Mathtrace – user guide

I. Equations, inequalities and expressions

Contents

I. Equations, inequalities and expressions

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1. Linear equations and inequalities

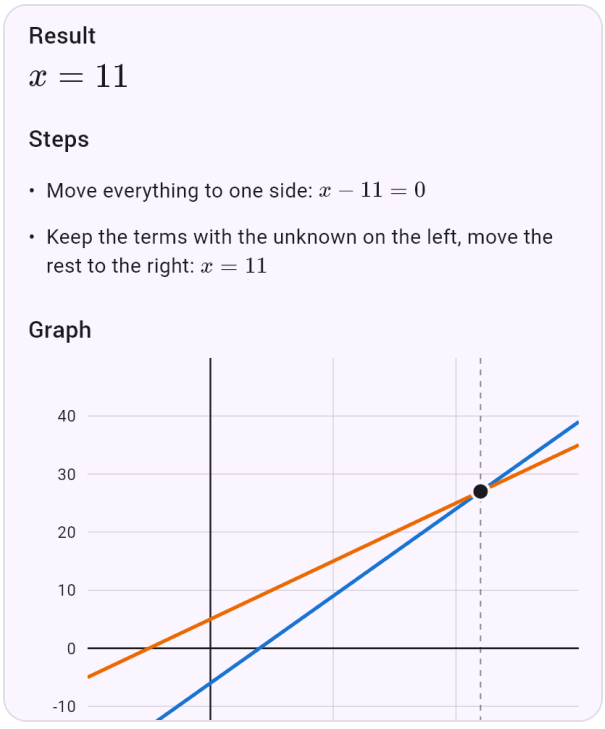
The app rearranges a linear equation the way it is done at school – it expands the brackets, moves the unknown to one side and divides. For an inequality it watches for the sign flip when multiplying by a negative number and writes the result as an interval; the graph shows both sides as lines and marks the solution on the x-axis.

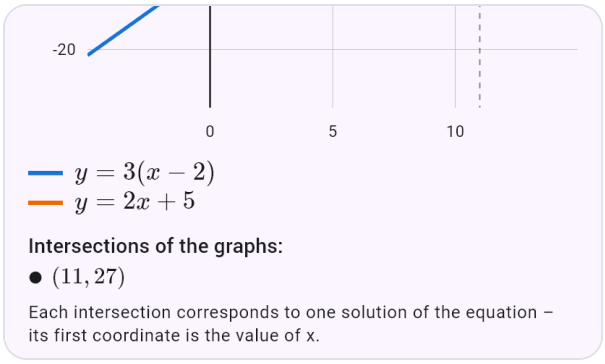
With the “Detailed” chip (or permanently with the “Detailed steps of equations” switch on the home screen) the steps follow the way equations are taught at primary school: the same operation on both sides (/ −4x, / :2), under it the intermediate line and the result. The unknown goes to the side where its coefficient comes out positive.

Examples

An equation with brackets.

Input: `3(x - 2) = 2x + 5`





An inequality – dividing by a negative number flips the sign.

Input: `2x - 3 < 5x + 6`

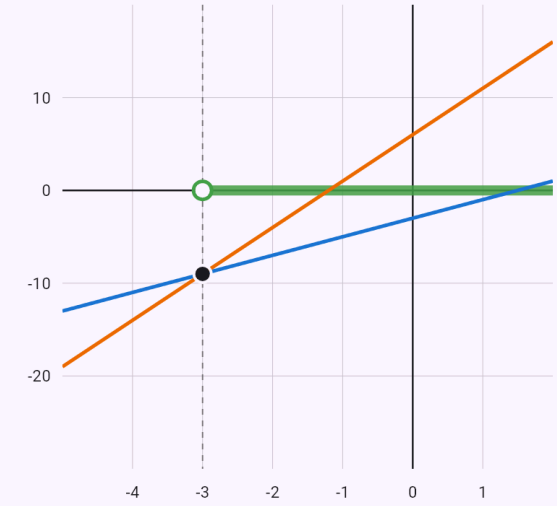
Result

$x \in (-3, \infty)$

Steps

- Keep the terms with the unknown on the left, move the rest to the right: $-3x < 9$
- Divide by the negative number -3, the inequality sign flips: $x > -3$

Graph



$y = 2x - 3$
$y = 5x + 6$

Solution of the inequality – where the relation between the two curves holds

Intersections of the graphs:

- $(-3, -9)$

Detailed steps – the same operation on both sides, written out as at school.

