot 1 + 1 \cdot 1 \end{pmatrix} = \begin{pmatrix}$$

- A 2×2 determinant is the product on the main diagonal minus the product on the other one

$$\det A^T A = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 2 \cdot 2 - 1 \cdot 1 = 3$$

- $\det A^T A = 3 \neq 0$, the inverse exists.
- For a 2×2 matrix swap the entries on the main diagonal, change the signs of the other two and divide by the determinant

$$\left(A^T A\right)^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

- Multiply row by column (sizes $2 \times 2 \cdot 2 \times 3 \to 2 \times 3$)

$$A^{+} = \begin{pmatrix} \frac{2}{3} \cdot 1 + (-\frac{1}{3}) \cdot 0 & \frac{2}{3} \cdot 0 + (-\frac{1}{3}) \cdot 1 & \frac{2}{3} \cdot 1 + (-\frac{1}{3} \\ (-\frac{1}{3}) \cdot 1 + \frac{2}{3} \cdot 0 & (-\frac{1}{3}) \cdot 0 + \frac{2}{3} \cdot 1 & (-\frac{1}{3}) \cdot 1 + \end{pmatrix}$$

The pseudoinverse of a singular matrix from its singular value decomposition.

Input: `A = [1 2; 2 4]`

A^+

Result

$$A^+ = \begin{pmatrix} \frac{1}{25} & \frac{2}{25} \\ \frac{2}{25} & \frac{4}{25} \end{pmatrix}$$

Steps

- The matrix hasn't full rank (rank 1), so the pseudoinverse comes from the singular value decomposition: $A^+ = V\Sigma^+U^T$
- The singular value decomposition $A = U\Sigma V^T$: the eigenvalues of A^TA are the squares of the singular values, its eigenvectors make up V. We compute A^TA
- Multiply row by column (sizes $2\times 2 \cdot 2\times 2 \to 2\times 2$)
  $A^TA = \begin{pmatrix} 1\cdot 1+2\cdot 2 & 1\cdot 2+2\cdot 4 \\ 2\cdot 1+4\cdot 2 & 2\cdot 2+4\cdot 4 \end{pmatrix} = \begin{pmatrix} 5 & 10 \\ 10 & 20 \end{pmatrix}$
- The eigenvalues are the roots of the characteristic equation det(A − λE) = 0 (E the identity matrix). We subtract λ on the diagonal
  $A^TA - \lambda E = \begin{pmatrix} 5-\lambda & 10 \\ 10 & 20-\lambda \end{pmatrix}$
- A 2×2 determinant is the product on the main diagonal minus the product on the other one
  $\det\left(A^TA-\lambda E\right) = \begin{vmatrix} 5-\lambda & 10 \\ 10 & 20-\lambda \end{vmatrix} = (5-\lambda)\cdot(20-$
- The characteristic polynomial, factored
  $\det\left(A^TA-\lambda E\right) = \lambda^2 - 25\lambda = \lambda(\lambda-25) = 0$
- The eigenvalues (a multiple root has more names):
  $\lambda_1 = 25,\ \lambda_2 = 0$
- The eigenvectors for $\lambda_1 = 25$ are the non-zero solutions of the homogeneous system; the matrix in the reduced row echelon form
  $\left(A^TA-\lambda_1 E\right)\mathbf{v} = \mathbf{o}:\ \begin{pmatrix} -20 & 10 \\ 10 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{pmatrix}$
- The eigenvectors for $\lambda_1 = 25$ (algebraic multiplicity 1, geometric 1 − the number of free unknowns)
  $\mathbf{v}_1 = t\begin{pmatrix} 1 \\ 2 \end{pmatrix},\ t \neq 0$
- The eigenvectors for $\lambda_2 = 0$ are the non-zero solutions of the homogeneous system; the matrix in the reduced row echelon form
  $\left(A^TA-\lambda_2 E\right)\mathbf{v} = \mathbf{o}:\ \begin{pmatrix} 5 & 10 \\ 10 & 20 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$
- The eigenvectors for $\lambda_2 = 0$ (algebraic multiplicity 1, geometric 1 − the number of free unknowns)
  $\mathbf{v}_2 = t\begin{pmatrix} 2 \\ -1 \end{pmatrix},\ t \neq 0$
- The singular values are the square roots of the eigenvalues, in descending order:
  $\sigma_1 = \sqrt{25} = 5,\ \sigma_2 = 0$. Σ has the size of A
  $\Sigma = \begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix}$
- The eigenvectors normalized (divided by their magnitude); a multiple eigenvalue's first made orthogonal by the Gram–Schmidt process. They are the columns of
  $\mathbf{v}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} \end{pmatrix},\ \mathbf{v}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{2\sqrt{5}}{5} \\ -\frac{\sqrt{5}}{5} \end{pmatrix}$
  $V = \begin{pmatrix} \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} & -\frac{\sqrt{5}}{5} \end{pmatrix}$
- The columns of U for the non-zero singular values, $\mathbf{u}_i = A\mathbf{v}_i/\sigma_i$
  $\mathbf{u}_1 = \frac{1}{\sigma_1}A\mathbf{v}_1 = \frac{1}{5}\begin{pmatrix} \sqrt{5} \\ 2\sqrt{5} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} \end{pmatrix}$
- The missing columns of U complete an orthonormal basis – normalized solutions of $A^T\mathbf{x} = \mathbf{o}$, orthogonal to the columns before
  $\mathbf{u}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{2\sqrt{5}}{5} \\ -\frac{\sqrt{5}}{5} \end{pmatrix}$
- The singular value decomposition
  $A = U\Sigma V^T = \begin{pmatrix} \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} & -\frac{\sqrt{5}}{5} \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} & -\frac{\sqrt{5}}{5} \end{pmatrix}$
- Σ^+ is Σ transposed with the non-zero singular values inverted
  $\Sigma^+ = \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & 0 \end{pmatrix}$
- The pseudoinverse
  $A^+ = V\Sigma^+U^T = \begin{pmatrix} \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} & -\frac{\sqrt{5}}{5} \end{pmatrix}\begin{pmatrix} \frac{1}{5} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{5}}{5} & \frac{2\sqrt{}}{5} \\ \frac{2\sqrt{5}}{5} & -\frac{\sqrt{}}{} \end{pmatrix}$

The pseudoinverse from the rank factorization A = BC.

Input: `[1 2 3; 4 5 6; 7 8 9]` Pseudoinverse

Result

$$A^{+} = \begin{pmatrix} -\frac{23}{36} & -\frac{1}{6} & \frac{11}{36} \\ -\frac{1}{18} & 0 & \frac{1}{18} \\ \frac{19}{36} & \frac{1}{6} & -\frac{7}{36} \end{pmatrix}$$

Steps

- The matrix hasn't full rank (rank 2) and its singular values would be nested radicals, so the pseudoinverse is computed exactly from the rank factorization A = BC – B the linearly independent columns of A, C the non-zero rows of its reduced row echelon form.
- We reduce the matrix to the reduced row echelon form; the pivots are in the columns $1, 2$, B consists of these columns of A and C of the non-zero rows of the reduced matrix

  $A \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$
- $A^{+} = C^{T}\left(CC^{T}\right)^{-1}\left(B^{T}B\right)^{-1}B^{T}$
- Multiply row by column (sizes $2 \times 3 \cdot 3 \times 2 \to 2 \times 2$)

  $CC^{T} = \begin{pmatrix} 1 \cdot 1 + 0 \cdot 0 + (-1) \cdot (-1) & 1 \cdot 0 + 0 \cdot 1 + (- \\ 0 \cdot 1 + 1 \cdot 0 + 2 \cdot (-1) & 0 \cdot 0 + 1 \cdot 1 + 2 \end{pmatrix}$
- A 2×2 determinant is the product on the main diagonal minus the product on the other one

  $\det CC^{T} = \begin{vmatrix} 2 & -2 \\ -2 & 5 \end{vmatrix} = 2 \cdot 5 - (-2) \cdot (-2) = 6$
- $\det CC^{T} = 6 \neq 0$, the inverse exists.
- For a 2×2 matrix swap the entries on the main diagonal, change the signs of the other two and divide by the determinant

  $\left(CC^{T}\right)^{-1} = \frac{1}{6}\begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} \frac{5}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$
- Multiply row by column (sizes $2 \times 3 \cdot 3 \times 2 \to 2 \times 2$)

  $B^{T}B = \begin{pmatrix} 1 \cdot 1 + 4 \cdot 4 + 7 \cdot 7 & 1 \cdot 2 + 4 \cdot 5 + 7 \cdot 8 \\ 2 \cdot 1 + 5 \cdot 4 + 8 \cdot 7 & 2 \cdot 2 + 5 \cdot 5 + 8 \cdot 8 \end{pmatrix} = \left($
- A 2×2 determinant is the product on the main diagonal minus the product on the other one

  $\det B^{T}B = \begin{vmatrix} 66 & 78 \\ 78 & 93 \end{vmatrix} = 66 \cdot 93 - 78 \cdot 78 = 54$
- $\det B^{T}B = 54 \neq 0$, the inverse exists.
- For a 2×2 matrix swap the entries on the main diagonal, change the signs of the other two and divide by the determinant

  $\left(B^{T}B\right)^{-1} = \frac{1}{54}\begin{pmatrix} 93 & -78 \\ -78 & 66 \end{pmatrix} = \begin{pmatrix} \frac{31}{18} & -\frac{13}{9} \\ -\frac{13}{9} & \frac{11}{9} \end{pmatrix}$
- The pseudoinverse

  $A^{+} = C^{T}\left(CC^{T}\right)^{-1}\left(B^{T}B\right)^{-1}B^{T} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} \frac{5}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$

A Jordan form with two blocks of one eigenvalue.

Input: `[1 1 0; -1 3 0; -1 1 2]` Jordan form

Result

$$A = PJP^{-1},\ P = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix},\ J$$

Steps

- The eigenvalues are the roots of the characteristic equation det(A − λE) = 0 (E the identity matrix). We subtract λ on the diagonal

$$A - \lambda E = \begin{pmatrix} 1-\lambda & 1 & 0 \\ -1 & 3-\lambda & 0 \\ -1 & 1 & 2-\lambda \end{pmatrix}$$

- Rule of Sarrus, the products along the main diagonals minus those along the other ones

$$\begin{aligned} \det(A-\lambda E) &= \begin{vmatrix} 1-\lambda & 1 & 0 \\ -1 & 3-\lambda & 0 \\ -1 & 1 & 2-\lambda \end{vmatrix} \\ &= (1-\lambda)\cdot(3-\lambda)\cdot(2-\lambda) + 1\cdot 0\cdot(-1 \\ &\quad - 0\cdot(3-\lambda)\cdot(-1) - (1-\lambda)\cdot 0\cdot 1 - \\ &= \left(-\lambda^3 + 6\lambda^2 - 11\lambda + 6\right) + 0 + 0 - 0 - \\ &= -\lambda^3 + 6\lambda^2 - 12\lambda + 8 \end{aligned}$$

- The characteristic polynomial, factored

$$\det(A-\lambda E) = -\lambda^3 + 6\lambda^2 - 12\lambda + 8 = -(\lambda-2)^3 =$$

- The eigenvalues (a multiple root has more names):

$$\lambda_1 = \lambda_2 = \lambda_3 = 2$$

- For $\lambda = 2$ (algebraic multiplicity 3) we compute the powers of $A - \lambda E$ and their ranks h until the rank has dropped by the algebraic multiplicity

$$A - 2E = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, h = 1$$

$$(A-2E)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad h = 0$$

- There are as many Jordan blocks for $\lambda = 2$ as independent eigenvectors: $n - h(A-\lambda E)$ = 2. The number of blocks of size at least k is $h_{k-1} - h_k$, so the block sizes are $2, 1$: $J_2(2),\ J_1(2)$

- A chain of length 2 for $\lambda = 2$: we choose its top vector in the kernel of $(A-\lambda E)^k$ outside the kernel of the lower power and the span of the previous chains, and multiply it by $A - \lambda E$ down to an eigenvector

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ \mathbf{v}_1 = (A-2E)\mathbf{v}_2 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

- A block of size 1 for $\lambda = 2$ gives an eigenvector

$$\mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

- The Jordan form: the columns of P are the chains' vectors (each from its eigenvector up), J has the eigenvalues on its diagonal and ones above it inside the blocks; $A = PJP^{-1}$

$$P = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix},\ J = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},\ P^{-1} = \begin{pmatrix} 0 & 0 \\ 1 & - \\ 0 & 1 \end{pmatrix}$$

The power A^n through the Jordan form.

Input: `[3 1; -1 1]` Power A^n

Result

$$A^n = \begin{pmatrix} 2^{n-1}(n+2) & n\cdot 2^{n-1} \\ -n\cdot 2^{n-1} & -2^{n-1}(n-2) \end{pmatrix}$$

Steps

- The eigenvalues are the roots of the characteristic equation det(A − λE) = 0 (E the identity matrix). We subtract λ on the diagonal

$$A-\lambda E = \begin{pmatrix} 3-\lambda & 1 \\ -1 & 1-\lambda \end{pmatrix}$$

- A 2×2 determinant is the product on the main diagonal minus the product on the other one

$$\det(A-\lambda E) = \begin{vmatrix} 3-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = (3-\lambda)\cdot(1-\lambda) - 1$$

- The characteristic polynomial, factored: $\det(A-\lambda E) = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2 = 0$
- The eigenvalues (a multiple root has more names): $\lambda_1 = \lambda_2 = 2$
- For $\lambda = 2$ (algebraic multiplicity 2) we compute the powers of $A-\lambda E$ and their ranks h until the rank has dropped by the algebraic multiplicity

$$A-2E = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, h = 1$$

$$(A-2E)^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad h = 0$$

- There are as many Jordan blocks for $\lambda = 2$ as independent eigenvectors: $n - h(A-\lambda E)$ = 1. The number of blocks of size at least k is $h_{k-1} - h_k$, so the block sizes are 2: $J_2(2)$
- A chain of length 2 for $\lambda = 2$: we choose its top vector in the kernel of $(A-\lambda E)^k$ outside the kernel of the lower power and the span of the previous chains, and multiply it by $A-\lambda E$ down to an eigenvector

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \mathbf{v}_1 = (A-2E)\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

- The Jordan form: the columns of P are the chains' vectors (each from its eigenvector up), J has the eigenvalues on its diagonal and ones above it inside the blocks; $A = PJP^{-1}$

$$P = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix},\ J = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix},\ P^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

- We compute the power through the Jordan form: $A^n = PJ^nP^{-1}$. A block's power comes from the binomial theorem – the element d places above the diagonal is $\binom{n}{d}\lambda^{n-d}$

$$J^n = \begin{pmatrix} 2^n & n\cdot 2^{n-1} \\ 0 & 2^n \end{pmatrix}$$

- We multiply the matrices

$$A^n = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} J^n \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2^{n-1}(n+2) & n\cdot \\ -n\cdot 2^{n-1} & -2^{n-1} \end{pmatrix}$$

The exponential e^{At} – the solution of the system x' = Ax.

Input: `A = [0 1; -2 -3]`

`e^(At)`

Result

$$e^{At} = \begin{pmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ 2(e^{-2t} - e^{-t}) & 2e^{-2t} - e^{-t} \end{pmatrix},$$

Steps

- The eigenvalues are the roots of the characteristic equation det(A − λE) = 0 (E the identity matrix). We subtract λ on the diagonal

  $A - \lambda E = \begin{pmatrix} -\lambda & 1 \\ -2 & -\lambda - 3 \end{pmatrix}$

- A 2×2 determinant is the product on the main diagonal minus the product on the other one

  $\det(A - \lambda E) = \begin{vmatrix} -\lambda & 1 \\ -2 & -\lambda - 3 \end{vmatrix} = (-\lambda) \cdot (-\lambda - 3) - 1$

- The characteristic polynomial, factored

  $\det(A - \lambda E) = \lambda^2 + 3\lambda + 2 = (\lambda + 2)(\lambda + 1) = 0$

- The eigenvalues (a multiple root has more names):

  $\lambda_1 = -2,\ \lambda_2 = -1$

- A block of size 1 for $\lambda = -2$ gives an eigenvector

  $(A + 2E)\mathbf{v} = \mathbf{o}:\ \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{pmatrix},\ \mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

- A block of size 1 for $\lambda = -1$ gives an eigenvector

  $(A + E)\mathbf{v} = \mathbf{o}:\ \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},\ \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

- All the Jordan blocks are of size 1, so J is diagonal and the matrix is diagonalizable. The columns of P are the eigenvectors and $A = PJP^{-1}$

  $P = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix},\ J = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix},\ P^{-1} = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}$

- We compute the exponential through the Jordan form: $e^{At} = Pe^{Jt}P^{-1}$. In a block, the element d places above the diagonal is $e^{\lambda t}\frac{t^d}{d!}$

  $e^{Jt} = \begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{pmatrix}$

- We multiply the matrices

  $e^{At} = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} e^{Jt} \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2e^{-t} - e^{-2t} & e \\ 2(e^{-2t} - e^{-t}) & 2 \end{pmatrix}$

- For t = 1

  $e^{A} = \begin{pmatrix} 2e^{-1} - e^{-2} & e^{-1} - e^{-2} \\ 2(e^{-2} - e^{-1}) & 2e^{-2} - e^{-1} \end{pmatrix}$

The minimal polynomial, the Cayley–Hamilton theorem and the inverse from it.

Input: `[2 1 0; 0 2 0; 0 0 3]` Minimal polynomial

Result

$$m(\lambda) = (\lambda-2)^2(\lambda-3) = \lambda^3 - 7\lambda^2 +$$

Steps

- The eigenvalues are the roots of the characteristic equation det(A − λE) = 0 (E the identity matrix). We subtract λ on the diagonal

$$A - \lambda E = \begin{pmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{pmatrix}$$

- Rule of Sarrus, the products along the main diagonals minus those along the other ones

$$\begin{aligned} \det(A-\lambda E) &= \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} \\ &= (2-\lambda)\cdot(2-\lambda)\cdot(3-\lambda) + 1\cdot 0\cdot 0 + \\ &\quad - 0\cdot(2-\lambda)\cdot 0 - (2-\lambda)\cdot 0\cdot 0 - 1\cdot \\ &= \left(-\lambda^3 + 7\lambda^2 - 16\lambda + 12\right) + 0 + 0 - 0 \\ &= -\lambda^3 + 7\lambda^2 - 16\lambda + 12 \end{aligned}$$

- The characteristic polynomial, factored

$$\det(A-\lambda E) = -\lambda^3 + 7\lambda^2 - 16\lambda + 12 = -(\lambda-2)^2(\lambda$$

- The eigenvalues (a multiple root has more names):
$\lambda_1 = \lambda_2 = 2,\ \lambda_3 = 3$
- The minimal polynomial divides the characteristic polynomial and has all its roots. The candidates from the lowest degree

$$(\lambda-2)(\lambda-3),\ (\lambda-2)^2(\lambda-3)$$

- We substitute the matrix – not the zero matrix, so this candidate isn't the minimal polynomial.

$$q(\lambda) = (\lambda-2)(\lambda-3):\ q(A) = (A-2E)(A-3E) =$$

- We substitute the matrix – the zero matrix; the lowest such candidate is the minimal polynomial.

$$q(\lambda) = (\lambda-2)^2(\lambda-3):\ q(A) = (A-2E)^2(A-3E)$$

- The Cayley–Hamilton theorem: every square matrix is a root of its characteristic polynomial
$p(\lambda) = \lambda^3 - 7\lambda^2 + 16\lambda - 12,\ p(A) = A^3 - 7A^2 + 16A$
- The matrix is invertible (the minimal polynomial's constant term isn't zero), so m(A) = O gives the inverse

$$A^{-1} = \frac{1}{12}\left(A^2 - 7A + 16E\right) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

Known limitations

- Matrices up to 6×6.
- Eigenvalues, diagonalization, the singular value decomposition, the Jordan form, the power A^n, the exponential and the minimal polynomial for matrices up to 4×4 and without a parameter; eigenvalues and diagonalization with one parameter up to 3×3 (no cubic factor with the parameter), the general case's eigenvectors only for eigenvalues without a root of the parameter.
- The Jordan form with its steps only for rational eigenvalues, otherwise just the result; a general power A^n and e^{At} need exact eigenvalues (a cubic factor without a rational root: only e^A approximately).

22. Vectors

Write a vector with semicolons, each on its own line: u = (1; 2; 3); its name may have an index (F_1 = (3; 4)). Write a point with square brackets: A[1; 2; 3]. Vectors in the plane have two coordinates, in space three.

A single point in the plane is put in its quadrant (or on an axis) by its coordinates' signs and drawn. With two points, beside the distance |AB| there's the “Midpoint” chip – the midpoint's coordinates are the averages of the end points'. Words work too: “midpoint AB, A[1; 2], B[5; 6]”, “distance between A[1; 2] and B[4; 6]”, “quadrant A[3; −2]”.

Without a further line the chip says what to compute: the magnitude, the sum (the resultant of forces with its magnitude and direction), the dot product (zero means perpendicular vectors), the angle (from its cosine, exact for the table values, otherwise in degrees and minutes), the cross product (by the determinant with i, j, k), the area of a parallelogram and a triangle, the volume of a parallelepiped and a tetrahedron by the triple product, or linear dependence. For points the app first builds the vectors AB, AC, AD from the first point: two points give the distance, three a triangle, four a tetrahedron.

On the last line you can write an expression: u + 2v, u·v (or u*v), u × v, |u|, (u × v)·w, |AB| and the vector AB from two points. The graph shows the vectors as arrows – in the plane with a grid, in space with axes you turn with your finger; an area is shaded and an angle marked by an arc.

How to enter it

`u = (1; 2; 3)`	A vector with semicolons, each on its own line.
`A[1; 2; 3]`	A point with square brackets.
`u + 2v, u·v, u × v, |AB|, (u × v)·w`	An expression with vectors on the last line.

Examples

A point's quadrant.

Input: `A[3; -2]`

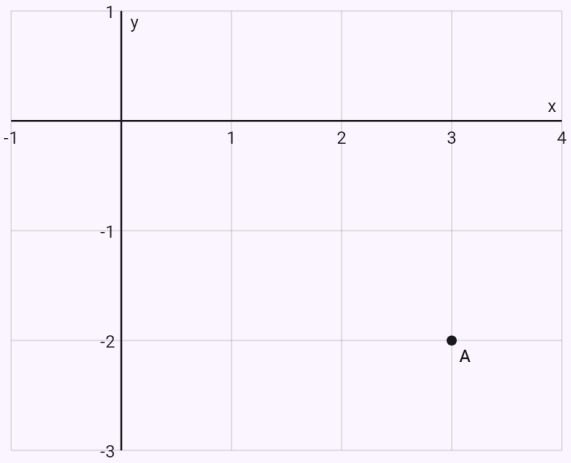
Result

$A \in \mathrm{IV}$

Steps

- The point $A[3;\ -2]$ has x positive and y negative – it lies in the 4th quadrant (bottom right).

Graph



The vectors as arrows – the given ones blue, the result red, helper lines dashed.

A segment's midpoint.

Input: `střed AB, A[1; 2], B[5; 6]`

Result

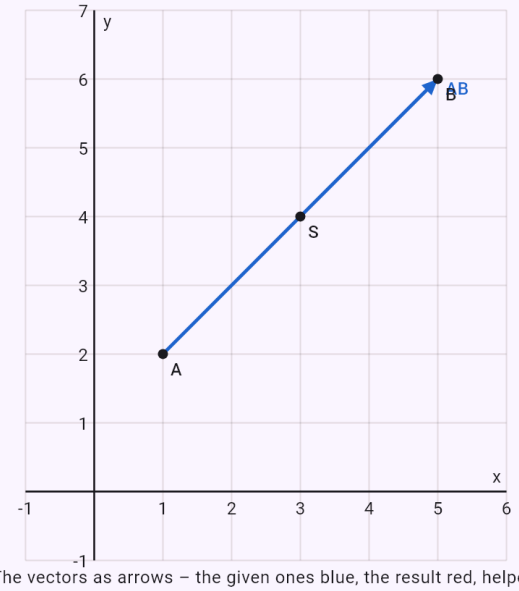
$$S_{AB} = [3;\ 4]$$

Steps

- The midpoint's coordinates are the averages of the end points' ones

$$S_{AB} = \left[\frac{x_A + x_B}{2};\ \frac{y_A + y_B}{2}\right] = \left[\frac{1+5}{2};\ \frac{2+6}{2}\right] = [3;\ 4]$$

Graph



The vectors as arrows – the given ones blue, the result red, helper lines dashed.

The magnitude of a vector.

Input: `u = (1; 2; 3)`

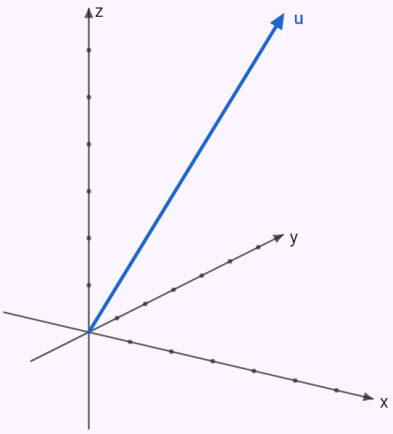
Result

$$|\vec{u}| = \sqrt{14} \approx 3.7417$$

Steps

- The magnitude of the vector: $|\vec{u}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14} \approx 3.7417$

Graph



The vectors as arrows – the given ones blue, the result red, helper lines dashed. Vectors in space – turn the graph by dragging it.