Input: `4x + 3 = 6x + 7` Detailed

Result

$x = -2$

Steps

- Subtract $4x$ from both sides

$$\begin{aligned} 4x + 3 &= 6x + 7 \quad / -4x \\ 4x + 3 - 4x &= 6x + 7 - 4x \\ 3 &= 2x + 7 \end{aligned}$$

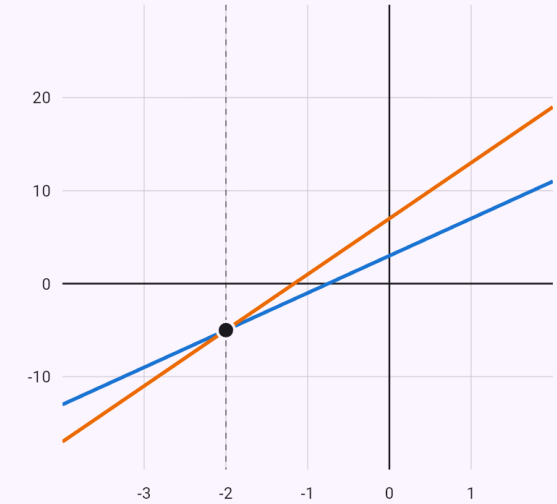
- Subtract 7 from both sides

$$\begin{aligned} 3 &= 2x + 7 \quad / -7 \\ 3 - 7 &= 2x + 7 - 7 \\ -4 &= 2x \end{aligned}$$

- Divide both sides by 2

$$\begin{aligned} -4 &= 2x \quad / : 2 \\ \frac{-4}{2} &= \frac{2x}{2} \\ -2 &= x \\ x &= -2 \end{aligned}$$

Graph



$y = 4x + 3$
$y = 6x + 7$

Intersections of the graphs:

- $(-2, -5)$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of x.

Fractions – both sides multiplied by the common denominator.

Input: `x/2 + x/3 = 10` Detailed

Result

$$x = 12$$

Steps

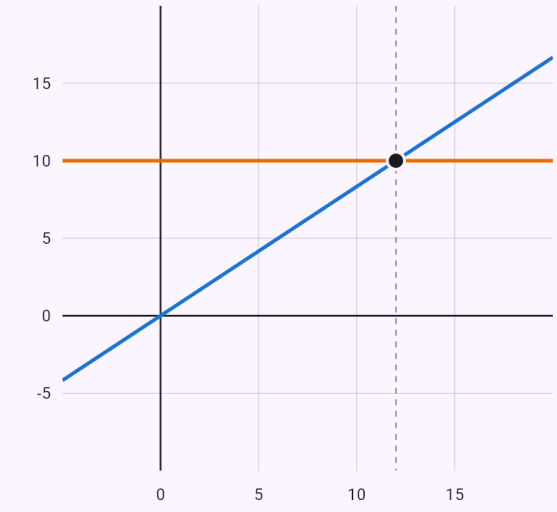
- Multiply both sides by 6

$$\frac{x}{2} + \frac{x}{3} = 10 \quad / \cdot 6$$
$$6 \cdot \frac{x}{2} + 6 \cdot \frac{x}{3} = 6 \cdot 10$$
$$3x + 2x = 60$$
$$5x = 60$$

- Divide both sides by 5

$$5x = 60 \quad / : 5$$
$$\frac{5x}{5} = \frac{60}{5}$$
$$x = 12$$

Graph



$y = \frac{x}{2} + \frac{x}{3}$

$y = 10$

Intersections of the graphs:

- $(12, 10)$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of x.

2. Quadratic equations

The app brings a quadratic equation to the form ax² + bx + c = 0 and solves it with the discriminant and the quadratic formula; it also recognizes a double root. If the equation has no real solution, it says so, and the “Show Complex Solution” button shows the complex roots.

Examples

Two real roots.