The angle of two vectors.

Input: `u = (1; 2; 2)`
`v = (2; -1; 2)`

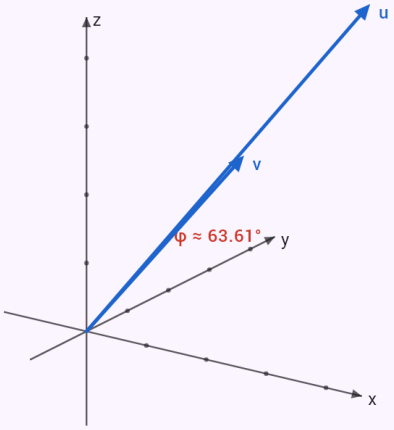
Result

$$\varphi \approx 63.61^\circ \approx 63^\circ 37'$$

Steps

- The dot product is the sum of the products of the matching coordinates:
  $\vec{u} \cdot \vec{v} = 1 \cdot 2 + 2 \cdot (-1) + 2 \cdot 2 = 4$
- The magnitude of the vector:
  $|\vec{u}| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3$
- The magnitude of the vector:
  $|\vec{v}| = \sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{9} = 3$
- The angle of the vectors from the dot product so
  $\varphi \approx 63.61^\circ \approx 63^\circ 37'$
  $\cos\varphi = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}| \cdot |\vec{v}|} = \dfrac{4}{3 \cdot 3} = \dfrac{4}{9}$

Graph



The vectors as arrows – the given ones blue, the result red, helper lines dashed. Vectors in space – turn the graph by dragging it.

A sum and a multiple of vectors.

Input: `u = (1; 2)`
`v = (3; 4)`
`u + 2v`

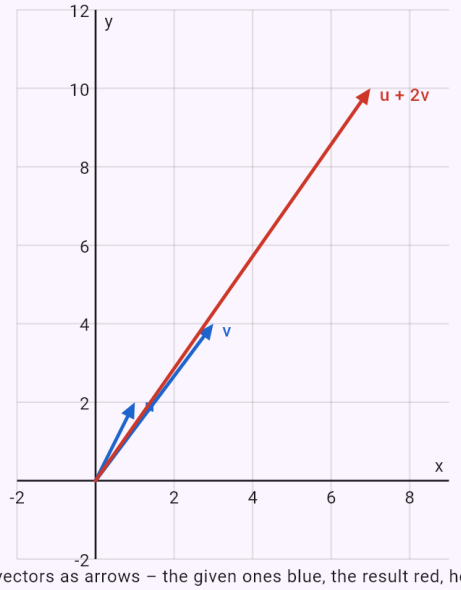
Result

$$\vec{u} + 2\vec{v} = (7;\ 10)$$

Steps

- A multiple of a vector – each coordinate times the number: $2\vec{v} = (2 \cdot 3;\ 2 \cdot 4) = (6;\ 8)$
- Vectors are added (subtracted) coordinate by coordinate: $\vec{u} + 2\vec{v} = (1 + 6;\ 2 + 8) = (7;\ 10)$

Graph



The vectors as arrows – the given ones blue, the result red, helper lines dashed.

The cross product.

Input: `u = (1; 2; 3)`
`v = (4; 5; 6)` Cross product

Result

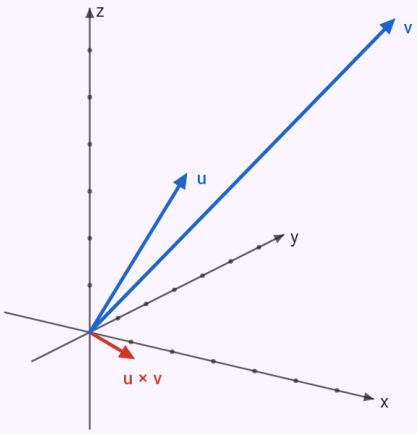
$$\vec{u} \times \vec{v} = (-3;\ 6;\ -3)$$

Steps

- The cross product by expanding the determinant along its first row

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$$
$$= (2 \cdot 6 - 3 \cdot 5;\ 3 \cdot 4 - 1 \cdot 6;\ 1 \cdot 5 - 2 \cdot 4) = (-3;$$

Graph



The vectors as arrows – the given ones blue, the result red, helper lines dashed. Vectors in space – turn the graph by dragging it.

The area of a triangle from three points.

Input: `A[1; 0; 2]`
`B[3; 1; 0]`

`C[0; 2; 1]`

Result

$$S_{\triangle} = \frac{5\sqrt{2}}{2} \approx 3.5355$$

Steps

- A vector from two points – the end point's coordinates minus the starting point's

  $\overrightarrow{AB} = B - A = (3 - 1; 1 - 0; 0 - 2) = (2; 1; -2)$

- A vector from two points – the end point's coordinates minus the starting point's

  $\overrightarrow{AC} = C - A = (0 - 1; 2 - 0; 1 - 2) = (-1; 2; -1)$

- The cross product by expanding the determinant along its first row

  $$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -2 \\ -1 & 2 & -1 \end{vmatrix}$$
  $$= (1 \cdot (-1) - (-2) \cdot 2; (-2) \cdot (-1) - 2 \cdot (-$$

- The magnitude of the vector

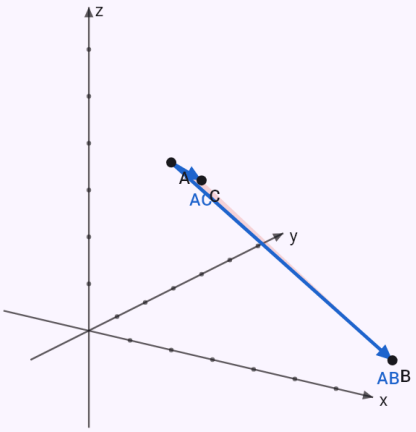
  $|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50} = 5\sqrt{2} \approx 7.0711$

- The parallelogram's area is the magnitude of the cross product (in the plane the absolute value of the determinant): $S = |\overrightarrow{AB} \times \overrightarrow{AC}| = 5\sqrt{2} \approx 7.0711$

- The triangle is half the parallelogram:

  $S_{\triangle} = \frac{1}{2} S = \frac{5\sqrt{2}}{2} \approx 3.5355$

Graph



The vectors as arrows – the given ones blue, the result red, helper lines dashed. Vectors in space – turn the graph by dragging it.

The volume of a parallelepiped and a tetrahedron by the triple product.

Input: `u = (1; 0; 2)`

`v = (2; 1; 0)`

`w = (0; 3; 1)`

Result

$$V = 13,\ V_4 = \frac{13}{6}$$

Steps

- The scalar triple product as the determinant of the vectors' coordinates

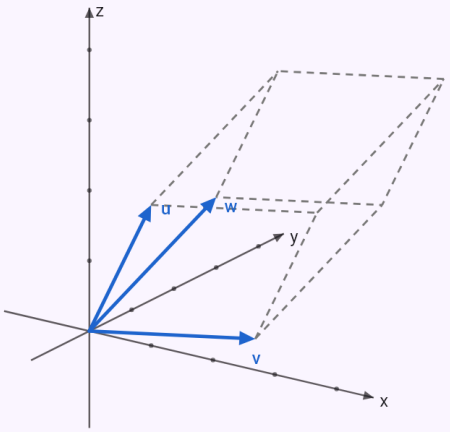
$$[\vec{u}, \vec{v}, \vec{w}] = (\vec{u} \times \vec{v}) \cdot \vec{w} = \begin{vmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 3 & 1 \end{vmatrix}$$

- Rule of Sarrus, the products along the main diagonals minus those along the other ones

$$\begin{vmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 3 & 1 \end{vmatrix} = 1 \cdot 1 \cdot 1 + 0 \cdot 0 \cdot 0 + 2 \cdot 2 \cdot 3 - 2 \cdot 1 \cdot 0 - 1 \cdot 0 \cdot 3 - 0 \cdot 2 \cdot 1 = 1 + 0 + 12 - 0 - 0 - 0 = 13$$

- The parallelepiped's volume is the absolute value of the triple product: $V = |[\vec{u}, \vec{v}, \vec{w}]| = 13$
- The tetrahedron has a sixth of the parallelepiped's volume: $V_4 = \frac{1}{6}|[\vec{u}, \vec{v}, \vec{w}]| = \frac{13}{6}$

Graph



The vectors as arrows – the given ones blue, the result red, helper lines dashed. Vectors in space – turn the graph by dragging it.

The resultant of two forces.

Input: `F_1 = (3; 1)`

`F_2 = (-1; 3)` Sum (resultant)

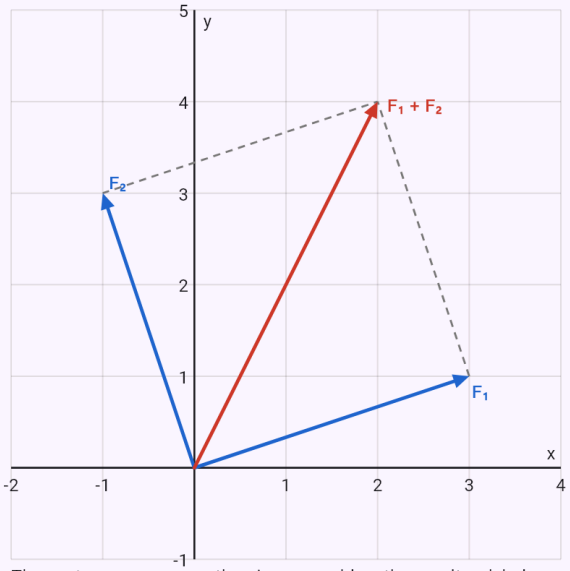
Result

$$\vec{F_1} + \vec{F_2} = (2;\ 4),\ |\vec{F_1} + \vec{F_2}| = 2\sqrt{5} \approx$$

Steps

- Vectors are added (subtracted) coordinate by coordinate $\vec{F_1} + \vec{F_2} = (3 + (-1);\ 1 + 3) = (2;\ 4)$
- The magnitude of the vector: $|\vec{F_1} + \vec{F_2}| = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5} \approx 4.4721$
- The resultant's direction – its angle with the positive x-axis: $\operatorname{tg} \alpha = \frac{4}{2},\ \alpha \approx 63.43^\circ$

Graph



The vectors as arrows – the given ones blue, the result red, helper lines dashed.

The linear dependence of three vectors.

Input: `u = (1; 2; 3)`
`v = (2; 1; 0)`
`w = (4; 5; 6)` Linear dependence

Result

The vectors are linearly dependent.

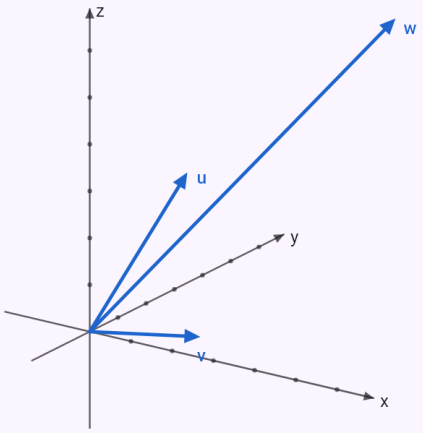
Steps

- Rule of Sarrus, the products along the main diagonals minus those along the other ones

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 4 & 5 & 6 \end{vmatrix}$$
$$= 1 \cdot 1 \cdot 6 + 2 \cdot 0 \cdot 4 + 3 \cdot 2 \cdot 5 - 3 \cdot 1 \cdot 4 - 1 \cdot 0 \cdot 5 - 2 \cdot 2 \cdot 6$$
$$= 6 + 0 + 30 - 12 - 0 - 24$$
$$= 0$$

- The determinant of the vectors' coordinates: $D = 0$. A non-zero determinant means linearly independent vectors, zero dependent ones.

Graph



The vectors as arrows – the given ones blue, the result red, helper lines dashed. Vectors in space – turn the graph by dragging it.

Known limitations

- The coordinates are numbers (no parameter), at most six; the graph only in the plane and in space.

23. Lines and planes

Analytic geometry in the plane and in space. Write a point as for vectors: A[1; 2]. Name a line or a plane and write its equation after a colon, each on its own line: general (p: 2x + 3y − 5 = 0), slope-intercept (p: y = 2x + 1), parametric (p: x = 1 + 2t, y = 3 − t) or vector (p: X = A + t·u). In space an equation in x, y, z is a plane (ρ: x + y + z = 1), so are parametric equations with two parameters; write a line in space parametrically or as the intersection of two planes (p: x + y + z = 1, x − y = 0). The keys p:, q: and ρ: are on the functions keyboard.

A chip says what to compute – the offer follows what you enter: one line or plane gets all forms of its equation, two get their relative position with the intersection, the angle and the distance, a point and a line (plane) the distance, the perpendicular and the parallel through the point, the projection and the reflected point. Two or three points alone get the line AB, the perpendicular bisector and a triangle's centroid with altitudes (in space the plane ABC) among the vectors' chips.

The app computes a distance by the formula or – with the chip “By the projection” – as at a seminar: the perpendicular through the point, its foot and the segment's length. For skew lines it also finds their common perpendicular. Angles are exact for the table values, otherwise in degrees and minutes. You can also write the task on the last line: angle(p, q), distance(A, p), intersection(p, q), perpendicular(A, p), parallel(A, p), projection(A, ρ), reflection(A, p), bisector(A, B), midpoint(A, B), centroid(A, B, C), line(A, B), plane(A, B, C). The graph shows the lines and planes; in space you turn it with your finger.

Geometric mappings in the plane: a translation by a vector, a rotation about a centre by an angle (in degrees, positive counterclockwise), a homothety with a centre and a coefficient, and point and line reflections. A point, a polygon (points in a row: translation(A, B, C, u) is a triangle), a line and a circle (k: (x − 1)^2 + (y − 2)^2 = 4) can be mapped. Write the task on the last line, or pick the chip “Translation”, “Rotation”, “Homothety” or “Reflection” – then the angle and the coefficient go on a line of their own (φ = 90, κ = 2) and the centre is the point S. The steps show the mapping in coordinates and every point’s image, two points’ images for a line and the centre and radius for a circle; the graph both the original and the image.

Conic sections in the axes’ position (no xy term): squares completed to the standard or vertex form, a circle’s centre and radius, an ellipse’s and a hyperbola’s centre, semi-axes, eccentricity and foci (a hyperbola’s asymptotes too), a parabola’s vertex, parameter, focus and directrix; the graph shows the curve with its points.

Write a tangent at a point of a conic in words – “tangent to x^2 + y^2 = 25 at [3; 4]” – and the app finds it by halving the equation. The position of a conic and a line (“position of x^2/9 + y^2/4 = 1 and y = x + 1”) comes from substitution and the discriminant: a secant, a tangent or a line missing the conic, with the intersections. A sphere (x² + y² + z² − 2x + 4y − 6z − 2 = 0) is completed to squares for its centre and

radius. The forms of a line include the intercept form x/p + y/q = 1. Tangents from a point outside ("tangents to the circle x^2 + y^2 = 25 from the point [7; 1]") are found among the lines through the point with a slope k – a tangent is one whose discriminant is zero; the vertical line is checked separately. Tangents parallel or perpendicular to a line ("… parallel to the line 3x + 4y = 0", "… perpendicular to the line y = x") have the same normal vector and an unknown constant c. Write a sphere with a name and a plane, a line or a point on the next line: with a plane the centre's distance is compared with the radius (the circle of the section with its centre and radius, a point of contact, or nothing), a line's parametric equations are substituted and the discriminant decides, at a point of the sphere the tangent plane is found.

How to enter it

`p: 2x + 3y - 5 = 0, q: x = 1 + 2t, y = 3 - t`	A line by a general, slope-intercept or parametric equation, each on its own line.
`p: X = A + t·u`	A line by a vector equation from a point and a vector.
`p: x + 2y - 2z + 3 = 0`	A plane by a general equation (in space).
`p: x + y + z = 1, x - y = 0`	A line in space as the intersection of two planes.
`angle(p, q), distance(A, p), centroid(A, B, C)`	The task on the last line.
`translation(A, B, C, u), rotation(A, S, 90), homothety(p, k, S, 2), reflection(k, q)`	A geometric mapping: the figures, then the vector, the centre and the angle or coefficient, or the centre or axis of a reflection.
`x^2 + y^2 - 4x + 6y - 3 = 0`	A conic – an equation in x and y of the second degree.

Examples

All forms of a line's equation from its parametric equations.

Input: `p: x = 1 + 2t, y = 3 - t`

Result

$$p: x + 2y - 7 = 0$$

$$p: x = 1 + 2t,\ y = 3 - t,\ t \in \mathbb{R}$$

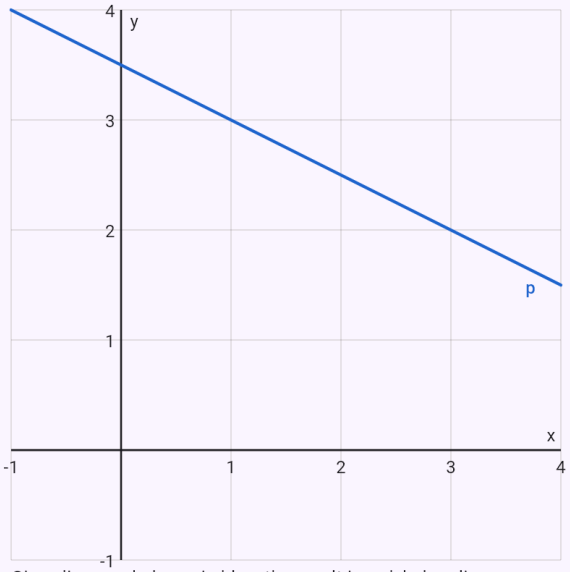
$$p: y = -\frac{1}{2}x + \frac{7}{2}$$

$$p: \frac{x}{7} + \frac{y}{\frac{7}{2}} = 1$$

Steps

- The parametric equations give a point (the constants) and the direction vector (the coefficients of the parameter): $P_p[1;\ 3],\ \vec{u}_p = (2;\ -1)$
- The line's normal vector is perpendicular to its direction – swap the coordinates and change one sign: $\vec{u}_p = (2;\ -1)\ \Rightarrow\ \vec{n}_p = (1;\ 2)$
- The general equation: the coefficients are the normal vector's coordinates, the constant follows from the point $p: x + 2y + c = 0,\ P_p[1;\ 3]:\ 1 + 2 \cdot 3 + c = 0\ \Rightarrow\ c =$
- The slope-intercept form – the general equation solved for y: $p: y = -\frac{1}{2}x + \frac{7}{2}$
- Intercept form – the denominators are the intercepts on the x and y axes: $p: \frac{x}{7} + \frac{y}{\frac{7}{2}} = 1$

Graph



Given lines and planes in blue, the result in red, helper lines dashed.

The intersection of two lines.

Input: `p: 2x + 3y - 5 = 0`

`q: x - y + 1 = 0`

Result

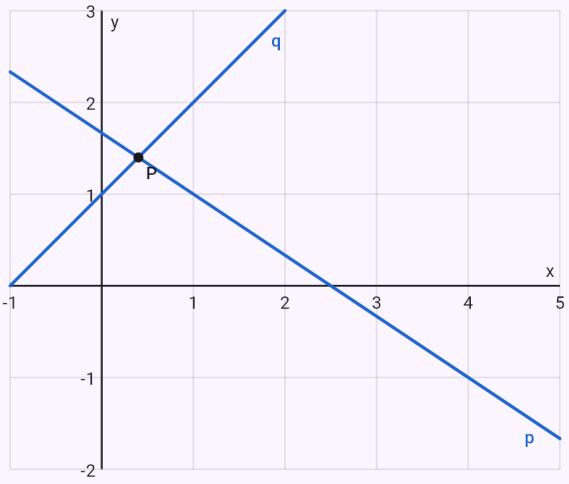
The lines intersect at:

$$P\left[\frac{2}{5};\ \frac{7}{5}\right]$$

Steps

- The general equation gives the normal vector – the coefficients of x, y (z):
  $p: 2x + 3y - 5 = 0 \ \Rightarrow \ \vec{n}_p = (2;\ 3)$
- The general equation gives the normal vector – the coefficients of x, y (z):
  $q: x - y + 1 = 0 \ \Rightarrow \ \vec{n}_q = (1;\ -1)$
- Neither vector is a multiple of the other – the two are not parallel
  $\vec{n}_p = (2;\ 3),\ \vec{n}_q = (1;\ -1),\ \vec{n}_q \neq k \cdot \vec{n}_p$
- The intersection solves the system of both lines' equations (Cramer's rule)
  $\begin{cases} 2x + 3y - 5 = 0 \\ x - y + 1 = 0 \end{cases} D = -5,\ D_x = -2,\ D_y = -7 \ \Rightarrow$:

Graph



Given lines and planes in blue, the result in red, helper lines dashed.

The distance of a point from a line by the formula.

Input: `A[3; 4]`

`p: 3x + 4y - 5 = 0`

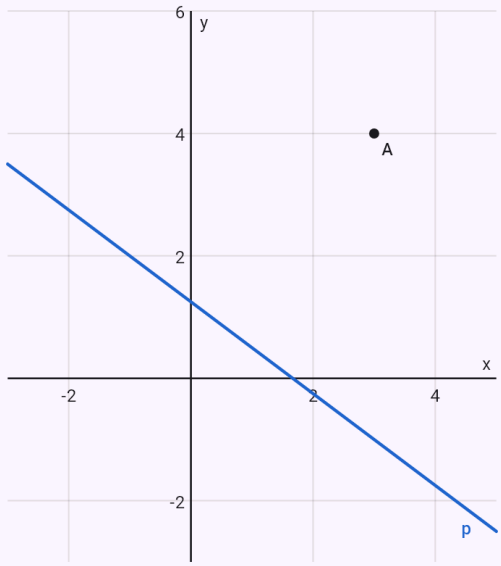
Result

$$v(A, p) = 4$$

Steps

- The general equation gives the normal vector – the coefficients of x, y (z):
  $p: 3x + 4y - 5 = 0 \ \Rightarrow \ \vec{n}_p = (3;\ 4)$
- The distance of the point from the line
  $v(A, p) = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}} = \frac{|3 \cdot 3 + 4 \cdot 4 - 5|}{\sqrt{3^2 + 4^2}} = \frac{20}{5} =$

Graph



Given lines and planes in blue, the result in red, helper lines dashed.

The same distance by the projection.

Input: `A[3; 4]`

`p: 3x + 4y - 5 = 0` By the projection

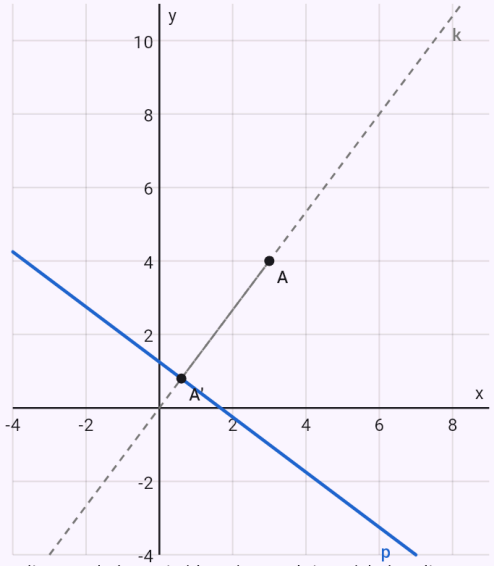
Result

$$v(A, p) = 4$$

Steps

- The general equation gives the normal vector – the coefficients of x, y (z):
  $p : 3x + 4y - 5 = 0 \;\Rightarrow\; \vec{n}_p = (3;\, 4)$
- The perpendicular through A to the line p – one line's direction vector is the other's normal
  $\vec{u}_k = \vec{n}_p = (3;\, 4),\; k : x = 3 + 3t,\; y = 4 + 4t,\; t \in \mathbb{R}$
- The foot of the perpendicular A' is where it meets p – substitute and compute t
  $3 \cdot (3 + 3t) + 4 \cdot (4 + 4t) - 5 = 0 \;\Rightarrow\; t = -\frac{4}{5},\; A'\left[\frac{3}{5};\right.$
- The distance of the point from the line
  $v(A, p) = |AA'| = \sqrt{\left(-\frac{12}{5}\right)^2 + \left(-\frac{16}{5}\right)^2} = 4$

Graph



Given lines and planes in blue, the result in red, helper lines dashed.

A triangle's centroid, altitudes and orthocenter.

Input: `A[1; 2]`

`B[5; 2]`

C[3; 6] Centroid and altitudes

Result

$$T\left[3;\ \frac{10}{3}\right]$$

$$v_a : x - 2y + 3 = 0$$

$$v_b : x + 2y - 9 = 0$$

$$v_c : x - 3 = 0$$

$$V[3;\ 3]$$

$$|v_a| = \frac{8\sqrt{5}}{5} \approx 3.5777,\ |v_b| = \frac{8\sqrt{5}}{5} \approx$$

Steps

- The centroid – the average of the vertices' coordinates

  $$T = \frac{A+B+C}{3} = \left[\frac{1+5+3}{3};\ \frac{2+2+6}{3}\right],\ T\left[3;\ \frac{10}{3}\right.$$

- The altitude from A is perpendicular to the opposite side – the side's vector is its normal, its length the vertex's distance from the side

  $$\overrightarrow{BC} = (-2;\ 4)\ \rightarrow\ (1;\ -2) = \vec{n}_{v_a},\ v_a : x - 2y + 3 = 0,$$

- The altitude from B is perpendicular to the opposite side – the side's vector is its normal, its length the vertex's distance from the side

  $$\overrightarrow{CA} = (-2;\ -4)\ \rightarrow\ (1;\ 2) = \vec{n}_{v_b},\ v_b : x + 2y - 9 = 0,$$

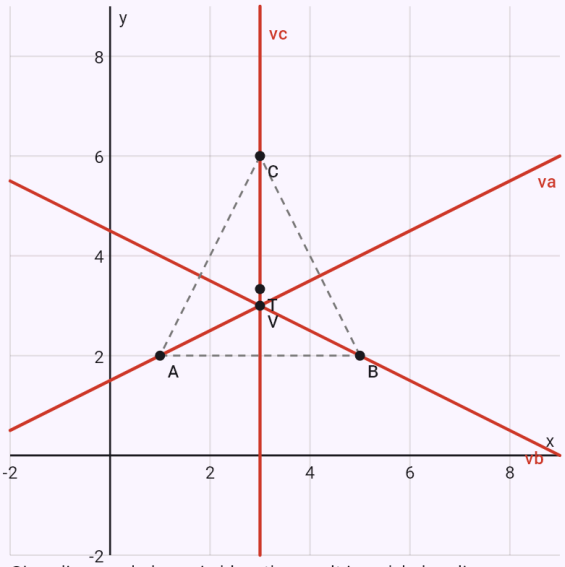
- The altitude from C is perpendicular to the opposite side – the side's vector is its normal, its length the vertex's distance from the side

  $$\overrightarrow{AB} = (4;\ 0)\ \rightarrow\ (1;\ 0) = \vec{n}_{v_c},\ v_c : x - 3 = 0,\ |v_c| = v$$

- The orthocenter – two altitudes intersected

  $$\begin{cases} x - 2y + 3 = 0 \\ x + 2y - 9 = 0 \end{cases} \Rightarrow V[3;\ 3]$$

Graph



Given lines and planes in blue, the result in red, helper lines dashed.

A point reflected in a line.

Input: A[1; 2]

`p: x - y = 0` Reflection

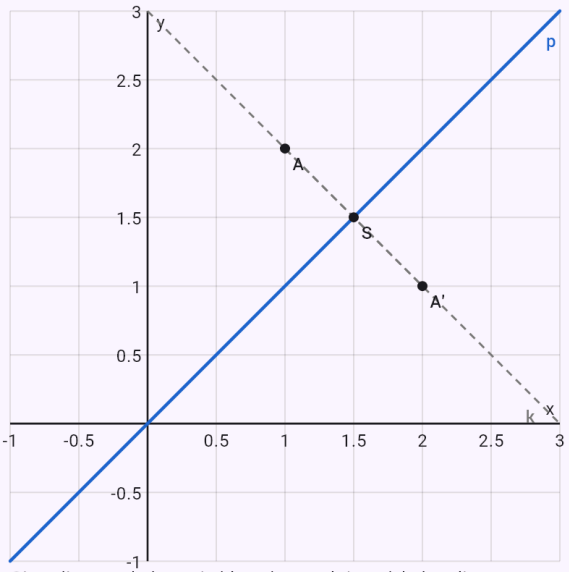
Result

$$A'[2;\ 1]$$

Steps

- The general equation gives the normal vector – the coefficients of x, y (z): $p: x - y = 0 \ \Rightarrow \ \vec{n}_p = (1;\ -1)$
- The perpendicular through A to the line p – one line's direction vector is the other's normal
  $\vec{u}_k = \vec{n}_p = (1;\ -1),\ k: x = 1 + t,\ y = 2 - t,\ t \in \mathbb{R}$
- The foot of the perpendicular S is where it meets p – substitute and compute t
  $(1 + t) - (2 - t) = 0 \ \Rightarrow \ t = \frac{1}{2},\ S\left[\frac{3}{2};\ \frac{3}{2}\right]$
- The reflected point: the centre (the perpendicular's foot) is the midpoint between the point and its image
  $A' = 2S - A = \left(2 \cdot \frac{3}{2} - 1;\ 2 \cdot \frac{3}{2} - 2\right),\ A'[2;\ 1]$

Graph



Given lines and planes in blue, the result in red, helper lines dashed.

A triangle translated by a vector.

Input: `A[1; 2]`
`B[4; 2]`
`C[2; 5]`
`u = (3; -1)`
`translation(A, B, C, u)`

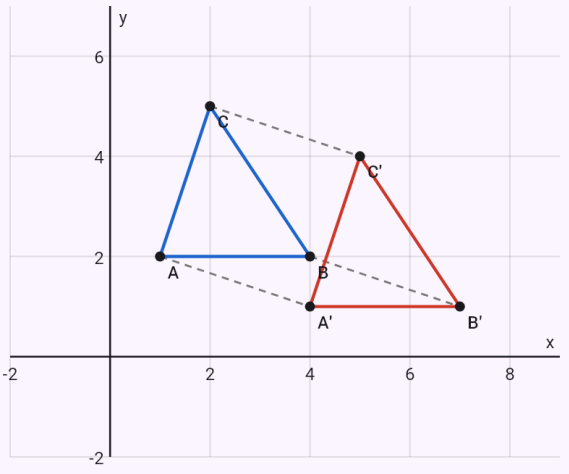
Result

$$A'[4;\ 1],\ B'[7;\ 1]$$
$$C'[5;\ 4]$$

Steps

- Translation by a vector: every point X moves by the same vector, X' = X + u: $X' = X + \vec{u},\ \vec{u} = (3;\ -1)$
- The image of a point: $A' = [1 + 3;\ 2 + (-1)] = [4;\ 1]$
- The image of a point: $B' = [4 + 3;\ 2 + (-1)] = [7;\ 1]$
- The image of a point: $C' = [2 + 3;\ 5 + (-1)] = [5;\ 4]$

Graph



Given lines and planes in blue, the result in red, helper lines dashed.

A triangle rotated by 90° about S.

Input: `A[3; 1]`

`B[5; 1]`

`C[3; 4]`

`S[1; 1]`

`rotation(A, B, C, S, 90)`

Result

$A'[1;\ 3],\ B'[1;\ 5]$

$C'[-2;\ 3]$

Steps

- Rotation about the centre S by the angle φ (positive counterclockwise): the vector X − S is turned by φ and added to the centre

  $X' = S + ((x - s_1)\cos\varphi - (y - s_2)\sin\varphi;\ (x - s_1)\sin$

- The image of a point

  $A - S = (2;\ 0),\ A' = [1 + 2 \cdot 0 - 0 \cdot 1;\ 1 + 2 \cdot 1 + 0 \cdot 0$

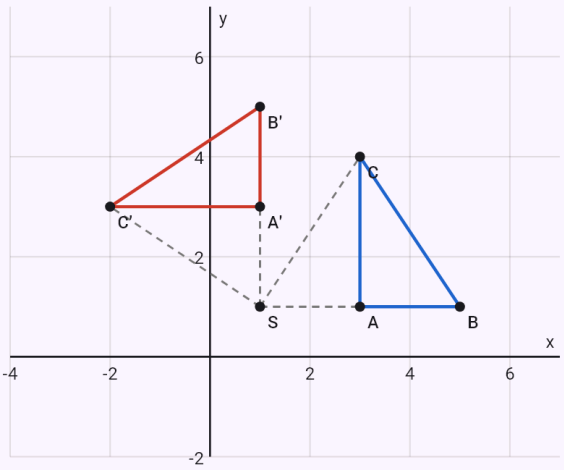
- The image of a point

  $B - S = (4;\ 0),\ B' = [1 + 4 \cdot 0 - 0 \cdot 1;\ 1 + 4 \cdot 1 + 0 \cdot 0$

- The image of a point

  $C - S = (2;\ 3),\ C' = [1 + 2 \cdot 0 - 3 \cdot 1;\ 1 + 2 \cdot 1 + 3 \cdot 0$

Graph



Given lines and planes in blue, the result in red, helper lines dashed.

A line and a circle by a homothety with the coefficient 2.

Input: `p: x + y - 3 = 0`

`k: (x - 2)^2 + (y - 1)^2 = 1`

```
S[0; 0]
homothety(p, k, S, 2)
```

Result

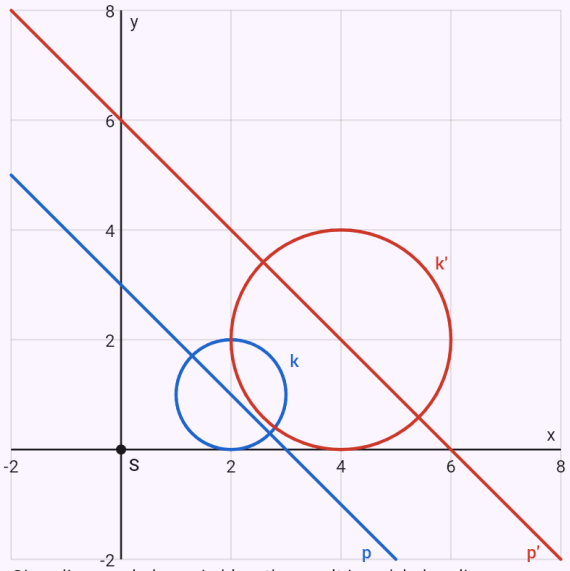
$$p' : x + y - 6 = 0$$

$$k' : (x-4)^2 + (y-2)^2 = 4$$

Steps

- The general equation gives the normal vector – the coefficients of x, y (z):
  $p : x + y - 3 = 0 \;\Rightarrow\; \vec{n}_p = (1;\, 1)$
- The circle k – its centre and radius from the standard form (completing the squares)
  $k : (x-2)^2 + (y-1)^2 = 1 \;\Rightarrow\; S_k[2;\, 1],\; r = 1$
- Homothety with the centre S and the coefficient κ: the vector X − S is multiplied by κ (a negative κ puts the image on the other side of the centre):
  $X' = S + \kappa\,(X - S),\; \kappa = 2$
- The image of the line p: two of its points mapped, the image is the line through theirs – parallel to the original
  $P_1[0;\, 3],\; P_2[1;\, 2] \in p \;\Rightarrow\; P_1'[0;\, 6],\; P_2'[2;\, 4] \in p'$
- The line's normal vector is perpendicular to its direction – swap the coordinates and change one sign:
  $\vec{u}_{p'} = (2;\, -2) \;\Rightarrow\; \vec{n}_{p'} = (1;\, 1)$
- The general equation: the coefficients are the normal vector's coordinates, the constant follows from the point
  $p' : x + y + c = 0,\; P_{p'}[0;\, 6] :\; 0 + 6 + c = 0 \;\Rightarrow\; c = -$
- The image of the circle k: the centre mapped, the radius multiplied by |κ|
  $S_k[2;\, 1] \;\to\; S_k'[4;\, 2],\; r' = |\kappa|\, r = |2| \cdot 1 = 2$

Graph



Given lines and planes in blue, the result in red, helper lines dashed.

A circle reflected in a line – the axis chosen by the chip.

Input: `k: (x - 1)^2 + (y - 3)^2 = 4`

q: y = x Reflection

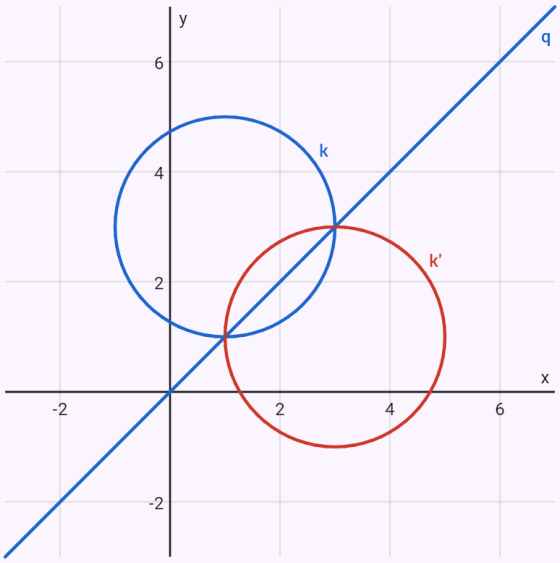
Result

$$k' : (x-3)^2 + (y-1)^2 = 4$$

Steps

- The circle k – its centre and radius from the standard form (completing the squares)
  $k : (x-1)^2 + (y-3)^2 = 4 \;\Rightarrow\; S_k[1;\,3],\; r = 2$
- The slope-intercept form rewritten as a general equation gives the normal vector: $q : x - y = 0 \;\Rightarrow\; \vec{n}_q = (1;\,-1)$
- Reflection in a line: the image lies on the perpendicular to the axis, which halves XX' – the point moves twice its distance from the axis along the normal vector (a; b):
  $X' = X - 2\,\dfrac{ax + by + c}{a^2 + b^2}\,(a;\,b),\; q : x - y = 0$
- The image of the circle k: the centre mapped, the radius stays (the mapping is an isometry):
  $S_k[1;\,3] \;\rightarrow\; S'_k[3;\,1],\; r' = r = 2$

Graph



Given lines and planes in blue, the result in red, helper lines dashed.

The intersection line of two planes.

Input: ρ: x + y + z = 1

σ: 2x - y + z = 3

Result

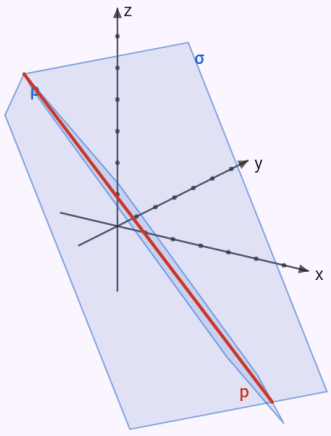
The planes intersect in the line:

$$p : x = \frac{4}{3} + 2t,\; y = -\frac{1}{3} + t,\; z = -3t$$

Steps

- The general equation gives the normal vector – the coefficients of x, y (z):
  $\rho : x + y + z - 1 = 0 \;\Rightarrow\; \vec{n}_\rho = (1;\,1;\,1)$
- The general equation gives the normal vector – the coefficients of x, y (z):
  $\sigma : 2x - y + z - 3 = 0 \;\Rightarrow\; \vec{n}_\sigma = (2;\,-1;\,1)$
- Neither vector is a multiple of the other – the two are not parallel
  $\vec{n}_\rho = (1;\,1;\,1),\; \vec{n}_\sigma = (2;\,-1;\,1),\; \vec{n}_\sigma \neq k \cdot \vec{n}_\rho$
- The intersection line's direction is perpendicular to both normal vectors – their cross product
  $\vec{u}_p = \vec{n}_\rho \times \vec{n}_\sigma = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix} = (2;\,1;\,-3)$
- A point of p: choose coordinates and compute the remaining one from the equation
  $z = 0 : \begin{cases} x + y - 1 = 0 \\ 2x - y - 3 = 0 \end{cases} \;\Rightarrow\; x = \frac{4}{3},\; y = -\frac{1}{3},\; P_p\left[\frac{4}{3};\right.$
- The intersection line – its point and direction vector:
  $p : x = \frac{4}{3} + 2t,\; y = -\frac{1}{3} + t,\; z = -3t,\; t \in \mathbb{R}$

Graph



Given lines and planes in blue, the result in red, helper lines dashed. Vectors in space – turn the graph by dragging it.

The angle of a line and a plane.

Input: `p: x = 1 + t, y = 2t, z = -1 + t`
`ρ: x + y + z = 1` Angle

Result

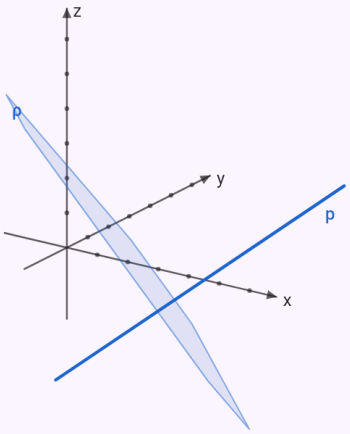
$$\varphi \approx 70.53^\circ \approx 70^\circ 32'$$

Steps

- The parametric equations give a point (the constants) and the direction vector (the coefficients of the parameter): $P_p[1;\,0;\,-1],\ \vec{u}_p = (1;\,2;\,1)$
- The general equation gives the normal vector – the coefficients of x, y (z): $\rho : x + y + z - 1 = 0 \ \Rightarrow\ \vec{n}_\rho = (1;\,1;\,1)$
- The angle of a line and a plane: the direction's and the normal's angle complements it to 90°, hence the sine so $\varphi \approx 70.53^\circ \approx 70^\circ 32'$

$$\sin\varphi = \frac{|\vec{u}_p \cdot \vec{n}_\rho|}{|\vec{u}_p| \cdot |\vec{n}_\rho|} = \frac{|1 \cdot 1 + 2 \cdot 1 + 1 \cdot 1|}{\sqrt{6} \cdot \sqrt{3}} = \frac{2\sqrt{2}}{3} \approx 0.9$$

Graph



Given lines and planes in blue, the result in red, helper lines dashed. Vectors in space – turn the graph by dragging it.

The distance and the common perpendicular of skew lines.

Input: `p: x = 1 + t, y = 2 - t, z = 3t`

`q: x = 2 + s, y = 1 + s, z = 1 - s` Distance

Result

$$v(p,q) = \frac{\sqrt{6}}{3} \approx 0.8165$$

$$o: x = \frac{3}{2} + t,\ y = \frac{3}{2} - 2t,\ z = \frac{3}{2} - t$$

Steps

- The parametric equations give a point (the constants) and the direction vector (the coefficients of the parameter): $P_p[1;\ 2;\ 0],\ \vec{u}_p = (1;\ -1;\ 3)$
- The parametric equations give a point (the constants) and the direction vector (the coefficients of the parameter): $P_q[2;\ 1;\ 1],\ \vec{u}_q = (1;\ 1;\ -1)$
- The distance of the skew lines

  $\vec{u}_p \times \vec{u}_q = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 3 \\ 1 & 1 & -1 \end{vmatrix} = (-2;\ 4;\ 2),\ \overrightarrow{P_pP_q} = (1;\ -1;$

  $v(p,q) = \frac{\left|\left[\overrightarrow{P_pP_q}, \vec{u}_p, \vec{u}_q\right]\right|}{|\vec{u}_p \times \vec{u}_q|} = \frac{|(-2)\cdot 1 + 4\cdot(-1) + 2\cdot}{2\sqrt{6}}$
- The common perpendicular: X on the first line and Y on the second with XY perpendicular to both directions – a system for t and s

  $X = P_p + t\vec{u}_p,\ Y = P_q + s\vec{u}_q,\ \overrightarrow{XY}\cdot\vec{u}_p = 0,\ \overrightarrow{XY}\cdot\vec{u}_q =$

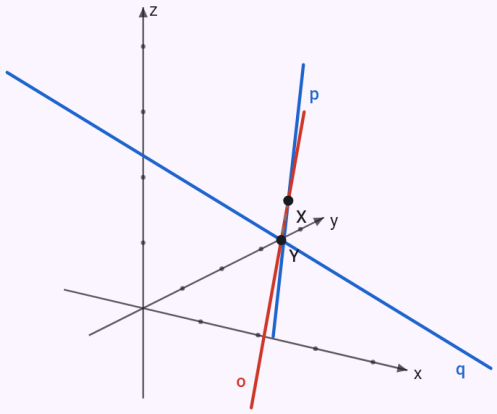
  $\begin{cases} 11t + 3s = 5 \\ -3t - 3s = -1 \end{cases} \Rightarrow\ t = \frac{1}{2},\ s = -\frac{1}{6}$

  $X\left[\frac{3}{2};\ \frac{3}{2};\ \frac{3}{2}\right],\ Y\left[\frac{11}{6};\ \frac{5}{6};\ \frac{7}{6}\right]$

- The parametric equations – a point plus the parameter times the direction vector:

  $o: x = \frac{3}{2} + t,\ y = \frac{3}{2} - 2t,\ z = \frac{3}{2} - t,\ t \in \mathbb{R}$

Graph



Given lines and planes in blue, the result in red, helper lines dashed. Vectors in space – turn the graph by dragging it.

A circle – squares completed, the centre and radius.

Input: `x^2 + y^2 - 4x + 6y - 3 = 0`

Result

$$S = [2;\ -3],\ r = 4$$

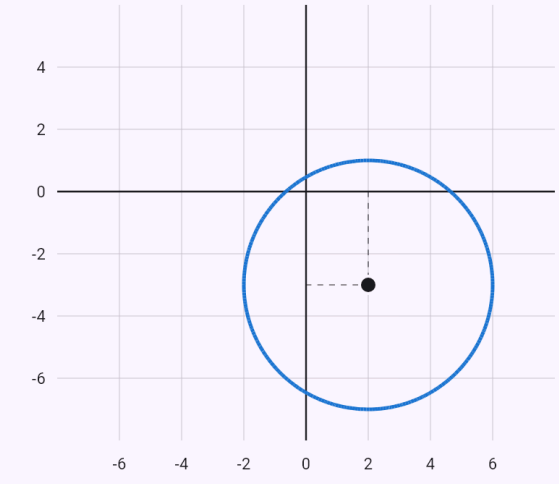
Steps

- We complete the squares

  $x^2 + y^2 - 4x + 6y - 3 = 0$

  $\Rightarrow\ (x-2)^2 + (y+3)^2 = 16$
- The circle's standard equation: $(x-2)^2 + (y+3)^2 = 16$
- Centre $S = [2;\ -3]$, radius $r = 4$

Graph



$x^2 + y^2 - 4x + 6y - 3 = 0$

The curve given by the equation in the x, y plane.

An ellipse – semi-axes, eccentricity and foci.

Input: `x^2/4 + y^2/9 = 1`

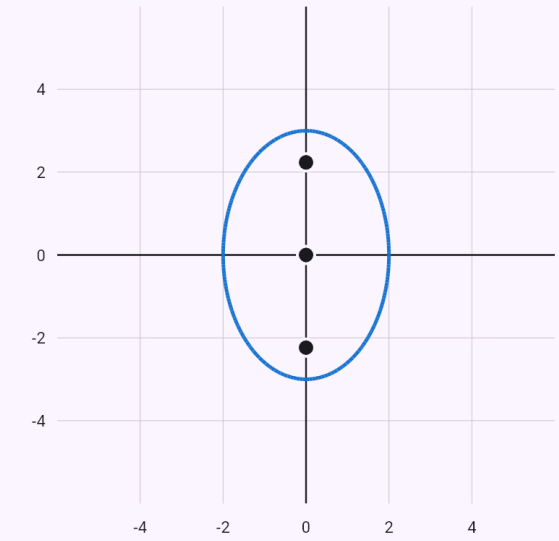
Result

$$S = [0;\ 0],\ a = 3,\ b = 2,\ e = \sqrt{5}$$

Steps

- The ellipse's standard equation: $\frac{x^2}{4} + \frac{y^2}{9} = 1$
- Centre $S = [0;\ 0]$, semi-axes $a = 3,\ b = 2$ (the major axis parallel to the y-axis), linear eccentricity $e = \sqrt{|a^2 - b^2|} = \sqrt{5}$, eccentricity $\varepsilon = \frac{e}{a} = \frac{\sqrt{5}}{3}$, foci $F_1 = \left[0;\ -\sqrt{5}\right],\ F_2 = \left[0;\ \sqrt{5}\right]$

Graph



$\frac{x^2}{4} + \frac{y^2}{9} = 1$

The curve given by the equation in the x, y plane.

A hyperbola – foci and asymptotes.

Input: `x^2/9 - y^2/16 = 1`

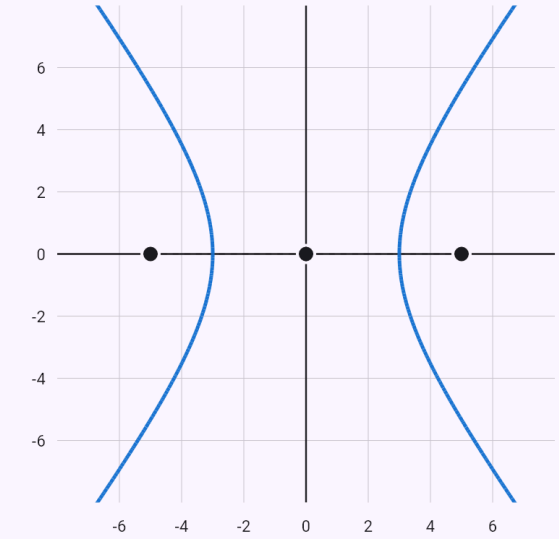
Result

$$S = [0;\ 0],\ a = 3,\ b = 4,\ e = 5$$

Steps

- The hyperbola's standard equation: $\frac{x^2}{9} - \frac{y^2}{16} = 1$
- Centre $S = [0;\ 0]$, semi-axes $a = 3,\ b = 4$ (the major axis parallel to the x-axis), linear eccentricity $e = \sqrt{a^2 + b^2} = 5$, eccentricity $\varepsilon = \frac{e}{a} = \frac{5}{3}$, foci $F_1 = [-5;\ 0],\ F_2 = [5;\ 0]$, asymptotes $y = \pm\frac{4}{3}x$

Graph



$\frac{x^2}{9} - \frac{y^2}{16} = 1$

The curve given by the equation in the x, y plane.

A parabola – vertex, focus and directrix.

Input: `y^2 = 8x`

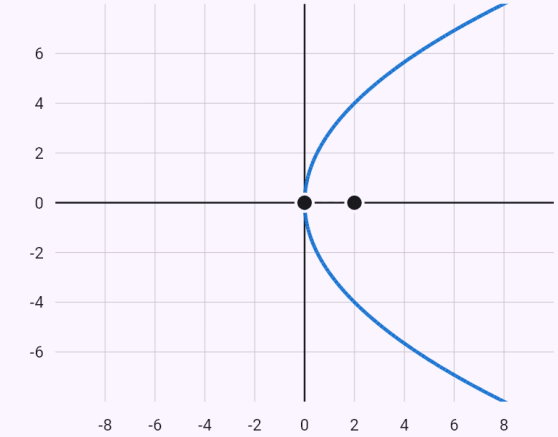
Result

$$V = [0;\ 0],\ p = 4,\ F = [2;\ 0]$$

Steps

- The parabola's vertex equation: $y^2 = 8x$
- Vertex $V = [0;\ 0]$, parameter $p = 4$, axis parallel to the x -axis, focus $F = [2;\ 0]$, directrix $x = -2$

Graph



$y^2 = 8x$

The curve given by the equation in the x, y plane.

The tangent to a circle at its point.

Input: `tangent to the circle x^2 + y^2 = 25 at [3; 4]`

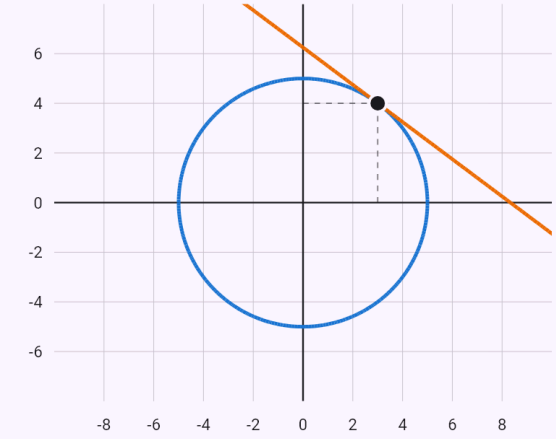
Result

$$t : 3x + 4y - 25 = 0$$

Steps

- The point $T = [3;\ 4]$ lies on the conic – substituted: $3^2 + 4^2 - 25 = 0$
- The tangent at the point $[x_0; y_0]$ of a conic by halving: $x^2 \to x_0 x,\ y^2 \to y_0 y,\ x \to \frac{x + x_0}{2},\ y \to \frac{y + y_0}{2}$: $3x + 4y - 25 = 0$
- The tangent: $t : 3x + 4y - 25 = 0$

Graph



$x^2 + y^2 = 25$

$3x + 4y - 25 = 0$

The conic and the lines; the marked points are the points of contact, the intersections and the given point.

Tangents from a point outside – a zero discriminant for the slope k.

Input: `tečny ke kružnici x^2 + y^2 = 25 z bodu [7; 1]`

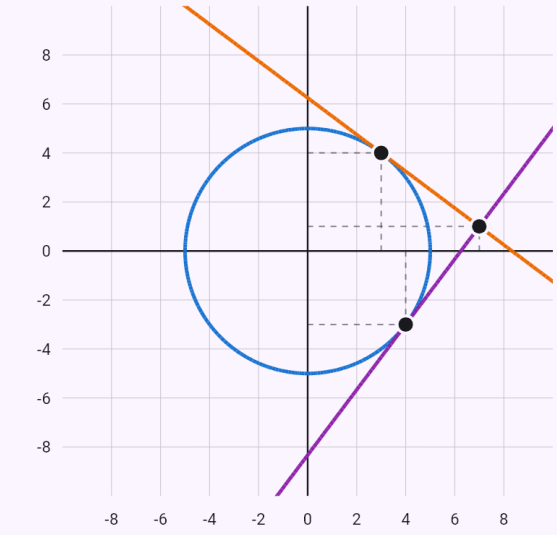
Result

$$t_1 : 3x + 4y - 25 = 0$$
$$t_2 : 4x - 3y - 25 = 0$$

Steps

- The point $M = [7;\ 1]$ does not lie on the conic – substituted: $7^2 + 1^2 - 25 = 25$, not 0.
- We look for the tangents among the lines through M with a slope k: $y - 1 = k(x - 7) \ \Rightarrow \ y = kx + 1 - 7k$
- We substitute for y in the conic's equation: $(k^2 + 1)x^2 + (-14k^2 + 2k)x + 49k^2 - 14k - 24 = 0$
- The line is a tangent exactly when the equation has a single (double) root – the discriminant is zero: $D = 12k^2 - 7k - 12 = 0 \ \Rightarrow \ k_1 = -\frac{3}{4},\ k_2 = \frac{4}{3}$
- The tangents: $t_1 : 3x + 4y - 25 = 0,\ t_2 : 4x - 3y - 25 = 0$; the points of contact: $T_1 = [3;\ 4],\ T_2 = [4;\ -3]$

Graph



$x^2 + y^2 = 25$
$3x + 4y - 25 = 0$
$4x - 3y - 25 = 0$

The conic and the lines; the marked points are the points of contact, the intersections and the given point.

Tangents parallel to a line.

Input: `tečny ke kružnici x^2 + y^2 = 25 rovnoběžné s přímkou 3x + 4y = 0`

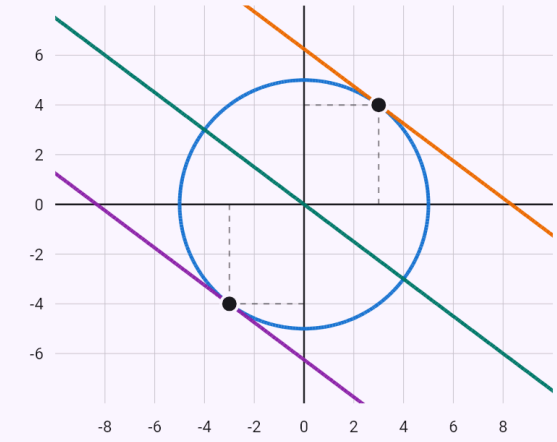
Result

$$t_1 : 3x + 4y - 25 = 0$$
$$t_2 : 3x + 4y + 25 = 0$$

Steps

- Tangents parallel to the line have its normal vector and differ only in the constant c: $t : 3x + 4y + c = 0$
- We substitute for y in the conic's equation: $y = \frac{-3x - c}{4} \ \Rightarrow \ 25x^2 + 6cx + c^2 - 400 = 0$
- The line is a tangent exactly when the equation has a single (double) root – the discriminant is zero: $D = c^2 - 625 = 0 \ \Rightarrow \ c_1 = -25,\ c_2 = 25$
- The tangents: $t_1 : 3x + 4y - 25 = 0,\ t_2 : 3x + 4y + 25 = 0$; the points of contact: $T_1 = [3;\ 4],\ T_2 = [-3;\ -4]$

Graph



$x^2 + y^2 = 25$
$3x + 4y - 25 = 0$
$3x + 4y + 25 = 0$
$3x + 4y = 0$

The conic and the lines; the marked points are the points of contact, the intersections and the given point.

The position of an ellipse and a line.

Input: `position of x^2/9 + y^2/4 = 1 and y = x + 1`

Result

$$P_1 = \left[-\frac{3\left(3+4\sqrt{3}\right)}{13};\ \frac{4\left(1-3\sqrt{3}\right)}{13}\right]$$

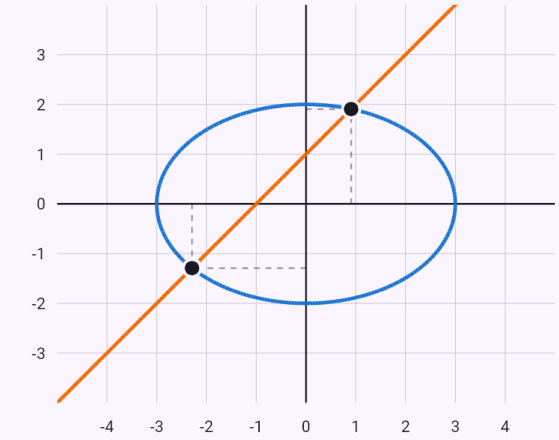
$$P_2 = \left[\frac{3\left(-3+4\sqrt{3}\right)}{13};\ \frac{4\left(1+3\sqrt{3}\right)}{13}\right]$$

Steps

- From the line we express y and substitute it into the conic: $y = x + 1 \ \Rightarrow\ 13x^2 + 18x - 27 = 0$
- The discriminant $D = 18^2 - 4 \cdot 13 \cdot (-27) = 1728$ is positive – the line is a secant with two common points.
- The intersections

$$P_1 = \left[-\frac{3\left(3+4\sqrt{3}\right)}{13};\ \frac{4\left(1-3\sqrt{3}\right)}{13}\right],\ P_2 = \left[\frac{3\left(-3+4\right.}{13}\right.$$

Graph



$\frac{x^2}{9} + \frac{y^2}{4} = 1$

$y = x + 1$

The conic and the lines; the marked points are the points of contact, the intersections and the given point.

A sphere – its centre and radius.

Input: `x^2 + y^2 + z^2 - 2x + 4y - 6z - 2 = 0`

Result

$$S = [1;\ -2;\ 3],\ r = 4$$

Steps

- We complete the squares

$$x^2 + y^2 + z^2 - 2x + 4y - 6z - 2 = 0$$
$$\Rightarrow\ (x-1)^2 + (y+2)^2 + (z-3)^2 = 16$$

- The sphere's centre form: $(x-1)^2 + (y+2)^2 + (z-3)^2 = 16$
- Centre $S = [1;\ -2;\ 3]$, radius $r = 4$

A sphere and a plane – the circle of the section.

Input: `k: x^2 + y^2 + z^2 - 2x + 4y - 6z - 2 = 0`

`ρ: x + y + z = 1`

Result

The plane cuts the sphere in a circle with the centre and the radius:

$$S'\left[\frac{2}{3};\ -\frac{7}{3};\ \frac{8}{3}\right],\ \varrho = \frac{\sqrt{141}}{3}$$

Steps

- Completing the squares turns the equation of the sphere k into the centre form – the centre and the radius

  $k : (x-1)^2 + (y+2)^2 + (z-3)^2 = 16 \ \Rightarrow\ S[1;\ -2;\ 3$

- The general equation gives the normal vector – the coefficients of x, y (z):
  $\rho : x + y + z - 1 = 0 \ \Rightarrow\ \vec{n}_\rho = (1;\ 1;\ 1)$

- The distance of the point from the plane

  $$v(S,\rho) = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2+b^2+c^2}} = \frac{|1+(-2)+3-1|}{\sqrt{1^2+1^2+1^2}}$$

- The centre's distance from the plane ρ is less than the radius – the plane cuts the sphere in a circle:
  $v = \frac{\sqrt{3}}{3} \approx 0.5774 < r = 4$

- The perpendicular through S to the plane ρ goes along its normal:
  $k : x = 1 + t,\ y = -2 + t,\ z = 3 + t,\ t \in \mathbb{R}$

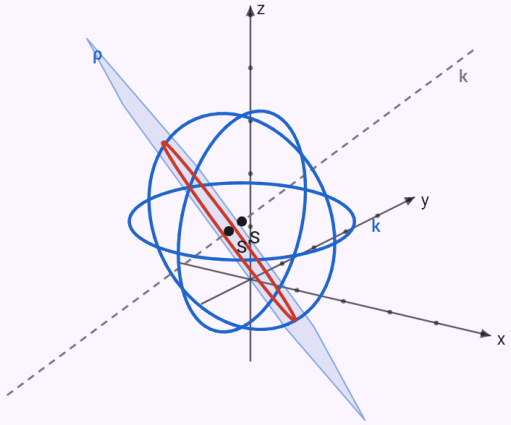
- The foot of the perpendicular S' is where it meets ρ – substitute and compute t

  $(1+t) + (-2+t) + (3+t) - 1 = 0 \ \Rightarrow\ t = -\frac{1}{3},\ S'[$

- The centre of the circle is the foot of the perpendicular from the centre, $S'\left[\frac{2}{3};\ -\frac{7}{3};\ \frac{8}{3}\right]$; the radius by Pythagoras

  $$\varrho = \sqrt{r^2 - v^2} = \sqrt{4^2 - \frac{1}{3}} = \frac{\sqrt{141}}{3} \approx 3.9581$$

Graph



Given lines and planes in blue, the result in red, helper lines dashed. Vectors in space – turn the graph by dragging it.

A sphere and a line – the intersections.

Input: `k: x^2 + y^2 + z^2 = 9`

```
p: x = 1 + t, y = 2t, z = -1 + t
```

Result

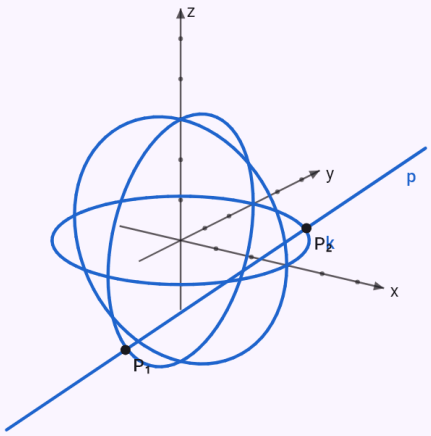
The line cuts the sphere in two points:

$$P_1\left[\frac{6-\sqrt{42}}{6};\ -\frac{\sqrt{42}}{3};\ \frac{-6-\sqrt{42}}{6}\right],\ P$$

Steps

- Completing the squares turns the equation of the sphere k into the centre form – the centre and the radius: $k: x^2+y^2+z^2=9\ \Rightarrow\ S[0;\,0;\,0],\ r=3$
- The parametric equations give a point (the constants) and the direction vector (the coefficients of the parameter): $P_p[1;\,0;\,-1],\ \vec{u}_p=(1;\,2;\,1)$
- We substitute the parametric equations of the line p into the sphere's equation $(t+1)^2+(2t)^2+(t-1)^2=3^2\ \Rightarrow\ 6t^2-7=0$
- The discriminant $D=0^2-4\cdot 6\cdot(-7)=168$ is positive – the line is a secant, with two common points.
- The intersections $t_1=-\frac{\sqrt{42}}{6}:\ P_1\left[\frac{6-\sqrt{42}}{6};\ -\frac{\sqrt{42}}{3};\ \frac{-6-\sqrt{42}}{6}\right],\quad t_2$

Graph



Given lines and planes in blue, the result in red, helper lines dashed. Vectors in space – turn the graph by dragging it.

Known limitations

- Coordinates and coefficients are numbers (no parameter); conics only in axial position (no xy term).
- Geometric mappings only in the plane; a circle is only mapped so far (no distance, position or tangents).
- A triangle's altitudes in space by their lengths only, without equations and their intersection.
- A line's or a plane's name is one letter (also with an index) – so “a: 2x + 3 = 5” is read as a line, not an equation.

24. Construction problems

Construction problems are entered in a form – in the “Applications” section of the home screen, the Geometry field (the “Construction problems” form). The basic task is a triangle ABC from three elements: three sides (SSS), two sides and the angle between them (SAS), a side and two angles (ASA – the third angle is computed from the sum of 180°), or two sides and the angle opposite one of them (SSA). Lengths are in centimetres, angles in degrees.

The solution follows school practice: the analysis (a sketch of the finished triangle with the given elements highlighted), the construction steps in symbolic notation (1. AB; |AB| = 6 cm, 2. k; k(A; 4 cm), 3. C; C ∈ k ∩ l …) and a discussion of the number of solutions. The picture under the result pages through the construction step by step with the arrows – the elements added in a step are highlighted in colour. The result also lists the computed sides and angles, so you can check your drawing.

The discussion explains why a problem has no solution (the triangle inequality, the sum of angles, a circle that doesn’t reach the ray) or why it has two (SSA with the angle opposite the shorter side).

A triangle can also be given by an altitude, a median, the circumradius or the inradius. A side is drawn first and the third vertex lies where two loci meet: a parallel at the altitude’s distance, a circle around the side’s midpoint with the median as radius, a circle around an end point, an angle’s arm, the arc of an angle over the side or the circumcircles. With two sides and the altitude from their common vertex, the altitude comes first; with the inradius, a side and an angle at it, the incircle and the tangent to it are constructed.

Quadrilaterals: a square, a rectangle, a rhombus, a parallelogram and a trapezoid (AB ∥ CD) from the sides a, b, c, d, the diagonals e, f, the angles α, β and the height v – the elements are picked in rows as for a triangle. Mostly the triangle ABC or ABD comes first and the fourth vertex completes it; otherwise the diagonals halving each other (perpendicular in a rhombus), the parallel at the height’s distance or, for a trapezoid, a helper triangle from shifting the side AD or the diagonal BD by the vector DC.

Loci are in the same form: the perpendicular bisector of a segment, the angle bisector, the Thales circle and the arcs from which a segment is seen at a given angle (two arcs symmetric about the segment). Each is constructed with compasses and a ruler step by step; the analysis explains why the points lie exactly there.

Circles and tangents: the tangents from a point to a circle (the points of contact on a Thales circle), the common tangents of two circles (via helper circles with the difference and the sum of the radii), the circle of a given radius touching an angle’s arms, the circles through a point touching a line and the circles through two points. The discussion counts the solutions by the position – two circles have four, three, two, one or no common tangents, for example.

Problems with relations: a given figure (a segment AB, a circle k and a point A, a line p and a point A, a circle k and a line p, two intersecting lines p, q) and two relations the point sought must satisfy – over a segment the vertex C of the triangle ABC, otherwise a point X. The relations are picked in rows: the segment is seen from the point at an angle (90° = the Thales circle), as far from two points or two lines, a distance from a point or a line, the ratio of the distances from A and B (the Apollonius circle), lies on a line or a circle, the line AX is a tangent to k, the centre of a circle of radius ϱ touching k, and A is the midpoint of XY with Y on k or p (point reflection). Each relation becomes a locus of points, the point sought is in their intersection; the discussion counts the solutions (over a segment in one half-plane, the other gives congruent mirror images). Under the result are the distances of the point from the given points and lines as a check.

General relations among more points (with a circle k or without): the points A to H are given by rows of relations – a point on k, a midpoint, a tangent to k, a length, a ratio of lengths, perpendicular, parallel, a point on a line, an angle and a triangle (only what to draw). The relations become equations in coordinates (the centre S at the origin, the first point on the x axis), solved exactly, with square roots. The result is the lengths between the points (with a circle also as multiples of r), the check and the number of solutions up to congruence. Where possible a school construction follows: each point on two loci – the given circle, a distance, a line, a tangent, a midpoint, a point reflection – and the computed lengths constructed as a right triangle's hypotenuse or leg or by the altitude theorem; the solution also shows the power of a point and the tangent-chord angle when they hold. Otherwise honestly only the computation and the picture.

Sections of solids: a cube, a cuboid, a regular prism and pyramid with a base of 3 to 6 sides (vertices ABC… and A'B'C'…, a pyramid's apex V) and the regular tetrahedron ABCD. The cutting plane is given by the points X, Y, Z, one per line: AE=2 is the point on the segment AE 2 cm from A – the segment may be an edge, a face's or the solid's diagonal or the axis SS' (S, S' the bases' centres), and a distance longer than it gives a point on its extension; X: AB = 1:2 is the point X dividing AB in the ratio 1 : 2. Auxiliary points can be used on other lines. The solid is in the oblique projection with the hidden edges dashed; a regular triangle, pentagon or hexagon in the base is first drawn beside it in its true shape and carried into the projection. The section is built side by side by the school's rules: points in one face are joined, parallel faces are cut in parallels, the line of a side meets an edge's line in a helper point T that lies in the neighbouring face too, and the line of a point inside the solid is carried to the base by an auxiliary vertical plane. Under the picture, the finished section in space can be turned with a finger.

Examples

A triangle from three sides (SSS).

Input: `construction(triangle; a=5; b=4; c=6)`

Result

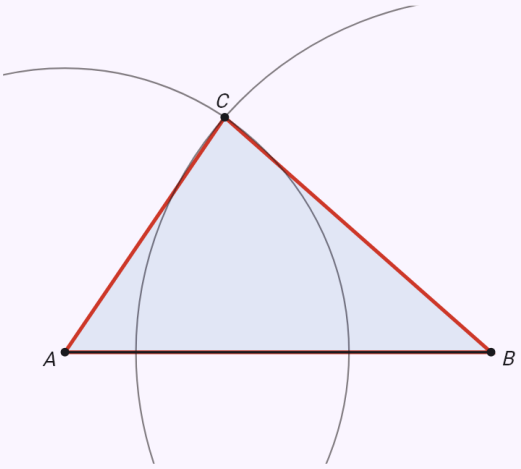
$a = 5\,\text{cm},\ b = 4\,\text{cm},\ c = 6\,\text{cm}$

$\alpha \approx 55.77^\circ,\ \beta \approx 41.41^\circ,\ \gamma \approx 82.82^\circ$

Steps

- Analysis: a sketch of the finished triangle with the given elements highlighted – $a = 5\,\text{cm},\ b = 4\,\text{cm},\ c = 6\,\text{cm}$
- 1. $AB;\ |AB| = 6\,\text{cm}$ – draw the side c
- 2. $k;\ k(A;\ 4\,\text{cm})$ – the circle k with centre A and radius b
- 3. $l;\ l(B;\ 5\,\text{cm})$ – the circle l with centre B and radius a
- 4. $C;\ C \in k \cap l$ – the third vertex is the intersection.
- 5. $\triangle ABC$ – join the vertices.
- Discussion: the triangle inequality $a + b > c$ holds, so the circles intersect. In one half-plane bounded by the first side the problem has one solution (in the other there is its congruent mirror image).

Graph



Step 5 of 5
5. $\triangle ABC$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

Two sides and the angle between them (SAS).

Input: `construction(triangle; b=4; c=6; alpha=60)`

Result

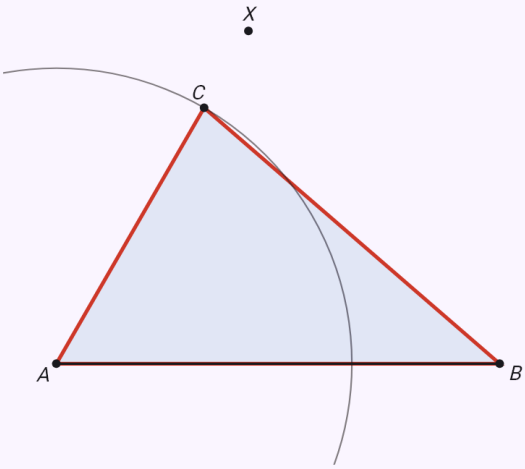
$a \approx 5.29\,\text{cm},\ b = 4\,\text{cm},\ c = 6\,\text{cm}$

$\alpha = 60^\circ,\ \beta \approx 40.89^\circ,\ \gamma \approx 79.11^\circ$

Steps

- Analysis: a sketch of the finished triangle with the given elements highlighted – $b = 4\,\text{cm},\ c = 6\,\text{cm},\ \alpha = 60^\circ$
- 1. $AB;\ |AB| = 6\,\text{cm}$ – draw the side c
- 2. $\angle BAX;\ |\angle BAX| = 60^\circ$ – construct the angle α; its arm is the ray $\overrightarrow{AX}$
- 3. $k;\ k(A;\ 4\,\text{cm})$ – the circle k with centre A and radius b
- 4. $C;\ C \in \overrightarrow{AX} \cap k$ – the third vertex is the intersection.
- 5. $\triangle ABC$ – join the vertices.
- Discussion: in one half-plane bounded by the first side the problem has one solution (in the other half-plane there is its congruent mirror image).

Graph



Step 5 of 5
5. $\triangle ABC$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

A side and two angles, one of them not at the side – the third angle is computed (ASA).

Input: `construction(triangle; c=6; alpha=50; gamma=70)`

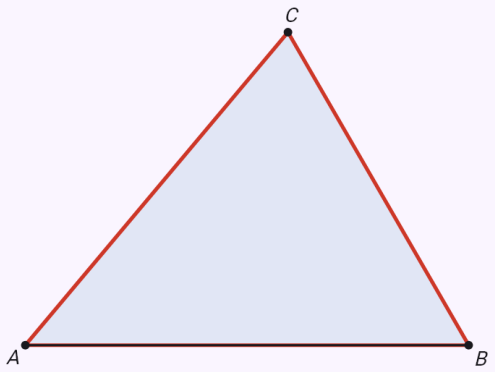
Result

$$a \approx 4.89\,\text{cm},\ b \approx 5.53\,\text{cm},\ c = 6\,\text{cm}$$
$$\alpha = 50^\circ,\ \beta = 60^\circ,\ \gamma = 70^\circ$$

Steps

- Analysis: a sketch of the finished triangle with the given elements highlighted – $c = 6\,\text{cm},\ \alpha = 50^\circ,\ \gamma = 70^\circ$
- The third angle from the sum of the angles in a triangle: $\beta = 180^\circ - \alpha - \gamma = 180^\circ - 50^\circ - 70^\circ = 60^\circ$
- 1. AB; $|AB| = 6\,\text{cm}$ – draw the side c
- 2. $\angle BAX$; $|\angle BAX| = 50^\circ$ – construct the angle α; its arm is the ray $\overrightarrow{AX}$
- 3. $\angle ABY$; $|\angle ABY| = 60^\circ$ – construct the angle β; its arm is the ray $\overrightarrow{BY}$
- 4. C; $C \in \overrightarrow{AX} \cap \overrightarrow{BY}$ – the third vertex is the intersection.
- 5. $\triangle ABC$ – join the vertices.
- Discussion: in one half-plane bounded by the first side the problem has one solution (in the other half-plane there is its congruent mirror image).

Graph



Step 5 of 5
5. $\triangle ABC$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

The angle opposite the shorter of two sides – two solutions (SSA).

Input: `construction(triangle; a=4; c=6; alpha=30)`

Result

$\triangle ABC_1$:

$a = 4\,\text{cm},\ b \approx 2.55\,\text{cm},\ c = 6\,\text{cm}$

$\alpha = 30^\circ,\ \beta \approx 18.59^\circ,\ \gamma \approx 131.41^\circ$

$\triangle ABC_2$:

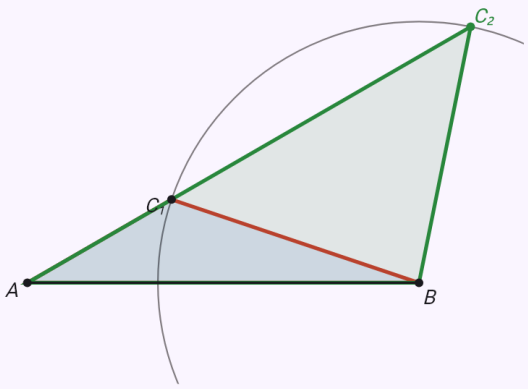
$a = 4\,\text{cm},\ b \approx 7.84\,\text{cm},\ c = 6\,\text{cm}$

$\alpha = 30^\circ,\ \beta \approx 101.41^\circ,\ \gamma \approx 48.59^\circ$

Steps

- Analysis: a sketch of the finished triangle with the given elements highlighted – $a = 4\,\text{cm},\ c = 6\,\text{cm},\ \alpha = 30^\circ$
- 1. $AB;\ |AB| = 6\,\text{cm}$ – draw the side c
- 2. $\angle BAX;\ |\angle BAX| = 30^\circ$ – construct the angle α; its arm is the ray $\overrightarrow{AX}$
- 3. $k;\ k(B;\ 4\,\text{cm})$ – the circle k with centre B and radius a
- 4. $C_1, C_2;\ C_1, C_2 \in \overrightarrow{AX} \cap k$ – there are two intersections, each the third vertex of one triangle.
- 5. $\triangle ABC_1,\ \triangle ABC_2$ – join the vertices.
- Discussion: the circle meets the ray at two points – in one half-plane bounded by the first side the problem has two solutions, two different triangles.

Graph



Step 5 of 5

5. $\triangle ABC_1,\ \triangle ABC_2$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

Sides that break the triangle inequality.

Input: `construction(triangle; a=2; b=3; c=6)`

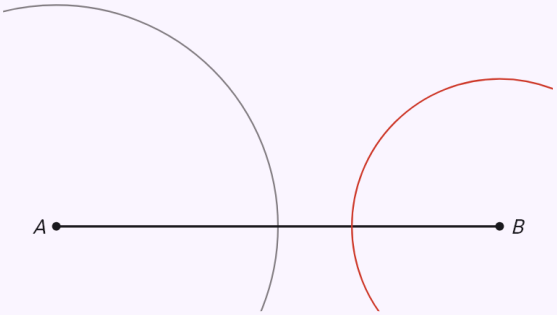
Result

No figure with the given elements can be constructed.

Steps

- Analysis: a sketch of the finished triangle with the given elements highlighted – $a = 2\,\mathrm{cm},\ b = 3\,\mathrm{cm},\ c = 6\,\mathrm{cm}$
- 1. $AB;\ |AB| = 6\,\mathrm{cm}$ – draw the side c
- 2. $k;\ k(A;\ 3\,\mathrm{cm})$ – the circle k with centre A and radius b
- 3. $l;\ l(B;\ 2\,\mathrm{cm})$ – the circle l with centre B and radius a
- Discussion: the triangle inequality fails ($a + b = 5\,\mathrm{cm} \leq c = 6\,\mathrm{cm}$) – the circles don't intersect and there is no triangle.

Graph



Step 3 of 3
3. $l;\ l(B;\ 2\,\mathrm{cm})$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

The side c, the altitude and the median to it – two solutions.

Input: `construction(triangle; c=6; vc=4; tc=5)`

Result

$\triangle ABC_1$:

$a \approx 7.21\,\text{cm},\ b = 4\,\text{cm},\ c = 6\,\text{cm}$

$\alpha = 90^\circ,\ \beta \approx 33.69^\circ,\ \gamma \approx 56.31^\circ$

$\triangle ABC_2$:

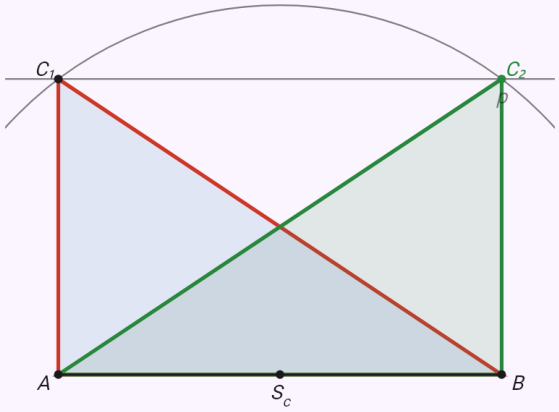
$a = 4\,\text{cm},\ b \approx 7.21\,\text{cm},\ c = 6\,\text{cm}$

$\alpha \approx 33.69^\circ,\ \beta = 90^\circ,\ \gamma \approx 56.31^\circ$

Steps

- Analysis: a sketch of the finished triangle with the given elements highlighted – $c = 6\,\text{cm},\ v_c = 4\,\text{cm},\ t_c = 5\,\text{cm}$
- 1. AB; $|AB| = 6\,\text{cm}$ – draw the side c
- 2. p; $p \parallel AB,\ |p, AB| = v_c = 4\,\text{cm}$ – the parallel p to AB at the distance v_c
- 3. S_c; $S_c \in AB,\ |AS_c| = |S_cB|$ – the midpoint S_c of the segment AB (where its bisector crosses it).
- 4. k; $k(S_c;\ 5\,\text{cm})$ – the circle k with centre S_c and radius t_c
- 5. C_1, C_2; $C_1, C_2 \in p \cap k$ – there are two intersections, each the third vertex of one triangle.
- 6. $\triangle ABC_1,\ \triangle ABC_2$ – join the vertices.
- Discussion: in the half-plane over the first side the loci have 2 common points – the problem has as many solutions (in the other half-plane there are their congruent mirror images).

Graph



Step 6 of 6

6. $\triangle ABC_1,\ \triangle ABC_2$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

The side c, the altitude to it and the opposite angle – the arc of the angle over the side.

Input: `construction(triangle; c=6; vc=3; gamma=60)`

Result

$\triangle ABC_1$:

$a \approx 6.91\,\text{cm},\ b \approx 3.01\,\text{cm},\ c = 6\,\text{cm}$

$\alpha \approx 94.26^\circ,\ \beta \approx 25.74^\circ,\ \gamma = 60^\circ$

$\triangle ABC_2$:

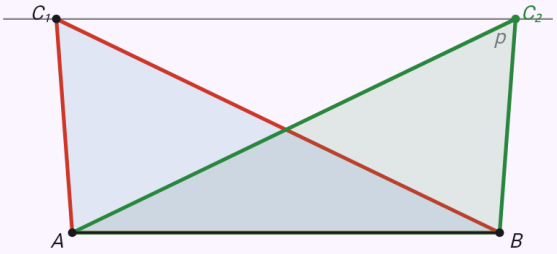
$a \approx 3.01\,\text{cm},\ b \approx 6.91\,\text{cm},\ c = 6\,\text{cm}$

$\alpha \approx 25.74^\circ,\ \beta \approx 94.26^\circ,\ \gamma = 60^\circ$

Steps

- Analysis: a sketch of the finished triangle with the given elements highlighted – $c = 6\,\text{cm},\ v_c = 3\,\text{cm},\ \gamma = 60^\circ$
- 1. AB; $|AB| = 6\,\text{cm}$ – draw the side c
- 2. p; $p \parallel AB,\ |p, AB| = v_c = 3\,\text{cm}$ – the parallel p to AB at the distance v_c
- 3. k; $|\angle AXB| = 60^\circ,\ X \in k$ – the arc k from which the segment AB is seen at the angle γ (constructed as in "Arcs of a given angle").
- 4. C_1, C_2; $C_1, C_2 \in p \cap k$ – there are two intersections, each the third vertex of one triangle.
- 5. $\triangle ABC_1,\ \triangle ABC_2$ – join the vertices.
- Discussion: in the half-plane over the first side the loci have 2 common points – the problem has as many solutions (in the other half-plane there are their congruent mirror images).

Graph



Step 5 of 5

5. $\triangle ABC_1,\ \triangle ABC_2$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

A side, the circumradius and another side – two circumcircles.

Input: `construction(triangle; c=6; r=4; a=5)`

Result

$\triangle ABC_1$:

$a = 5\,\text{cm},\ b \approx 1.38\,\text{cm},\ c = 6\,\text{cm}$

$\alpha \approx 38.68^\circ,\ \beta \approx 9.91^\circ,\ \gamma \approx 131.41^\circ$

$\triangle ABC_2$:

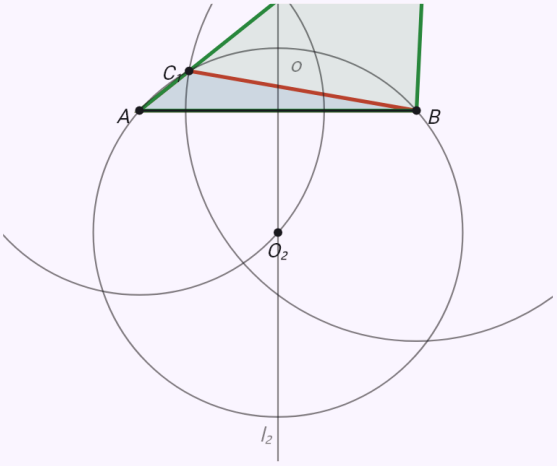
$a = 5\,\text{cm},\ b \approx 7.99\,\text{cm},\ c = 6\,\text{cm}$

$\alpha \approx 38.68^\circ,\ \beta \approx 92.73^\circ,\ \gamma \approx 48.59^\circ$

Steps

- Analysis: a sketch of the finished triangle with the given elements highlighted – $c = 6\,\text{cm},\ r = 4\,\text{cm},\ a = 5\,\text{cm}$
- 1. AB; $|AB| = 6\,\text{cm}$ – draw the side c
- 2. o; $o \perp AB,\ S_{AB} \in o$ – the perpendicular bisector o of the segment AB, constructed as in "Perpendicular bisector".
- 3. k; $k(A;\, 4\,\text{cm})$ – the circle k with centre A and radius r
- 4. O_1, O_2; $O_1, O_2 \in o \cap k$ – the intersections O_1, O_2
- 5. l_1, l_2; $l_1(O_1;\, r),\ l_2(O_2;\, r)$ – the circumcircle l_1, l_2 with centre O_1, O_2
- 6. m; $m(B;\, 5\,\text{cm})$ – the circle m with centre B and radius a
- 7. C_1, C_2; $C_1, C_2 \in (l_1 \cup l_2) \cap m$ – there are two intersections, each the third vertex of one triangle.
- 8. $\triangle ABC_1,\ \triangle ABC_2$ – join the vertices.
- Discussion: in the half-plane over the first side the loci have 2 common points – the problem has as many solutions (in the other half-plane there are their congruent mirror images).

Graph



Step 8 of 8

8. $\triangle ABC_1,\ \triangle ABC_2$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

Two sides and the altitude from their common vertex – the altitude drawn first.

Input: `construction(triangle; a=5; b=4; vc=3)`

Result

$\triangle AB_1C$:

$a = 5\,\text{cm},\ b = 4\,\text{cm},\ c \approx 6.65\,\text{cm}$

$\alpha \approx 48.59^\circ,\ \beta \approx 36.87^\circ,\ \gamma \approx 94.54^\circ$

$\triangle AB_2C$:

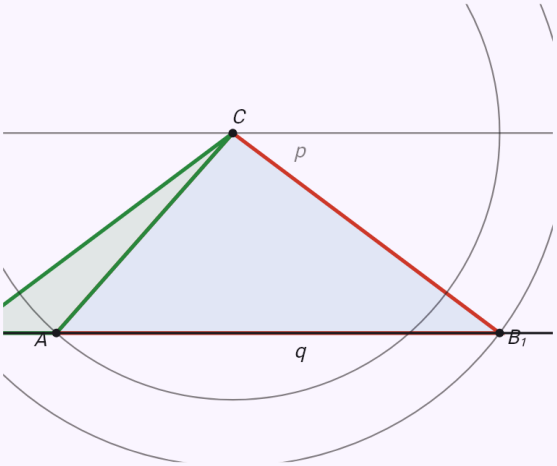
$a = 5\,\text{cm},\ b = 4\,\text{cm},\ c \approx 1.35\,\text{cm}$

$\alpha \approx 131.41^\circ,\ \beta \approx 36.87^\circ,\ \gamma \approx 11.72^\circ$

Steps

- Analysis: a sketch of the finished triangle with the given elements highlighted – $a = 5\,\text{cm},\ b = 4\,\text{cm},\ v_c = 3\,\text{cm}$
- 1. q; $A, B \in q$ – any line q for the side AB to lie on.
- 2. p; $p \parallel q,\ |p, q| = v_c = 3\,\text{cm}$ – the parallel p to q at the distance v_c
- 3. C; $C \in p$ – any point C of the line p
- 4. k; $k(C;\ 4\,\text{cm})$ – the circle k with centre C and radius b
- 5. A; $A \in k \cap q$ – one of the intersections, A (the other gives a congruent triangle).
- 6. l; $l(C;\ 5\,\text{cm})$ – the circle l with centre C and radius a
- 7. B_1, B_2; $B_1, B_2 \in l \cap q$ – there are two intersections, each the third vertex of one triangle.
- 8. $\triangle AB_1C,\ \triangle AB_2C$ – join the vertices.
- Discussion: the circle l meets the line q at two points – the problem has two solutions, two different triangles (the other intersection of the circle k gives only congruent ones).

Graph



Step 8 of 8

8. $\triangle AB_1C,\ \triangle AB_2C$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

A side, an angle at it and the inradius – the tangent via a Thales circle.

Input: `construction(triangle; c=6; alpha=60; rho=1)`

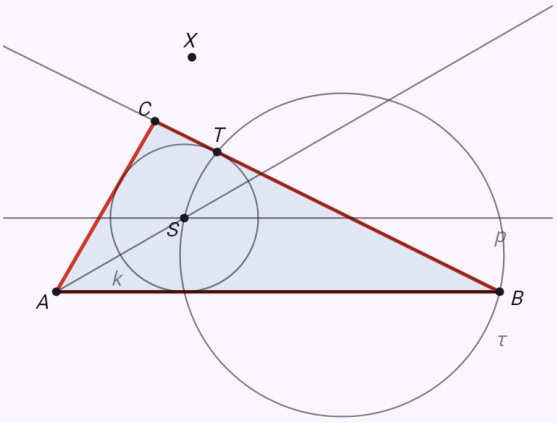
Result

$$a \approx 5.21\,\text{cm},\ b \approx 2.67\,\text{cm},\ c = 6\,\text{cm}$$
$$\alpha = 60^\circ,\ \beta \approx 26.37^\circ,\ \gamma \approx 93.63^\circ$$

Steps

- Analysis: a sketch of the finished triangle with the given elements highlighted – $c = 6\,\text{cm},\ \alpha = 60^\circ,\ \varrho = 1\,\text{cm}$
- 1. $AB;\ |AB| = 6\,\text{cm}$ – draw the side c
- 2. $\angle BAX;\ |\angle BAX| = 60^\circ$ – construct the angle α; its arm is the ray $\overrightarrow{AX}$
- 3. $o;\ o = \overrightarrow{AY},\ |\angle BAY| = \frac{\alpha}{2}$ – the bisector o of the angle α (constructed as in "Angle bisector"); the incentre lies on it.
- 4. $p;\ p \parallel AB,\ |p, AB| = \varrho = 1\,\text{cm}$ – the parallel p to AB at the distance ϱ
- 5. $S;\ S \in o \cap p$ – the intersection S
- 6. $k;\ k(S;\ \varrho)$ – the incircle k with centre S
- 7. $\tau;\ \tau(S_1;\ \frac{|BS|}{2})$ – the Thales circle τ over the segment BS
- 8. $T;\ T \in \tau \cap k$ – the point of contact T of the tangent from B (the other intersection is the point of contact on the first side).
- 9. $\overrightarrow{BT}$ – the tangent from the point B to the incircle.
- 10. $C;\ C \in \overrightarrow{AX} \cap \overrightarrow{BT}$ – the third vertex is the intersection.
- 11. $\triangle ABC$ – join the vertices.
- Discussion: in one half-plane bounded by the first side the problem has one solution (in the other half-plane there is its congruent mirror image).

Graph



Step 11 of 11
11. $\triangle ABC$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

A rectangle from a side and the diagonal – the right angle at B.

Input: `construction(rectangle; a=5; e=6.5)`

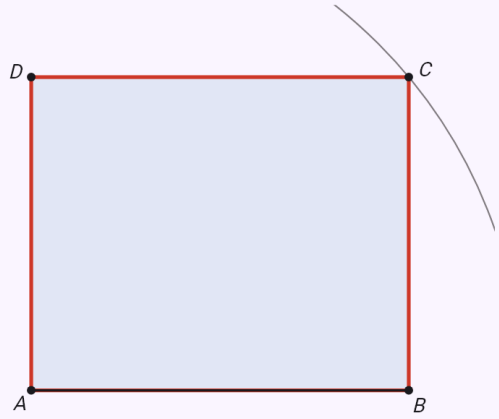
Result

$$a = 5\,\text{cm},\ b \approx 4.15\,\text{cm}$$
$$c = 5\,\text{cm},\ d \approx 4.15\,\text{cm}$$
$$\alpha = 90^\circ,\ \beta = 90^\circ$$
$$\gamma = 90^\circ,\ \delta = 90^\circ$$
$$e = 6.5\,\text{cm},\ f = 6.5\,\text{cm}$$

Steps

- Analysis ($a = 5\,\text{cm},\ e = 6.5\,\text{cm}$): the side AB with the other given elements determines the triangle ABC or ABD, constructed as a triangle problem. The fourth vertex completes it – a parallelogram's opposite sides are equal and parallel.
- 1. AB; $|AB| = 5\,\text{cm}$ – draw the side a
- 2. $\angle ABY$; $|\angle ABY| = 90^\circ$ – construct the angle β; its arm is the ray $\overrightarrow{BY}$
- 3. k; $k(A;\ 6.5\,\text{cm})$ – the circle k with centre A and radius e
- 4. C; $C \in \overrightarrow{BY} \cap k$ – the intersection C
- 5. D; $|CD| = |AB|,\ |AD| = |BC|$ – the fourth vertex D: opposite sides are equal, so it lies on the circles with the centres C, A and the radii $|AB|, |BC|$
- 6. $ABCD$ – join the vertices.
- Discussion: the problem has one solution – any other such quadrilateral is congruent to it.

Graph



Step 6 of 6
6. $ABCD$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

A rhombus from its diagonals – the vertices on the bisector of AC.

Input: `construction(rhombus; e=7; f=4)`

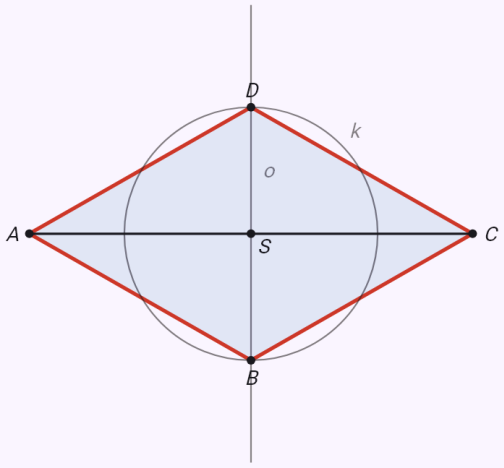
Result

$a \approx 4.03\,\text{cm},\ b \approx 4.03\,\text{cm}$
$c \approx 4.03\,\text{cm},\ d \approx 4.03\,\text{cm}$
$\alpha \approx 59.49^\circ,\ \beta \approx 120.51^\circ$
$\gamma \approx 59.49^\circ,\ \delta \approx 120.51^\circ$
$e = 7\,\text{cm},\ f = 4\,\text{cm}$

Steps

- Analysis ($e = 7\,\text{cm},\ f = 4\,\text{cm}$): the diagonals of a square and a rhombus are perpendicular and halve each other. The vertices B and D therefore lie on the perpendicular bisector of AC, half the other diagonal away from its midpoint S.
- 1. AC; $|AC| = 7\,\text{cm}$ – draw the diagonal AC
- 2. o; $o \perp AC,\ S \in o$ – the perpendicular bisector o of the segment AC, constructed as in "Perpendicular bisector".
- 3. k; $k(S;\ 2\,\text{cm})$ – the circle k with centre S and radius $\frac{f}{2}$
- 4. B, D; $B, D \in o \cap k$ – the intersections B, D
- 5. $ABCD$ – join the vertices.
- Discussion: the problem has one solution – any other such quadrilateral is congruent to it.

Graph



Step 5 of 5
5. $ABCD$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

A parallelogram from two sides and a diagonal – the triangle ABC first.

Input: `construction(parallelogram; a=6; b=4; e=8)`

Result

$a = 6\,\text{cm},\ b = 4\,\text{cm}$

$c = 6\,\text{cm},\ d = 4\,\text{cm}$

$\alpha \approx 75.52^\circ,\ \beta \approx 104.48^\circ$

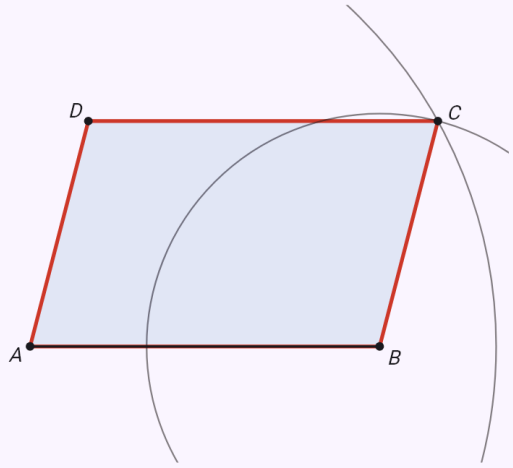
$\gamma \approx 75.52^\circ,\ \delta \approx 104.48^\circ$

$e = 8\,\text{cm},\ f \approx 6.32\,\text{cm}$

Steps

- Analysis ($a = 6\,\text{cm},\ b = 4\,\text{cm},\ e = 8\,\text{cm}$): the side AB with the other given elements determines the triangle ABC or ABD, constructed as a triangle problem. The fourth vertex completes it – a parallelogram's opposite sides are equal and parallel.
- 1. AB; $|AB| = 6\,\text{cm}$ – draw the side a
- 2. k; $k(A; 8\,\text{cm})$ – the circle k with centre A and radius e
- 3. l; $l(B; 4\,\text{cm})$ – the circle l with centre B and radius b
- 4. C; $C \in k \cap l$ – the intersection C
- 5. D; $|CD| = |AB|,\ |AD| = |BC|$ – the fourth vertex D: opposite sides are equal, so it lies on the circles with the centres C, A and the radii $|AB|, |BC|$
- 6. $ABCD$ – join the vertices.
- Discussion: the problem has one solution – any other such quadrilateral is congruent to it.

Graph



Step 6 of 6
6. $ABCD$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

A trapezoid from four sides – the helper triangle EBC.

Input: `construction(trapezoid; a=7; b=4; c=3; d=4)`

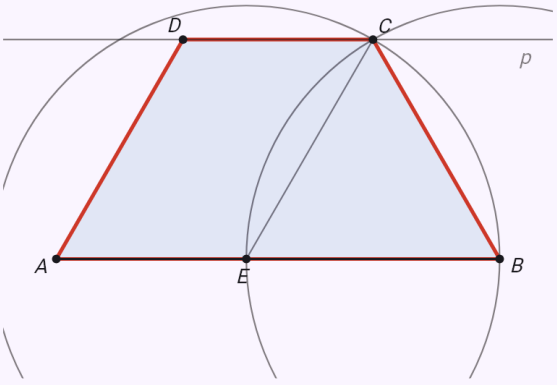
Result

$a = 7\,\text{cm},\ b = 4\,\text{cm}$
$c = 3\,\text{cm},\ d = 4\,\text{cm}$
$\alpha = 60^\circ,\ \beta = 60^\circ$
$\gamma = 120^\circ,\ \delta = 120^\circ$
$e \approx 6.08\,\text{cm},\ f \approx 6.08\,\text{cm}$

Steps

- Analysis ($a = 7\,\text{cm},\ b = 4\,\text{cm},\ c = 3\,\text{cm},\ d = 4\,\text{cm}$): shifting the side AD by the vector DC gives the triangle EBC with E on AB and |AE| = c. Its sides are |EB| = a − c, |BC| = b, |EC| = d and its angles at E and B are α and β. It is constructed first and D completed on the parallel to AB.
- 1. AB; $|AB| = 7\,\text{cm}$ – draw the side a
- 2. E; $E \in AB$, $|AE| = c = 3\,\text{cm}$ – the point E at the distance $3\,\text{cm}$ from A
- 3. k; $k(E;\ 4\,\text{cm})$ – the circle k with centre E and radius d
- 4. l; $l(B;\ 4\,\text{cm})$ – the circle l with centre B and radius b
- 5. C; $C \in k \cap l$ – the intersection C
- 6. D; $D \in p,\ p \parallel AB,\ C \in p,\ |CD| = 3\,\text{cm}$ – the vertex D on the parallel to AB through C, at the distance $3\,\text{cm}$ from it.
- 7. $ABCD$ – join the vertices.
- Discussion: the problem has one solution – any other such quadrilateral is congruent to it.

Graph



Step 7 of 7
7. $ABCD$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

A trapezoid from three sides and an angle – two solutions.

Input: `construction(trapezoid; a=7; b=3.5; c=3; alpha=60)`

Result

ABC_1D_1 :

$a = 7\,\text{cm},\ b = 3.5\,\text{cm}$

$c = 3\,\text{cm},\ d = 1.5\,\text{cm}$

$\alpha = 60^\circ,\ \beta \approx 21.79^\circ$

$\gamma \approx 158.21^\circ,\ \delta = 120^\circ$

$e \approx 3.97\,\text{cm},\ f \approx 6.38\,\text{cm}$

ABC_2D_2 :

$a = 7\,\text{cm},\ b = 3.5\,\text{cm}$

$c = 3\,\text{cm},\ d = 2.5\,\text{cm}$

$\alpha = 60^\circ,\ \beta \approx 38.21^\circ$

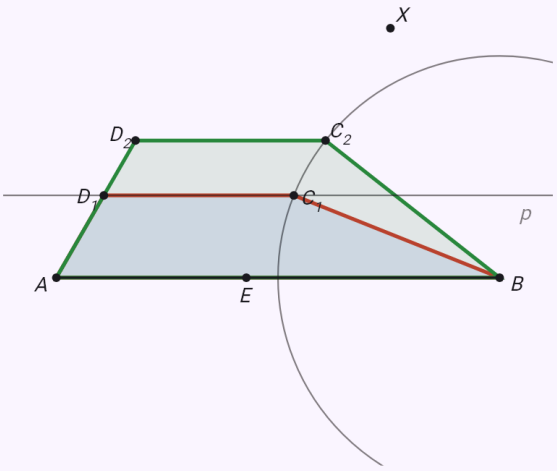
$\gamma \approx 141.79^\circ,\ \delta = 120^\circ$

$e \approx 4.77\,\text{cm},\ f \approx 6.14\,\text{cm}$

Steps

- Analysis ($a = 7\,\text{cm},\ b = 3.5\,\text{cm},\ c = 3\,\text{cm},\ \alpha = 60^\circ$): shifting the side AD by the vector DC gives the triangle EBC with E on AB and |AE| = c. Its sides are |EB| = a − c, |BC| = b, |EC| = d and its angles at E and B are α and β. It is constructed first and D completed on the parallel to AB.
- 1. $AB;\ |AB| = 7\,\text{cm}$ – draw the side a
- 2. $E;\ E \in AB,\ |AE| = c = 3\,\text{cm}$ – the point E at the distance $3\,\text{cm}$ from A
- 3. $\angle BEX;\ |\angle BEX| = 60^\circ$ – construct the angle α; its arm is the ray $\overrightarrow{EX}$
- 4. $k;\ k(B;\ 3.5\,\text{cm})$ – the circle k with centre B and radius b
- 5. $C_1, C_2;\ C_1, C_2 \in \overrightarrow{EX} \cap k$ – the intersections C_1, C_2
- 6. $D_1;\ D_1 \in p,\ p \parallel AB,\ C_1 \in p,\ |C_1D_1| = 3\,\text{cm}$ – the vertex D_1 on the parallel to AB through C_1, at the distance $3\,\text{cm}$ from it.
- 7. $D_2;\ D_2 \in p,\ p \parallel AB,\ C_2 \in p,\ |C_2D_2| = 3\,\text{cm}$ – the vertex D_2 on the parallel to AB through C_2, at the distance $3\,\text{cm}$ from it.
- 8. $ABC_1D_1,\ ABC_2D_2$ – join the vertices.
- Discussion: there are two intersections – the problem has two solutions, two different quadrilaterals.

Graph



Step 8 of 8

8. $ABC_1D_1,\ ABC_2D_2$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

The perpendicular bisector of a 6 cm segment AB.

Input: `construction(segment_bisector; d=6)`

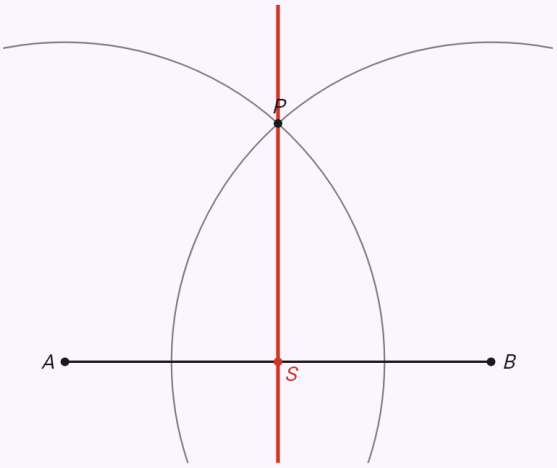
Result

$$o = \{X;\ |AX| = |BX|\}$$
$$|AS| = |SB| = 3\,\text{cm}$$

Steps

- Analysis ($|AB| = 6\,\text{cm}$): a point X equidistant from A and B is the apex of an isosceles triangle ABX – it lies on the perpendicular to AB through its midpoint, the perpendicular bisector.
- 1. $AB;\ |AB| = 6\,\text{cm}$ – draw the segment AB
- 2. $k;\ k(A;\ r),\ r = 4.5\,\text{cm}$ – the circle k with centre A and radius $r > \frac{|AB|}{2}$
- 3. $l;\ l(B;\ r)$ – the circle l with centre B and the same radius.
- 4. $P, Q;\ P, Q \in k \cap l$ – the intersections P, Q
- 5. $o;\ o = \overleftrightarrow{PQ}$ – the line o is the perpendicular bisector: the set of all points equidistant from A and B.

Graph



Step 5 of 5

5. $o;\ o = \overleftrightarrow{PQ}$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

The bisector of a 70° angle.

Input: `construction(angle_bisector; phi=70)`

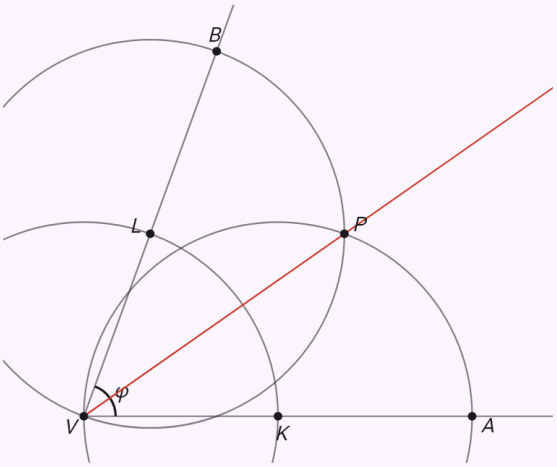
Result

$$o = \overrightarrow{VP}$$
$$|\angle AVP| = |\angle PVB| = 35^\circ$$

Steps

- Analysis ($|\angle AVB| = 70^\circ$): a point inside the angle equidistant from both arms lies on the ray that halves the angle – the angle bisector.
- 1. $\angle AVB;\ |\angle AVB| = 70^\circ$ – draw the angle $\angle AVB$
- 2. $k;\ k(V;\ 3\,\text{cm})$ – the circle k with centre V and any radius.
- 3. $K, L;\ K \in k \cap \overrightarrow{VA},\ L \in k \cap \overrightarrow{VB}$ – the intersections K, L
- 4. $m;\ m(K;\ s),\ s = 3\,\text{cm}$ – the circle m with centre K and radius $s > \frac{|KL|}{2}$
- 5. $n;\ n(L;\ s)$ – the circle n with centre L and the same radius.
- 6. $P;\ P \in m \cap n$ – the intersection P
- 7. $o;\ o = \overrightarrow{VP}$ – the ray o is the angle bisector: it halves the angle and its points are equidistant from both arms.

Graph



Step 7 of 7

7. $o;\ o = \overrightarrow{VP}$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

The Thales circle over a 5 cm segment.

Input: `construction(thales; d=5)`

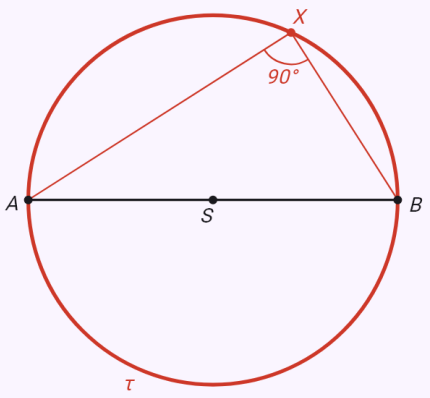
Result

$$\tau(S;\ 2.5\,\mathrm{cm}) \setminus \{A, B\}$$

Steps

- Analysis ($|AB| = 5\,\mathrm{cm}$): by Thales' theorem the angle AXB is right exactly when X lies on the circle with diameter AB (except for A and B).
- 1. $AB;\ |AB| = 5\,\mathrm{cm}$ – draw the segment AB
- 2. $S;\ S \in AB,\ |AS| = |SB|$ – the midpoint S of the segment AB (where its bisector crosses it).
- 3. $\tau;\ \tau(S;\ 2.5\,\mathrm{cm})$ – the Thales circle τ with centre S and radius $\dfrac{|AB|}{2} = 2.5\,\mathrm{cm}$; without A and B it is the set sought.

Graph



Step 3 of 3

3. $\tau;\ \tau(S;\ 2.5\,\mathrm{cm})$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

The points from which a 6 cm segment is seen at 60°.

Input: `construction(angle_arc; d=6; phi=60)`

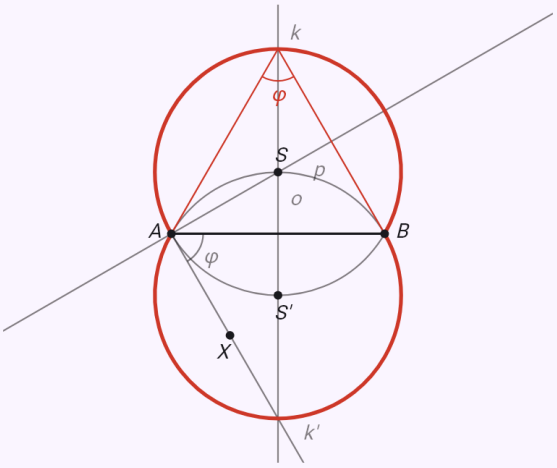
Result

$$r = \frac{|AB|}{2\sin\varphi} \approx 3.46\,\text{cm}$$

Steps

- Analysis ($|AB| = 6\,\text{cm},\ \varphi = 60°$): the points X from which AB is seen at the angle φ lie on two arcs symmetric about AB. The inscribed angle over the chord AB equals the angle between AB and the tangent at A – the arc's centre lies on the bisector of AB and on the perpendicular to the tangent at A.
- 1. AB; $|AB| = 6\,\text{cm}$ – draw the segment AB
- 2. o; $o \perp AB,\ S_{AB} \in o$ – the perpendicular bisector o of the segment AB, constructed as in "Perpendicular bisector".
- 3. $\angle BAX$; $|\angle BAX| = 60°$ – construct the angle φ; its arm is the ray $\overrightarrow{AX}$
- 4. p; $p \perp AX,\ A \in p$ – the perpendicular p to the ray $\overrightarrow{AX}$ at the point A
- 5. S; $S \in o \cap p$ – the intersection S
- 6. k; $k(S; |SA|)$ – the circle k with centre S and radius $|SA|$
- 7. k'; $k' = O_{AB}(k)$ – the circle k', the mirror image of k in the line AB
- 8. $\overset{\frown}{AB} \subset k,\ \overset{\frown}{AB} \subset k'$ – the arcs over the segment AB (without A and B) are the set sought: from each of their points AB is seen at the angle φ

Graph



Step 8 of 8

8. $\overset{\frown}{AB} \subset k,\ \overset{\frown}{AB} \subset k'$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

The tangents from a point 5 cm away to a circle of radius 3 cm.

Input: `construction(tangents_point; r=3; d=5)`

Result

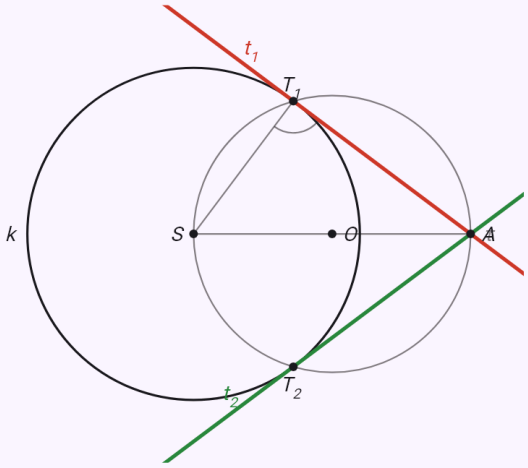
$$|AT_1| = |AT_2| = 4\,\mathrm{cm}$$

$$|AT| = \sqrt{|SA|^2 - r^2}$$

Steps

- Analysis ($r = 3\,\mathrm{cm},\ |SA| = 5\,\mathrm{cm}$): a tangent is perpendicular to the radius at the point of contact T, so the angle STA is right. The points of contact therefore lie on the Thales circle over SA.
- 1. $k;\ k(S;\ 3\,\mathrm{cm})$ – the circle k with centre S and radius $3\,\mathrm{cm}$
- 2. $A;\ |SA| = 5\,\mathrm{cm}$ – the point A at the distance $5\,\mathrm{cm}$ from S
- 3. $O;\ O \in SA,\ |SO| = |OA|$ – the midpoint O of the segment SA (where its bisector crosses it).
- 4. $\tau;\ \tau(O;\ |OA|)$ – the Thales circle τ over the segment SA
- 5. $T_1, T_2;\ T_1, T_2 \in k \cap \tau$ – the points of contact T_1, T_2 are where the circles k and τ meet.
- 6. $t_1, t_2;\ t_1 = \overleftrightarrow{AT_1},\ t_2 = \overleftrightarrow{AT_2}$ – the tangents t_1, t_2 from the point A
- Discussion: the point A is outside the circle – two tangents, symmetric about the line SA, pass through it.

Graph



Step 6 of 6

6. $t_1, t_2;\ t_1 = \overleftrightarrow{AT_1},\ t_2 = \overleftrightarrow{AT_2}$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

Four common tangents of two circles lying apart.

Input: `construction(common_tangents; r1=3; r2=1.5; d=6)`

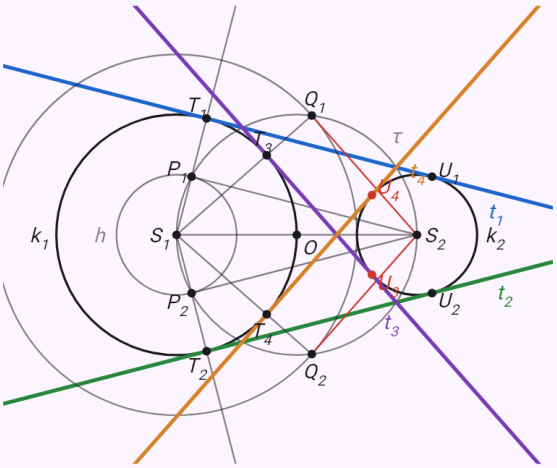
Result

$$|T_1U_1| = |T_2U_2| \approx 5.81\,\text{cm}$$
$$|T_3U_3| = |T_4U_4| \approx 3.97\,\text{cm}$$

Steps

- Analysis ($r_1 = 3\,\text{cm},\ r_2 = 1.5\,\text{cm},\ |S_1S_2| = 6\,\text{cm}$): moving an outer tangent parallel towards the centres by the smaller radius, it touches a helper circle around the bigger circle's centre with the radius $|r_1 - r_2|$ and passes through the smaller one's centre. For an inner tangent the helper circle's radius is $r_1 + r_2$. The tangents from the centre to the helper circle are found with the Thales circle over the line of centres.
- 1. $k_1;\ k_1(S_1;\ 3\,\text{cm})$ – the circle k_1 with centre S_1 and radius $3\,\text{cm}$
- 2. $k_2;\ k_2(S_2;\ 1.5\,\text{cm}),\ |S_1S_2| = 6\,\text{cm}$ – the circle k_2 with centre S_2 and radius $1.5\,\text{cm}$
- 3. $O;\ O \in S_1S_2,\ |S_1O| = |OS_2|$ – the midpoint O of the segment S_1S_2 (where its bisector crosses it).
- 4. $\tau;\ \tau(O;\ |OS_1|)$ – the Thales circle τ over the segment S_1S_2
- 5. $h;\ h(S_1;\ |r_1 - r_2|),\ |r_1 - r_2| = 1.5\,\text{cm}$ – the helper circle h with the centre S_1 and the difference of the radii as its radius, $1.5\,\text{cm}$
- 6. $P_1, P_2;\ P_1, P_2 \in h \cap \tau$ – the intersections P_1, P_2
- 7. $T_1, T_2;\ T_1 \in \overrightarrow{S_1P_1},\ T_2 \in \overrightarrow{S_1P_2}$ – the points of contact T_1, T_2 are where $\overrightarrow{S_1P_1}, \overrightarrow{S_1P_2}$ meet the circle k_1
- 8. $t_1, t_2;\ T_1 \in t_1 \parallel S_2P_1,\ T_2 \in t_2 \parallel S_2P_2$ – the outer common tangents t_1, t_2 through the points T_1, T_2, parallel to S_2P_1, S_2P_2; they touch the other circle too.
- 9. $h';\ h'(S_1;\ r_1 + r_2),\ r_1 + r_2 = 4.5\,\text{cm}$ – the helper circle h' with the centre S_1 and the sum of the radii as its radius, $4.5\,\text{cm}$
- 10. $Q_1, Q_2;\ Q_1, Q_2 \in h' \cap \tau$ – the intersections Q_1, Q_2
- 11. $T_3, T_4;\ T_3 \in S_1Q_1 \cap k_1,\ T_4 \in S_1Q_2 \cap k_1$ – the points of contact T_3, T_4 are where S_1Q_1, S_1Q_2 meet the circle k_1
- 12. $t_3, t_4;\ T_3 \in t_3 \parallel S_2Q_1,\ T_4 \in t_4 \parallel S_2Q_2$ – the inner common tangents t_3, t_4 through the points T_3, T_4, parallel to S_2Q_1, S_2Q_2
- Discussion: the circles lie apart ($|S_1S_2| = 6\,\text{cm} > r_1 + r_2 = 4.5\,\text{cm}$) – they have two outer and two inner common tangents.

Graph



Step 12 of 12

12. $t_3, t_4;\ T_3 \in t_3 \parallel S_2Q_1,\ T_4 \in t_4 \parallel S_2Q_2$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

A circle of radius 2 cm touching the arms of a 60° angle.

Input: `construction(circle_angle; phi=60; r=2)`

Result

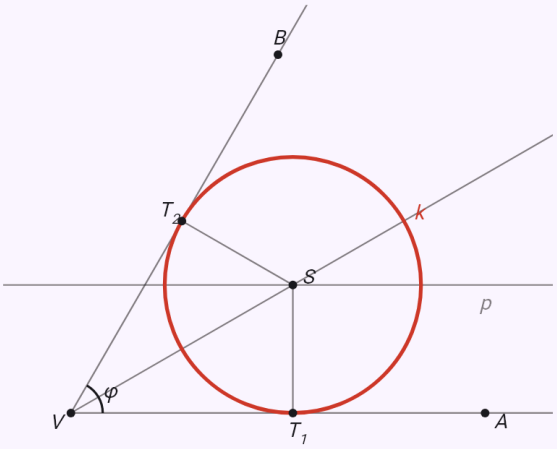
$$|VS| = \frac{r}{\sin\frac{\varphi}{2}} = 4\,\text{cm}$$

$$|VT_1| = |VT_2| = \frac{r}{\tan\frac{\varphi}{2}} \approx 3.46\,\text{cm}$$

Steps

- Analysis ($\varphi = 60°,\ r = 2\,\text{cm}$): the centre is r away from both arms – on the angle's bisector and on the parallel to an arm at the distance r. The points of contact are the feet of the perpendiculars from the centre to the arms.
- 1. $\angle AVB;\ |\angle AVB| = 60°$ – draw the angle $\angle AVB$
- 2. $o;\ o = \overrightarrow{VX},\ |\angle AVX| = \frac{\varphi}{2}$ – the bisector o of the angle $\angle AVB$ (constructed as in "Angle bisector"); its points are equally far from both arms.
- 3. $p;\ p \parallel VA,\ |p, VA| = r = 2\,\text{cm}$ – the parallel p to VA at the distance r
- 4. $S;\ S \in o \cap p$ – the intersection S
- 5. $T_1, T_2;\ ST_1 \perp VA,\ ST_2 \perp VB$ – the points of contact T_1, T_2 are the feet of the perpendiculars from the centre S to VA, VB
- 6. $k;\ k(S;\ 2\,\text{cm})$ – the circle k sought, with the centre S and the radius $2\,\text{cm}$
- Discussion: the bisector and the parallel always meet in a single point – the problem has one solution.

Graph



Step 6 of 6
6. $k;\ k(S;\ 2\,\text{cm})$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

Circles through a point touching a line – two solutions.

Input: `construction(circle_point_line; v=1.5; r=2.5)`

Result

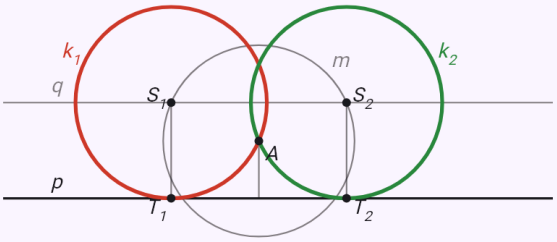
$$k_1(S_1;\ 2.5\,\text{cm}),\ k_2(S_2;\ 2.5\,\text{cm})$$

$$|T_1T_2| = 2\sqrt{v(2r-v)} \approx 4.58\,\text{cm}$$

Steps

- Analysis ($|Ap| = v = 1.5\,\text{cm},\ r = 2.5\,\text{cm}$): the centre is r away from the line p, so it lies on the parallel q to p at the distance r on A's side. It is also r away from A, so it lies on the circle m(A; r) too.
- 1. p – any line p
- 2. $A;\ |Ap| = 1.5\,\text{cm}$ – the point A at the distance $1.5\,\text{cm}$ from p
- 3. $q;\ q \parallel p,\ |q,p| = r = 2.5\,\text{cm}$ – the parallel q to p at the distance r
- 4. $m;\ m(A;\ 2.5\,\text{cm})$ – the circle m with centre A and radius $2.5\,\text{cm}$
- 5. $S_1, S_2;\ S_1, S_2 \in q \cap m$ – the intersections S_1, S_2
- 6. $T_1, T_2;\ S_1T_1 \perp p,\ S_2T_2 \perp p$ – the points of contact T_1, T_2 are the feet of the perpendiculars from the centre S_1, S_2 to p
- 7. $k_1, k_2;\ k_1(S_1;\ 2.5\,\text{cm}),\ k_2(S_2;\ 2.5\,\text{cm})$ – the circles k_1, k_2 sought, with the centres S_1, S_2 and the radius $2.5\,\text{cm}$
- Discussion: $|Ap| = 1.5\,\text{cm} < 2r = 5\,\text{cm}$ – the circle m meets the parallel q in two points, the problem has two solutions.

Graph



Step 7 of 7

7. $k_1, k_2;\ k_1(S_1;\ 2.5\,\text{cm}),\ k_2(S_2;\ 2.5\,\text{cm})$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

Circles of a given radius through two points – two solutions, symmetric about AB.

Input: `construction(circle_two_points; d=6; r=5)`

Result

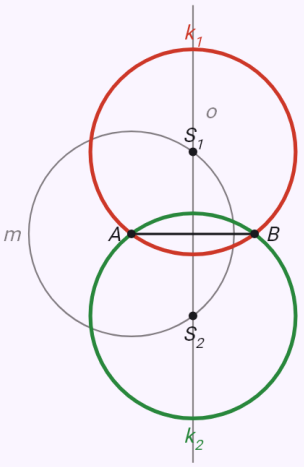
$$k_1(S_1;\ 5\,\mathrm{cm}),\ k_2(S_2;\ 5\,\mathrm{cm})$$

$$|S_1S_2| = 2\sqrt{r^2 - \frac{|AB|^2}{4}} = 8\,\mathrm{cm}$$

Steps

- Analysis ($|AB| = 6\,\mathrm{cm},\ r = 5\,\mathrm{cm}$): the centre is r away from both A and B – on the perpendicular bisector of AB and on the circle m(A; r).
- 1. $AB;\ |AB| = 6\,\mathrm{cm}$ – draw the segment AB
- 2. $o;\ o \perp AB,\ S_{AB} \in o$ – the perpendicular bisector o of the segment AB, constructed as in "Perpendicular bisector".
- 3. $m;\ m(A;\ 5\,\mathrm{cm})$ – the circle m with centre A and radius $5\,\mathrm{cm}$
- 4. $S_1, S_2;\ S_1, S_2 \in o \cap m$ – the intersections S_1, S_2
- 5. $k_1, k_2;\ k_1(S_1;\ 5\,\mathrm{cm}),\ k_2(S_2;\ 5\,\mathrm{cm})$ – the circles k_1, k_2 sought, with the centres S_1, S_2 and the radius $5\,\mathrm{cm}$
- Discussion: $|AB| = 6\,\mathrm{cm} < 2r = 10\,\mathrm{cm}$ – the circle m meets the bisector in two points, the problem has two solutions, symmetric about AB.

Graph



Step 5 of 5

5. $k_1, k_2;\ k_1(S_1;\ 5\,\mathrm{cm}),\ k_2(S_2;\ 5\,\mathrm{cm})$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

The vertex C from which AB is seen at 60°, at the height 3 cm – arcs and parallels.

Input: `construction(relations_segment; d=6; angle_AB=60; dist_AB=3)`

Result

$\triangle ABC_1$:

$a \approx 6.91\,\text{cm},\ b \approx 3.01\,\text{cm},\ c = 6\,\text{cm}$

$\alpha \approx 94.26^\circ,\ \beta \approx 25.74^\circ,\ \gamma = 60^\circ$

$\triangle ABC_2$:

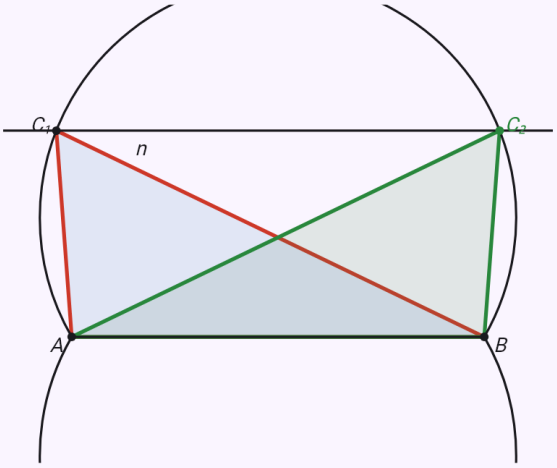
$a \approx 3.01\,\text{cm},\ b \approx 6.91\,\text{cm},\ c = 6\,\text{cm}$

$\alpha \approx 25.74^\circ,\ \beta \approx 94.26^\circ,\ \gamma = 60^\circ$

Steps

- Analysis ($|AB| = 6\,\text{cm}$): a sketch of the finished triangle ABC with the given relations highlighted. Each of the two relations gives a locus of points the vertex C lies on – so C is in their intersection.
- 1. AB; $|AB| = 6\,\text{cm}$ – draw the given segment AB.
- 2. m, m'; $m \cup m' = \{C;\ |\angle ACB| = 60^\circ\}$ – the points from which the segment AB is seen at the angle 60° lie on two arcs m, m' symmetric about AB (constructed as in "Arcs of a given angle").
- 3. n, n'; $n \parallel AB,\ n' \parallel AB,\ |nAB| = |n'AB| = 3\,\text{cm}$ – the points at the distance $3\,\text{cm}$ from the line AB lie on two lines n, n' parallel to it, one on each side.
- 4. C_1, C_2; $C_1, C_2 \in (m \cup m') \cap (n \cup n')$ – the points sought are the intersections of both loci.
- 5. $\triangle ABC_1,\ \triangle ABC_2$ – join the vertices.
- Discussion: in the half-plane above AB the loci have 2 common points – the problem has 2 solutions (in the opposite half-plane their congruent mirror images).

Graph



Step 5 of 5

5. $\triangle ABC_1,\ \triangle ABC_2$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

The ratio |AC| : |BC| = 2 and |BC| = 3 cm – the Apollonius circle.

Input: `construction(relations_segment; d=6; ratio_AB=2; dist_B=3)`

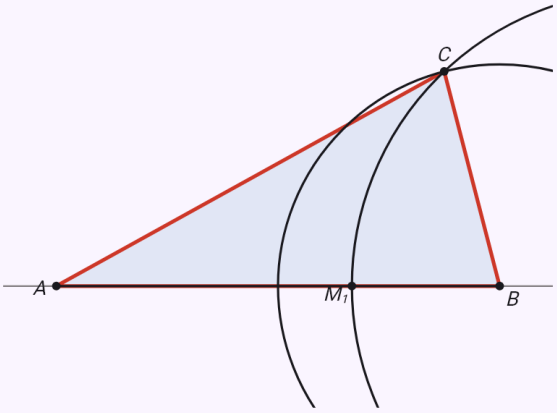
Result

$$a = 3\,\text{cm},\ b = 6\,\text{cm},\ c = 6\,\text{cm}$$
$$\alpha \approx 28.96^\circ,\ \beta \approx 75.52^\circ,\ \gamma \approx 75.52^\circ$$

Steps

- Analysis ($|AB| = 6\,\text{cm}$): a sketch of the finished triangle ABC with the given relations highlighted. Each of the two relations gives a locus of points the vertex C lies on – so C is in their intersection.
- 1. AB; $|AB| = 6\,\text{cm}$ – draw the given segment AB.
- 2. M_1, M_2; $M_1, M_2 \in \overleftrightarrow{AB}$, $|AM_i| : |BM_i| = 2$ – the points whose distances from the ends of the segment AB are in the ratio 2 lie on the Apollonius circle. First the points M_1 (inside the segment) and M_2 (on its extension) dividing AB in this ratio.
- 3. m; $m(S_{M_1M_2};\ \frac{|M_1M_2|}{2})$ – the Apollonius circle m is the Thales circle with the diameter M_1M_2.
- 4. n; $n(B;\ 3\,\text{cm})$ – the points at the distance $3\,\text{cm}$ from the point B lie on the circle n with the centre B
- 5. C; $C \in m \cap n$ – the point sought is the intersection of both loci.
- 6. $\triangle ABC$ – join the vertices.
- Discussion: in the half-plane above AB the loci have one common point – the problem has one solution (in the opposite half-plane its congruent mirror image).

Graph



Step 6 of 6
6. $\triangle ABC$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

The points of contact of the tangents from A – AX a tangent and X on the circle.

Input: `construction(relations_circle_point; r=3; d=5; tangent_A=1; on_k=1)`

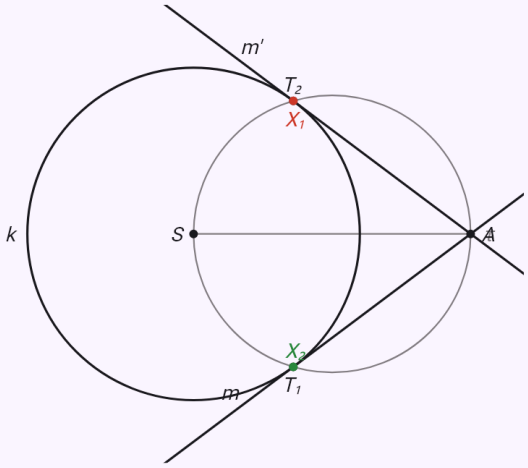
Result

$$X_1: \ |SX_1| = 3\,\text{cm}, \ |AX_1| = 4\,\text{cm}$$
$$X_2: \ |SX_2| = 3\,\text{cm}, \ |AX_2| = 4\,\text{cm}$$

Steps

- Analysis ($k(S;\,3\,\text{cm}),\ |SA| = 5\,\text{cm}$): a sketch with the given circle k, the point A and the point X sought, the given relations highlighted. Each of the two relations gives a locus of points – X lies in their intersection.
- 1. $k, A;\ k(S;\,3\,\text{cm}),\ |SA| = 5\,\text{cm}$ – draw the given circle k and the point A.
- 2. $\tau;\ \tau(S_{SA};\,\frac{|SA|}{2})$ – the line AX is to be a tangent to k: the points of contact lie on the Thales circle τ with the diameter SA.
- 3. $m, m';\ m = \overleftrightarrow{AT_1},\ m' = \overleftrightarrow{AT_2}$ – τ meets k at the points of contact T_1, T_2, the tangents m, m' pass through them.
- The point X is to lie on the circle k – the locus is the circle k itself, already drawn.
- 4. $X_1, X_2;\ X_1, X_2 \in (m \cup m') \cap n$ – the points sought are the intersections of both loci.
- Discussion: the loci have 2 common points – the problem has 2 solutions.

Graph



Step 4 of 4

4. $X_1, X_2;\ X_1, X_2 \in (m \cup m') \cap n$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

The centres of the circles of radius 1 cm touching both the circle k and the line p.

Input: `construction(relations_circle_line; r=2; d=3; touch_k=1; dist_p=1)`

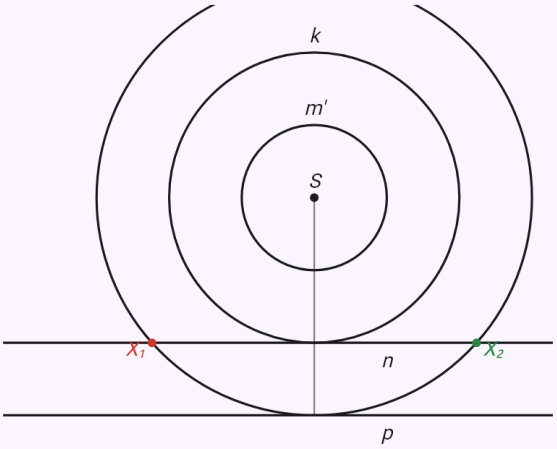
Result

$$X_1 : \ |SX_1| = 3\,\text{cm},\ |X_1p| = 1\,\text{cm}$$
$$X_2 : \ |SX_2| = 3\,\text{cm},\ |X_2p| = 1\,\text{cm}$$

Steps

- Analysis ($k(S;\,2\,\text{cm}),\ |Sp| = 3\,\text{cm}$): a sketch with the given circle k, the line p and the point X sought, the given relations highlighted. Each of the two relations gives a locus of points – X lies in their intersection.
- 1. $k, p;\ k(S;\,2\,\text{cm}),\ |Sp| = 3\,\text{cm}$ – draw the given circle k and the line p.
- 2. $m, m';\ m(S;\,3\,\text{cm}),\ m'(S;\,1\,\text{cm})$ – a circle of radius $1\,\text{cm}$ touches k from outside when its centre is r + ϱ from S, and from inside when it is |r − ϱ| from S: the centres lie on the circles m, m'
- 3. $n, n';\ n \parallel p,\ n' \parallel p,\ |np| = |n'p| = 1\,\text{cm}$ – the points at the distance $1\,\text{cm}$ from the line p lie on two lines n, n' parallel to it, one on each side.
- 4. $X_1, X_2;\ X_1, X_2 \in (m \cup m') \cap (n \cup n')$ – the points sought are the intersections of both loci.
- Discussion: the loci have 2 common points – the problem has 2 solutions.

Graph



Step 4 of 4

4. $X_1, X_2;\ X_1, X_2 \in (m \cup m') \cap (n \cup n')$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

The centres of the circles of radius 1.5 cm touching two intersecting lines – four solutions.

Input: `construction(relations_lines; phi=60; dist_p=1.5; dist_q=1.5)`

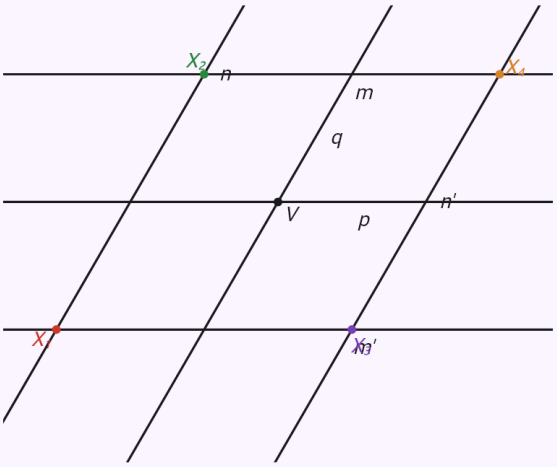
Result

$X_1 : \ |VX_1| = 3\,\text{cm}$
$|X_1p| = 1.5\,\text{cm}, \ |X_1q| = 1.5\,\text{cm}$
$X_2 : \ |VX_2| \approx 1.73\,\text{cm}$
$|X_2p| = 1.5\,\text{cm}, \ |X_2q| = 1.5\,\text{cm}$
$X_3 : \ |VX_3| \approx 1.73\,\text{cm}$
$|X_3p| = 1.5\,\text{cm}, \ |X_3q| = 1.5\,\text{cm}$
$X_4 : \ |VX_4| = 3\,\text{cm}$
$|X_4p| = 1.5\,\text{cm}, \ |X_4q| = 1.5\,\text{cm}$

Steps

- Analysis ($|\angle(p,q)| = 60°$): a sketch with the given lines p, q and the point X sought, the given relations highlighted. Each of the two relations gives a locus of points – X lies in their intersection.
- 1. $p, q;\ p \cap q = V,\ |\angle(p,q)| = 60°$ – draw the given lines p, q meeting at V.
- 2. $m, m';\ m \parallel p,\ m' \parallel p,\ |mp| = |m'p| = 1.5\,\text{cm}$ – the points at the distance $1.5\,\text{cm}$ from the line p lie on two lines m, m' parallel to it, one on each side.
- 3. $n, n';\ n \parallel q,\ n' \parallel q,\ |nq| = |n'q| = 1.5\,\text{cm}$ – the points at the distance $1.5\,\text{cm}$ from the line q lie on two lines n, n' parallel to it, one on each side.
- 4. $X_1, X_2, X_3, X_4;\ X_1, X_2, X_3, X_4 \in (m \cup m') \cap (n \cup n')$ – the points sought are the intersections of both loci.
- Discussion: the loci have 4 common points – the problem has 4 solutions.

Graph



Step 4 of 4

4. $X_1, X_2, X_3, X_4;\ X_1, X_2, X_3, X_4 \in (m \cup m') \cap$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

A circle k(S; 2 cm), AC a tangent, k the circumcircle of ABD, D the midpoint of BC, |AC| = 2|AD| – |AD| = r√7/2.

Input: `construction(general_circle; r=2; on_A=1; on_B=1; on_D=1; mid_DBC=1; tangent_AC=1;`

ratio_ACAD=2; triangle_ABC=1)

Result

$$|AB| = \sqrt{14}\,\text{cm} = \frac{\sqrt{14}}{2}\,r$$

$$|AD| = \sqrt{7}\,\text{cm} = \frac{\sqrt{7}}{2}\,r$$

$$|AC| = 2\sqrt{7}\,\text{cm} = \sqrt{7}\,r$$

$$|BD| = \sqrt{14}\,\text{cm} = \frac{\sqrt{14}}{2}\,r$$

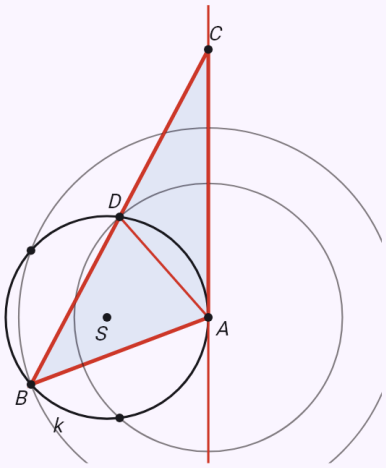
$$|BC| = 2\sqrt{14}\,\text{cm} = \sqrt{14}\,r$$

$$|DC| = \sqrt{14}\,\text{cm} = \frac{\sqrt{14}}{2}\,r$$

Steps

- Analysis ($k(S;\,2\,\text{cm})$): we write the relations in coordinates. The centre S at the origin and the point A on the positive x axis, A = [r; 0] – any solution can be turned into this position. The unknowns are the other points' coordinates.
- The point lies on the circle k ($A \in k$): $x_A^2 + y_A^2 = 4$
- The point lies on the circle k ($B \in k$): $x_B^2 + y_B^2 = 4$
- The point lies on the circle k ($D \in k$): $x_D^2 + y_D^2 = 4$
- The midpoint ($D = S_{BC}$): $x_D = \frac{x_B + x_C}{2}$, $y_D = \frac{y_B + y_C}{2}$
- The line $\overleftrightarrow{AC}$ is a tangent to k – its distance from the centre S is r: $(x_A y_C - x_C y_A)^2 = r^2\,|AC|^2$
- The ratio of lengths ($|AC| = 2\,|AD|$): $|AC|^2 = 4\,|AD|^2$
- $\triangle ABC$ is the resulting figure – its vertices must not be collinear.
- We solve the system exactly (no rounding). It has 2 different positions of the points (none coinciding), 1 up to congruence.
- Check: with the solution substituted all the relations hold.
- From the solution: the power of the point C with respect to k – the tangent squared equals the product of the secant's segments: $|CA|^2 = |CD| \cdot |CB| = 28$
- From the solution: the tangent-chord angle of the chord AD equals the inscribed angle over it: $|\angle CAD| = |\angle ABD| \approx 41.41°$
- 1. k; $k(S;\,2\,\text{cm})$ – draw the given circle k.
- 2. A; $A \in k$ – any point A of the circle k.
- The length $|AB| = \sqrt{14}\,\text{cm}$ is constructed as a leg of a right triangle with a given hypotenuse: $\sqrt{(4.5)^2 - (2.5)^2}$
- 3. B; $B \in k \cap m(A;\,\sqrt{14}\,\text{cm})$ – the point B is one of the loci's intersections (their number: 2).
- The length $|AD| = \sqrt{7}\,\text{cm}$ is constructed as a leg of a right triangle with a given hypotenuse: $\sqrt{4^2 - 3^2}$
- 4. D; $D \in k \cap m(A;\,\sqrt{7}\,\text{cm})$ – the point D is one of the loci's intersections (their number: 2).
- 5. C; $C = S_D(B)$ – the point C symmetric to B about the centre D (point reflection).
- 6. $\triangle ABC$ – the finished figure.
- Discussion: 2 positions of the points satisfy the relations, all congruent (symmetric) – up to congruence the problem has one solution.

Graph



Step 6 of 6
6. $\triangle ABC$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

A right triangle without a circle: |AB| = 5 cm, |AC| = 3 cm, the right angle at C.

Input: `construction(general; len_AB=5; len_AC=3; angle_ACB=90; triangle_ABC=1)`

Result

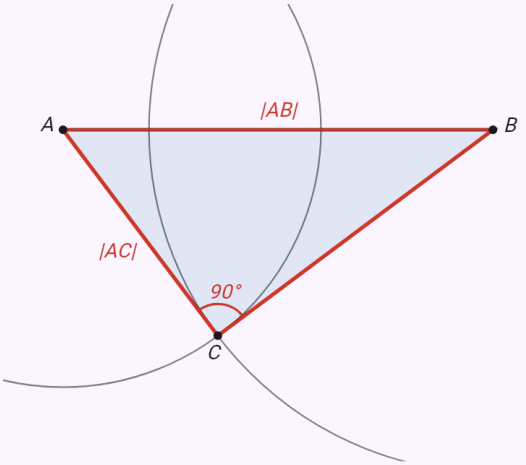
$$|AB| = 5\,\text{cm}$$
$$|AC| = 3\,\text{cm}$$
$$|BC| = 4\,\text{cm}$$

Steps

- Analysis: we write the relations in coordinates. The point A at the origin and the point B on the positive x axis – any solution can be moved and turned into this position. The unknowns are the other points' coordinates.
- The length ($|AB| = 5\,\text{cm}$): $|AB|^2 = (x_B - x_A)^2 + (y_B - y_A)^2 = 25$
- The length ($|AC| = 3\,\text{cm}$): $|AC|^2 = (x_C - x_A)^2 + (y_C - y_A)^2 = 9$
- The angle ($|\angle ACB| = 90^\circ$): $\overrightarrow{CA} \cdot \overrightarrow{CB} = |CA|\,|CB| \cos 90^\circ$
- $\triangle ABC$ is the resulting figure – its vertices must not be collinear.
- We solve the system exactly (no rounding). It has 2 different positions of the points (none coinciding), 1 up to congruence.
- Check: with the solution substituted all the relations hold.
- 1. A – any point A
- 2. B; $|AB| = 5\,\text{cm}$ – the point B at the computed distance from A
- 3. C; $C \in m(A;\,3\,\text{cm}) \cap m(B;\,4\,\text{cm})$ – the point C is one of the loci's intersections (their number: 2).
- 4. $\triangle ABC$ – the finished figure.

- Discussion: 2 positions of the points satisfy the relations, all congruent (symmetric) – up to congruence the problem has one solution.

Graph



Step 4 of 4
4. $\triangle ABC$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

A cube's section – the opposite faces in parallels.

Input: `construction(cube_section; a=6; AE=2; BF=4; CG=5)`

Result

$XYZK$

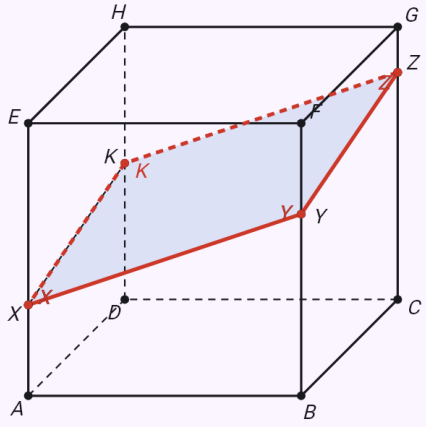
$|XY| \approx 6.32\,\text{cm},\ |YZ| \approx 6.08\,\text{cm}$

$|ZK| \approx 6.32\,\text{cm},\ |KX| \approx 6.08\,\text{cm}$

Steps

- Analysis ($a = 6\,\text{cm}$): the cutting plane is given by the points X, Y, Z. The section is constructed side by side by three rules: two points of the section in one face are joined; parallel faces are cut in parallel segments; a side's line extended meets an edge's line in a point lying in the planes of both faces at that edge.
- 1. $ABCDEFGH;\ a = 6\,\text{cm}$ – draw the cube $ABCDEFGH$ in the oblique projection (the depth at 45°, halved), the hidden edges dashed.
- 2. $X;\ X \in AE,\ |AX| = 2\,\text{cm}$ – the point X on the edge AE at the distance $2\,\text{cm}$ from the vertex A
- 3. $Y;\ Y \in BF,\ |BY| = 4\,\text{cm}$ – the point Y on the edge BF at the distance $4\,\text{cm}$ from the vertex B
- 4. $Z;\ Z \in CG,\ |CZ| = 5\,\text{cm}$ – the point Z on the edge CG at the distance $5\,\text{cm}$ from the vertex C
- 5. XY – the points X, Y lie in the face $ABFE$, join them.
- 6. YZ – the points Y, Z lie in the face $BCGF$, join them.
- 7. $ZK;\ ZK \parallel YX,\ K \in DH$ – the face $CDHG$ is parallel to the face $ABFE$, so the section cuts it in the parallel to YX through Z; it meets the edge DH at K
- 8. XK – the points X, K lie in the face $ADHE$, join them.
- 9. $XYZK$ – the section is the quadrilateral $XYZK$

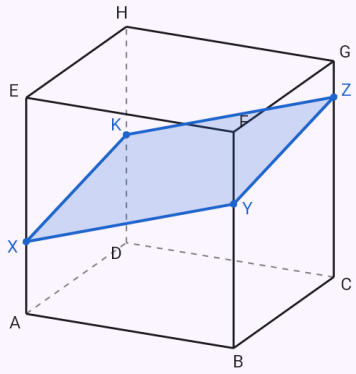
Graph



Step 9 of 9
9. $XYZK$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

The section in space – turn it with a finger



The regular hexagon through the midpoints of edges – the helper point T_1.

Input: `construction(cube_section; a=6; AB=3; BC=3; GH=3)`

Result

$XYMZKL$

$|XY| \approx 4.24\,\text{cm},\ |YM| \approx 4.24\,\text{cm}$

$|MZ| \approx 4.24\,\text{cm},\ |ZK| \approx 4.24\,\text{cm}$

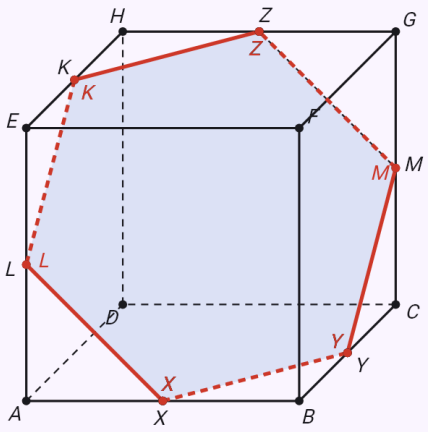
$|KL| \approx 4.24\,\text{cm},\ |LX| \approx 4.24\,\text{cm}$

Steps

- Analysis ($a = 6\,\text{cm}$): the cutting plane is given by the points X, Y, Z. The section is constructed side by side by three rules: two points of the section in one face are joined; parallel faces are cut in parallel segments; a side's line extended meets an edge's line in a point lying in the planes of both faces at that edge.
- 1. $ABCDEFGH;\ a = 6\,\text{cm}$ – draw the cube $ABCDEFGH$ in the oblique projection (the depth at 45°, halved), the hidden edges dashed.
- 2. $X;\ X \in AB,\ |AX| = 3\,\text{cm}$ – the point X on the edge AB at the distance $3\,\text{cm}$ from the vertex A
- 3. $Y;\ Y \in BC,\ |BY| = 3\,\text{cm}$ – the point Y on the edge BC at the distance $3\,\text{cm}$ from the vertex B
- 4. $Z;\ Z \in GH,\ |GZ| = 3\,\text{cm}$ – the point Z on the edge GH at the distance $3\,\text{cm}$ from the vertex G
- 5. XY – the points X, Y lie in the face $ABCD$, join them.
- 6. $ZK;\ ZK \parallel XY,\ K \in EH$ – the face $EFGH$ is parallel to the face $ABCD$, so the section cuts it in the parallel to XY through Z; it meets the edge EH at K
- 7. $T_1;\ T_1 \in \overleftrightarrow{ZK} \cap \overleftrightarrow{EF}$ – extend the side ZK and the edge EF; their common point T_1 lies in the plane of the face $EFGH$ and of the face $ABFE$
- 8. $XL;\ L \in \overleftrightarrow{XT_1} \cap AE$ – the line XT_1 lies in the plane of the face $ABFE$; it meets the edge AE at L
- 9. KL – the points K, L lie in the face $ADHE$, join them.
- 10. $ZM;\ ZM \parallel XL,\ M \in CG$ – the face $CDHG$ is parallel to the face $ABFE$, so the section cuts it in the parallel to XL through Z; it meets the edge CG at M
- 11. YM – the points Y, M lie in the face $BCGF$, join them.

- 12. $XYMZKL$ – the section is the hexagon $XYMZKL$

Graph

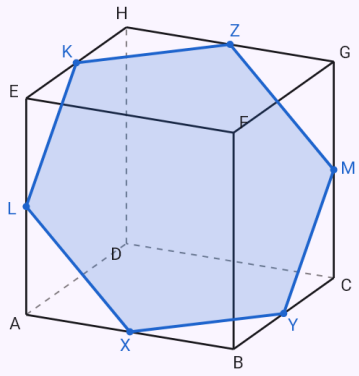


Step 12 of 12

12. $XYMZKL$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

The section in space – turn it with a finger



A pyramid cut by the plane of the midpoints of two lateral edges and a base edge.

Input: `construction(pyramid_section; a=6; v=6; AB=3; BC=3; DV=4)`

Result

$XYKZL$

$|XY| \approx 4.24\,\text{cm},\ |YK| \approx 2.87\,\text{cm}$

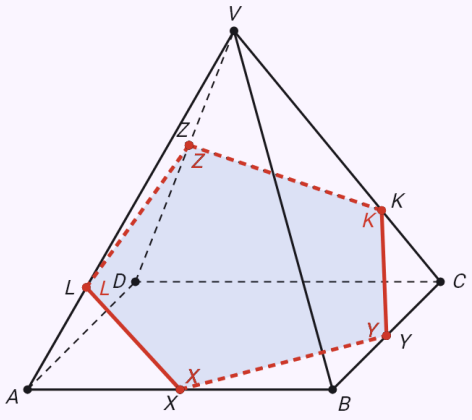
$|KZ| \approx 3.92\,\text{cm},\ |ZL| \approx 3.92\,\text{cm}$

$|LX| \approx 2.87\,\text{cm}$

Steps

- Analysis ($a = 6\,\text{cm},\ v = 6\,\text{cm}$): the cutting plane is given by the points X, Y, Z. The section is constructed side by side: two points of the section in one face are joined; a side parallel to another face's edge means the plane cuts that face in a parallel to it; a side's line extended meets an edge's line in a point lying in the planes of both faces at that edge.
- 1. $ABCDV;\ a = 6\,\text{cm},\ v = 6\,\text{cm}$ – draw the regular square pyramid $ABCDV$ in the oblique projection (the depth at 45°, halved), the hidden edges dashed.
- 2. $X;\ X \in AB,\ |AX| = 3\,\text{cm}$ – the point X on the edge AB at the distance $3\,\text{cm}$ from the vertex A
- 3. $Y;\ Y \in BC,\ |BY| = 3\,\text{cm}$ – the point Y on the edge BC at the distance $3\,\text{cm}$ from the vertex B
- 4. $Z;\ Z \in DV,\ |DZ| = 4\,\text{cm}$ – the point Z on the edge DV at the distance $4\,\text{cm}$ from the vertex D
- 5. XY – the points X, Y lie in the face $ABCD$, join them.
- 6. $T_1;\ T_1 \in \overleftrightarrow{XY} \cap \overleftrightarrow{CD}$ – extend the side XY and the edge CD; their common point T_1 lies in the plane of the face $ABCD$ and of the face CDV
- 7. $ZK;\ K \in \overleftrightarrow{ZT_1} \cap CV$ – the line ZT_1 lies in the plane of the face CDV; it meets the edge CV at K
- 8. YK – the points Y, K lie in the face BCV, join them.
- 9. $T_2;\ T_2 \in \overleftrightarrow{YK} \cap \overleftrightarrow{BV}$ – extend the side YK and the edge BV; their common point T_2 lies in the plane of the face BCV and of the face ABV
- 10. $XL;\ L \in \overleftrightarrow{XT_2} \cap AV$ – the line XT_2 lies in the plane of the face ABV; it meets the edge AV at L
- 11. ZL – the points Z, L lie in the face DAV, join them.
- 12. $XYKZL$ – the section is the pentagon $XYKZL$

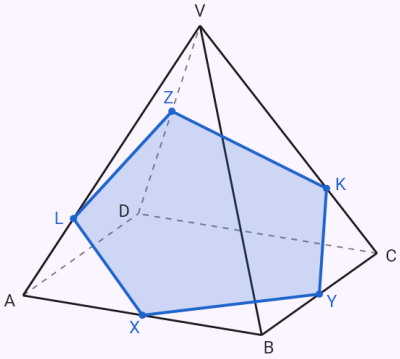
Graph



Step 12 of 12
12. $XYKZL$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

The section in space – turn it with a finger



A point on an edge's extension – Y 2 cm past the vertex B.

Input: `construction(pyramid_section; n=4; a=4; v=6; AV=1; AB=6; CD=2)`

Result

$XLKZM$

$|XL| \approx 3.64\,\text{cm},\ |LK| \approx 1.92\,\text{cm}$

$|KZ| \approx 2.83\,\text{cm},\ |ZM| \approx 1.92\,\text{cm}$

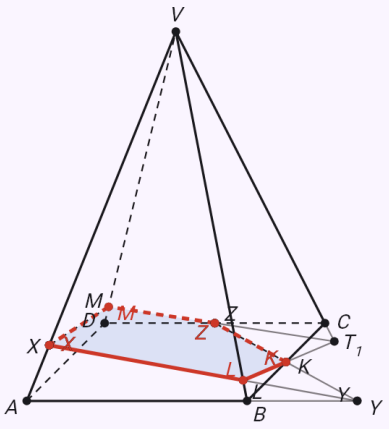
$|MX| \approx 3.64\,\text{cm}$

Steps

- Analysis ($a = 4\,\text{cm},\ v = 6\,\text{cm}$): the cutting plane is given by the points X, Y, Z. The section is constructed side by side: two points of the section in one face are joined; a side parallel to another face's edge means the plane cuts that face in a parallel to it; a side's line extended meets an edge's line in a point lying in the planes of both faces at that edge.
- 1. $ABCDV$; $a = 4\,\text{cm},\ v = 6\,\text{cm}$ – draw the regular square pyramid $ABCDV$ in the oblique projection (the depth at 45°, halved), the hidden edges dashed.
- 2. X; $X \in AV,\ |AX| = 1\,\text{cm}$ – the point X on the edge AV at the distance $1\,\text{cm}$ from the vertex A
- 3. Y; $Y \in\!\mapsto AB,\ |AY| = 6\,\text{cm}$ – the point Y on the ray AB at the distance $6\,\text{cm}$ from the vertex A, past the vertex B – outside the solid.
- 4. Z; $Z \in CD,\ |CZ| = 2\,\text{cm}$ – the point Z on the edge CD at the distance $2\,\text{cm}$ from the vertex C
- 5. ZK; $ZK \subset \overleftrightarrow{YZ}$ – the points YZ lie in the plane of the face $ABCD$; their line cuts the face in the segment ZK
- 6. XL; $XL \subset \overleftrightarrow{XY}$ – the points XY lie in the plane of the face ABV; their line cuts the face in the segment XL
- 7. KL – the points K, L lie in the face BCV, join them.
- 8. T_1; $T_1 \in \overleftrightarrow{KL} \cap \overleftrightarrow{CV}$ – extend the side KL and the edge CV; their common point T_1 lies in the plane of the face BCV and of the face CDV
- 9. ZM; $M \in \overleftrightarrow{ZT_1} \cap DV$ – the line ZT_1 lies in the plane of the face CDV; it meets the edge DV at M
- 10. XM – the points X, M lie in the face DAV, join them.

- 11. $XLKZM$ – the section is the pentagon $XLKZM$

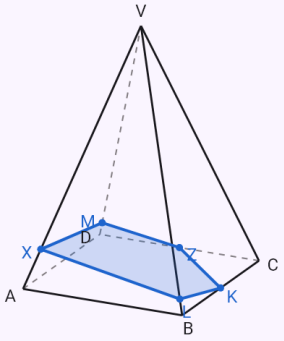
Graph



Step 11 of 11
11. $XLKZM$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

The section in space – turn it with a finger



A point inside the solid on its diagonal – an auxiliary vertical plane.

Input: `construction(cube_section; a=6; AG=5; BF=1; DH=2)`

Result

$QPYMNZ$

$|QP| \approx 7.45\,\text{cm},\ |PY| \approx 5.06\,\text{cm}$

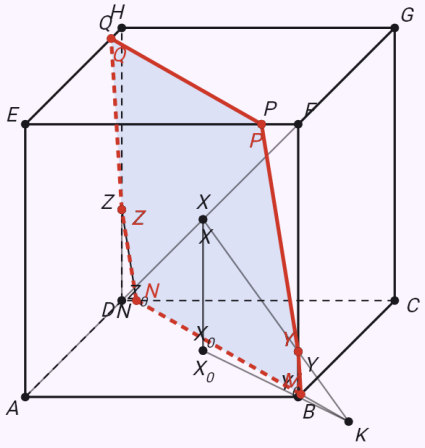
$|YM| \approx 1.01\,\text{cm},\ |MN| \approx 8.14\,\text{cm}$

$|NZ| \approx 2.03\,\text{cm},\ |ZQ| \approx 4.05\,\text{cm}$

Steps

- Analysis ($a = 6\,\text{cm}$): the cutting plane is given by the points X, Y, Z. The section is constructed side by side by three rules: two points of the section in one face are joined; parallel faces are cut in parallel segments; a side's line extended meets an edge's line in a point lying in the planes of both faces at that edge.
- 1. $ABCDEFGH;\ a = 6\,\text{cm}$ – draw the cube $ABCDEFGH$ in the oblique projection (the depth at 45°, halved), the hidden edges dashed.
- 2. $X;\ X \in AG,\ |AX| = 5\,\text{cm}$ – the point X on the segment AG at the distance $5\,\text{cm}$ from the point A
- 3. $Y;\ Y \in BF,\ |BY| = 1\,\text{cm}$ – the point Y on the edge BF at the distance $1\,\text{cm}$ from the vertex B
- 4. $Z;\ Z \in DH,\ |DZ| = 2\,\text{cm}$ – the point Z on the edge DH at the distance $2\,\text{cm}$ from the vertex D
- 5. $K;\ K \in \overleftrightarrow{XY} \cap \overleftrightarrow{X_0Y_0}$ – the point X lies inside the solid, so we use the auxiliary vertical plane of the line XY: we project the points into the base (X_0Y_0), and the lines XY and X_0Y_0 meet at K, where the line XY pierces the plane of the base $ABCD$
- 6. $L;\ L \in \overleftrightarrow{XZ} \cap \overleftrightarrow{X_0Z_0}$ – the point X lies inside the solid, so we use the auxiliary vertical plane of the line XZ: we project the points into the base (X_0Z_0), and the lines XZ and X_0Z_0 meet at L, where the line XZ pierces the plane of the base $ABCD$
- 7. $MN;\ MN \subset \overleftrightarrow{KL}$ – the points KL lie in the plane of the face $ABCD$; their line cuts the face in the segment MN
- 8. ZN – the points Z, N lie in the face $CDHG$, join them.
- 9. YM – the points Y, M lie in the face $BCGF$, join them.
- 10. $YP;\ YP \parallel NZ,\ P \in EF$ – the face $ABFE$ is parallel to the face $CDHG$, so the section cuts it in the parallel to NZ through Y; it meets the edge EF at P
- 11. $PQ;\ PQ \parallel MN,\ Q \in EH$ – the face $EFGH$ is parallel to the face $ABCD$, so the section cuts it in the parallel to MN through P; it meets the edge EH at Q
- 12. ZQ – the points Z, Q lie in the face $ADHE$, join them.
- 13. $QPYMNZ$ – the section is the hexagon $QPYMNZ$

Graph

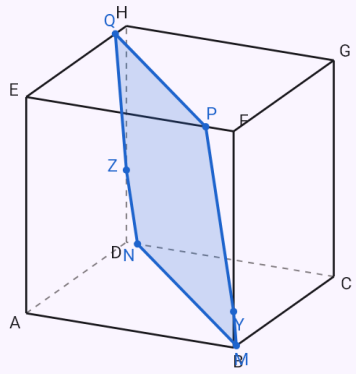


Step 13 of 13

13. $QPYMNZ$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

The section in space – turn it with a finger



The point X divides the edge AB in the ratio 1 : 2.

Input: `construction(cube_section; a=6; X: AB=1:2; BF=3; CG=4)`

Result

$XYZK$

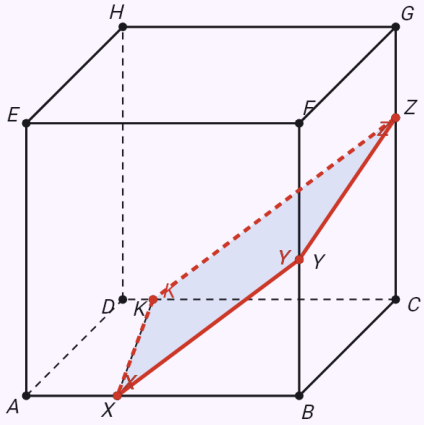
$|XY| = 5\,\text{cm},\ |YZ| \approx 6.08\,\text{cm}$

$|ZK| \approx 6.67\,\text{cm},\ |KX| \approx 6.15\,\text{cm}$

Steps

- Analysis ($a = 6\,\text{cm}$): the cutting plane is given by the points X, Y, Z. The section is constructed side by side by three rules: two points of the section in one face are joined; parallel faces are cut in parallel segments; a side's line extended meets an edge's line in a point lying in the planes of both faces at that edge.
- 1. $ABCDEFGH;\ a = 6\,\text{cm}$ – draw the cube $ABCDEFGH$ in the oblique projection (the depth at 45°, halved), the hidden edges dashed.
- 2. $X;\ |AX| : |XB| = 1 : 2$ – the point X divides the segment AB in the ratio $1 : 2$; this gives the point X
- 3. $Y;\ Y \in BF,\ |BY| = 3\,\text{cm}$ – the point Y on the edge BF at the distance $3\,\text{cm}$ from the vertex B
- 4. $Z;\ Z \in CG,\ |CZ| = 4\,\text{cm}$ – the point Z on the edge CG at the distance $4\,\text{cm}$ from the vertex C
- 5. XY – the points X, Y lie in the face $ABFE$, join them.
- 6. YZ – the points Y, Z lie in the face $BCGF$, join them.
- 7. $ZK;\ ZK \parallel XY,\ K \in CD$ – the face $CDHG$ is parallel to the face $ABFE$, so the section cuts it in the parallel to XY through Z; it meets the edge CD at K
- 8. XK – the points X, K lie in the face $ABCD$, join them.
- 9. $XYZK$ – the section is the quadrilateral $XYZK$

Graph

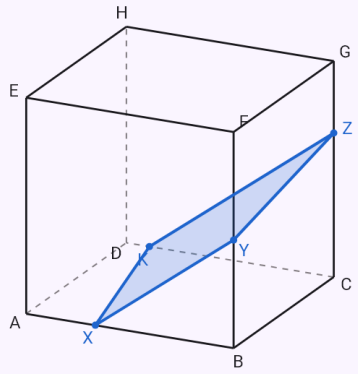


Step 9 of 9

9. $XYZK$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

The section in space – turn it with a finger



A section of a regular hexagonal prism – the base first in its true shape.

Input: `construction(prism_section; n=6; a=2; v=4; AA'=1; CC'=2; EE'=3)`

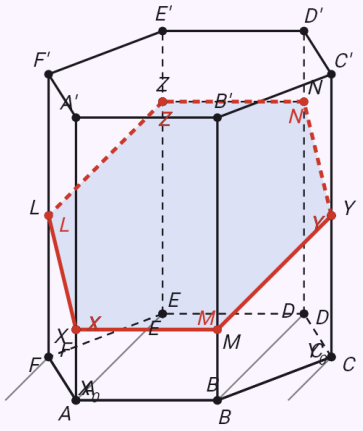
Result

$$XMYNZL$$
$$|XM| = 2\,\text{cm},\ |MY| \approx 2.24\,\text{cm}$$
$$|YN| \approx 2.24\,\text{cm},\ |NZ| = 2\,\text{cm}$$
$$|ZL| \approx 2.24\,\text{cm},\ |LX| \approx 2.24\,\text{cm}$$

Steps

- Analysis ($a = 2\,\text{cm},\ v = 4\,\text{cm}$): the cutting plane is given by the points X, Y, Z. The section is constructed side by side by three rules: two points of the section in one face are joined; parallel faces are cut in parallel segments; a side's line extended meets an edge's line in a point lying in the planes of both faces at that edge.
- 1. $ABCDEF$; $a = 2\,\text{cm}$ – beside it, draw the base $ABCDEF$ in its true shape: the regular hexagon with the side a, the side AB on the x axis (points A_0, B_0, …).
- 2. $ABCDEF$; $y' = \frac{1}{2}\,y$ – from the vertices C, D, E, F draw the perpendiculars to the line AB; carry their feet onto the x axis unchanged and from them draw half the perpendicular's length (the depth) at 45° – the base $ABCDEF$ in the oblique projection.
- 3. $ABCDEFA'B'C'D'E'F'$; $a = 2\,\text{cm},\ v = 4\,\text{cm}$ – draw the regular hexagonal prism $ABCDEFA'B'C'D'E'F'$ in the oblique projection (the depth at 45°, halved), the hidden edges dashed.
- 4. X; $X \in AA'$, $|AX| = 1\,\text{cm}$ – the point X on the edge AA' at the distance $1\,\text{cm}$ from the vertex A
- 5. Y; $Y \in CC'$, $|CY| = 2\,\text{cm}$ – the point Y on the edge CC' at the distance $2\,\text{cm}$ from the vertex C
- 6. Z; $Z \in EE'$, $|EZ| = 3\,\text{cm}$ – the point Z on the edge EE' at the distance $3\,\text{cm}$ from the vertex E
- 7. K; $K \in \overleftrightarrow{XY} \cap \overleftrightarrow{X_0Y_0}$ – the points XY are not in one face, so we use the auxiliary vertical plane of the line XY: we project the points into the base (X_0Y_0), and the lines XY and X_0Y_0 meet at K, where the line XY pierces the plane of the base $ABCDEF$
- 8. ZL; $ZL \subset \overleftrightarrow{ZK}$ – the points ZK lie in the plane of the face $EFF'E'$; their line cuts the face in the segment ZL
- 9. XL – the points X, L lie in the face $FAA'F'$, join them.
- 10. YM; $YM \parallel LZ$, $M \in BB'$ – the face $BCC'B'$ is parallel to the face $EFF'E'$, so the section cuts it in the parallel to LZ through Y; it meets the edge BB' at M
- 11. XM – the points X, M lie in the face $ABB'A'$, join them.
- 12. YN; $YN \parallel XL$, $N \in DD'$ – the face $CDD'C'$ is parallel to the face $FAA'F'$, so the section cuts it in the parallel to XL through Y; it meets the edge DD' at N
- 13. ZN – the points Z, N lie in the face $DEE'D'$, join them.
- 14. $XMYNZL$ – the section is the hexagon $XMYNZL$

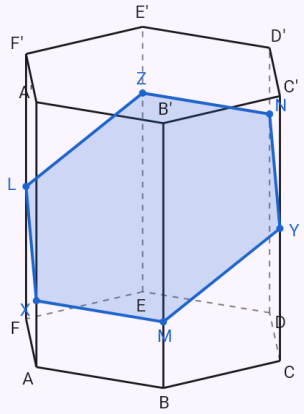
Graph



Step 14 of 14
14. $XMYNZL$

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

The section in space – turn it with a finger



A section of the regular tetrahedron through three edges' midpoints.

Input: `construction(tetrahedron_section; a=4; AB=2; AC=2; AD=2)`

Result

XYZ

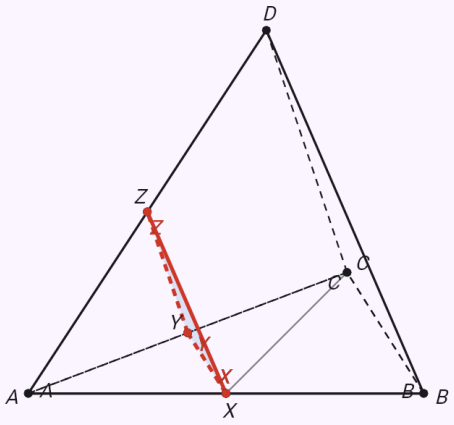
$|XY| = 2\,\text{cm},\ |YZ| = 2\,\text{cm}$

$|ZX| = 2\,\text{cm}$

Steps

- Analysis ($a = 4\,\text{cm}$): the cutting plane is given by the points X, Y, Z. The section is constructed side by side: two points of the section in one face are joined; a side parallel to another face's edge means the plane cuts that face in a parallel to it; a side's line extended meets an edge's line in a point lying in the planes of both faces at that edge.
- 1. ABC; $a = 4\,\text{cm}$ – beside it, draw the base ABC in its true shape: the regular triangle with the side a, the side AB on the x axis (points A_0, B_0, ...).
- 2. ABC; $y' = \frac{1}{2}\,y$ – from the vertices C draw the perpendiculars to the line AB; carry their feet onto the x axis unchanged and from them draw half the perpendicular's length (the depth) at 45° – the base ABC in the oblique projection.
- 3. $ABCD$; $a = 4\,\text{cm}$ – draw the regular tetrahedron $ABCD$ in the oblique projection (the depth at 45°, halved), the hidden edges dashed.
- 4. X; $X \in AB$, $|AX| = 2\,\text{cm}$ – the point X on the edge AB at the distance $2\,\text{cm}$ from the vertex A
- 5. Y; $Y \in AC$, $|AY| = 2\,\text{cm}$ – the point Y on the edge AC at the distance $2\,\text{cm}$ from the vertex A
- 6. Z; $Z \in AD$, $|AZ| = 2\,\text{cm}$ – the point Z on the edge AD at the distance $2\,\text{cm}$ from the vertex A
- 7. XY – the points X, Y lie in the face ABC, join them.
- 8. XZ – the points X, Z lie in the face ABD, join them.
- 9. YZ – the points Y, Z lie in the face CAD, join them.
- 10. XYZ – the section is the triangle XYZ

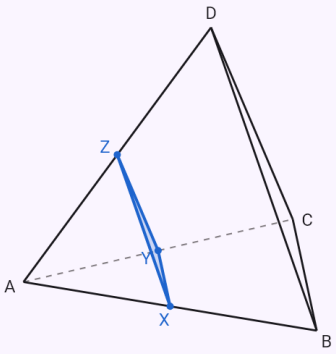
Graph



Step 10 of 10
10. XYZ

The analysis shows a sketch of the finished construction with the given elements. Then use the arrows to go through the construction step by step – the elements added in a step are highlighted.

The section in space – turn it with a finger



Known limitations

- So far a triangle (sides, angles, the altitude and median to the given side, the radii r and ϱ in the combinations above), four basic loci, five circle and tangent problems, quadrilaterals in the combinations of the form's hint and sections of a cube, a cuboid, regular prisms and pyramids (a base of 3 to 6 sides) and the regular tetrahedron; a general quadrilateral, altitudes and medians to other sides and further solids will follow.
- Problems with relations take exactly two relations to one of the five given figures; relations among more points are given as general relations.

- General relations: at most 8 points and 10 relations, a construction only from the catalogue of moves; an analysis like “similar triangles” isn’t automated.
- Entered only in the form – not in words or from a photo.

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