Input: `x^2 - 5x + 6 = 0`

Result

$$x_1 = 3$$
$$x_2 = 2$$

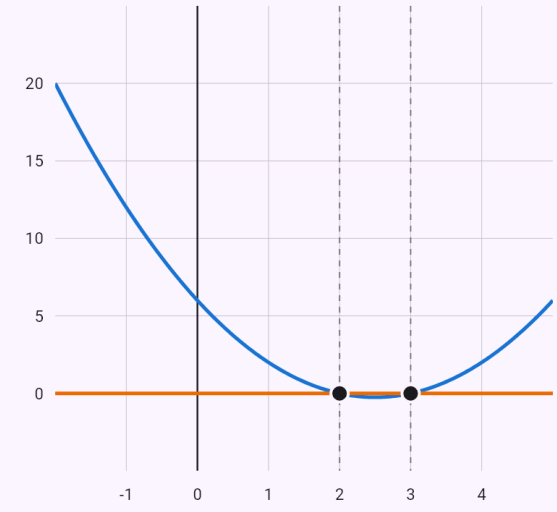
Solve by a numerical method:

Σ Newton (tangents) | Σ Bisection | Σ Secant | Σ Regula falsi | Σ Fixed point

Steps

- Compute the discriminant:
  $D = b^2 - 4ac = (-5)^2 - 4 \cdot 1 \cdot 6 = 1$
- Substitute into the quadratic formula:
  $x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{5 \pm 1}{2}$

Graph



$y = x^2 - 5x + 6$
$y = 0$

Intersections of the graphs:

● $(2, 0)$ ● $(3, 0)$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of x.

A negative discriminant – complex roots.

Input: `x^2 + 2x + 5 = 0` Show Complex Solution

Result

$$x_1 = -1 + 2i$$
$$x_2 = -1 - 2i$$

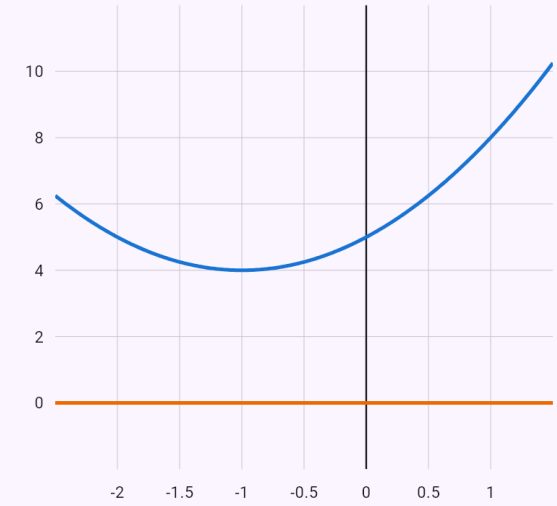
Solve by a numerical method:

Σ Newton (tangents) | Σ Bisection | Σ Secant | Σ Regula falsi | Σ Fixed point

Steps

- Compute the discriminant:
  $D = b^2 - 4ac = 2^2 - 4 \cdot 1 \cdot 5 = -16$
- The discriminant is negative – over the complex numbers: $\sqrt{D} = \sqrt{-16} = 4i$
- Substitute into the quadratic formula:
  $x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{-2 \pm 4i}{2}$

Graph



$y = x^2 + 2x + 5$
$y = 0$

The graphs don't intersect – the equation has no real solution.

3. Higher-degree equations

The app turns a biquadratic equation into a quadratic one by substitution. A general higher-degree equation is factored with synthetic division (Horner's scheme): roots are looked for among the divisors of the constant term and each root found is divided out until a quadratic equation is left.

Examples

A biquadratic equation by the substitution y = x².

Input: `x^4 - 5x^2 + 4 = 0`

Result

$$x_1 = -2$$
$$x_2 = -1$$
$$x_3 = 1$$
$$x_4 = 2$$

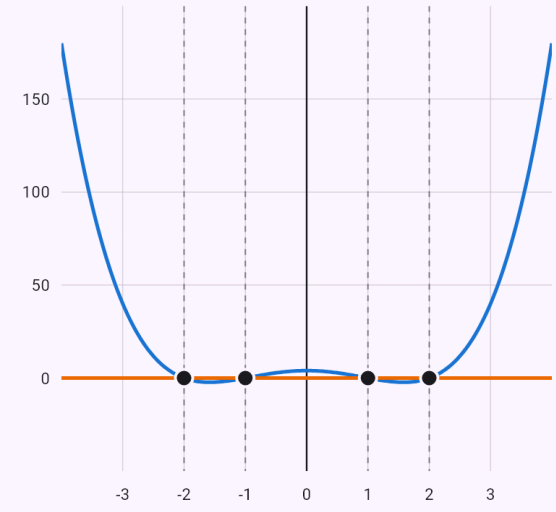
Solve by a numerical method:

Σ Newton (tangents) | Σ Bisection | Σ Secant | Σ Regula falsi | Σ Fixed point

Steps

- The substitution $u = x^2$ turns the equation into a quadratic one: $u^2 - 5u + 4 = 0$
- Compute the discriminant: $D = b^2 - 4ac = (-5)^2 - 4 \cdot 1 \cdot 4 = 9$
- Substitute into the quadratic formula: $u_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{5 \pm 3}{2}$
- Back to the original unknown: $x^2 = 1 \Rightarrow x = \pm 1$
- Back to the original unknown: $x^2 = 4 \Rightarrow x = \pm 2$

Graph



$y = x^4 - 5x^2 + 4$
$y = 0$

Intersections of the graphs:

● $(-2, 0)$ ● $(-1, 0)$ ● $(1, 0)$ ● $(2, 0)$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of x.

Synthetic division – roots among the divisors of the constant term.

Input: `x^3 - 6x^2 + 11x - 6 = 0`

Result

$$x_1 = 1$$
$$x_2 = 2$$
$$x_3 = 3$$

Solve by a numerical method:

Σ Newton (tangents) | Σ Bisection | Σ Secant | Σ Regula falsi | Σ Fixed point

Steps

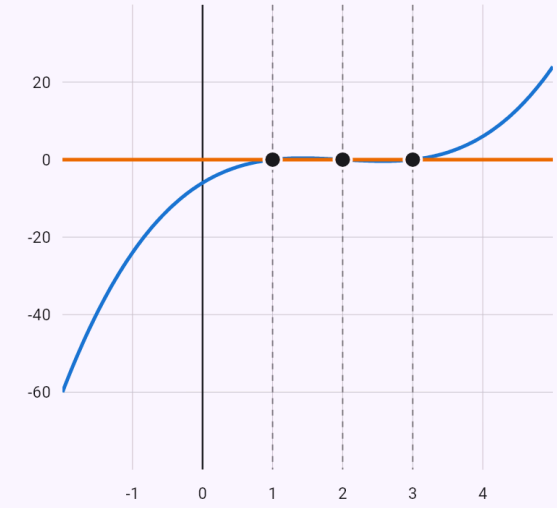
- Rational roots can only be divisors of the constant term -6: $\pm 1,\ \pm 2,\ \pm 3,\ \pm 6$
- Horner's scheme confirms the root $x = 1$ and divides the polynomial by it:

	1	-6	11	-6
1	1	-5	6	$\boxed{0}$

$$P(x) = (x-1)\left(x^2 - 5x + 6\right)$$

- Compute the discriminant:
$D = b^2 - 4ac = (-5)^2 - 4 \cdot 1 \cdot 6 = 1$
- Substitute into the quadratic formula:
$x_{1,2} = \dfrac{-b \pm \sqrt{D}}{2a} = \dfrac{5 \pm 1}{2}$

Graph



$y = x^3 - 6x^2 + 11x - 6$
$y = 0$

Intersections of the graphs:

● $(1, 0)$ ● $(2, 0)$ ● $(3, 0)$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of x.

4. Rational equations

For an equation with the unknown in a denominator the app first states the conditions, then multiplies the equation by the common denominator and solves it. It compares the roots with the conditions and discards those that do not satisfy them.

Examples

Conditions, common denominator, a quadratic equation.

Input: `(x+3)/(x-1) = 2/(x-1) + x`

Result

$$x_1 = 1 - \sqrt{2}$$
$$x_2 = 1 + \sqrt{2}$$

Conditions: $x \neq 1$

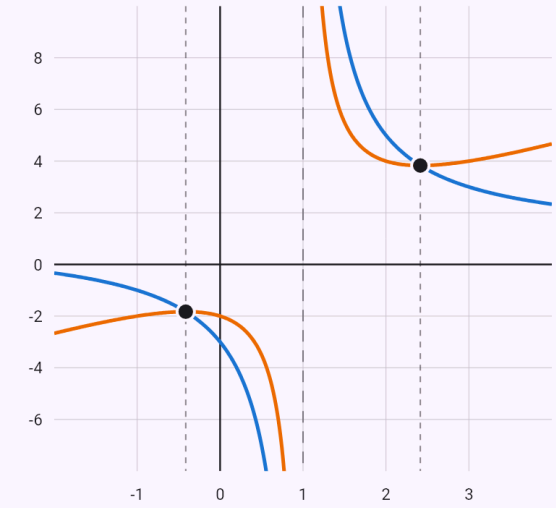
Solve by a numerical method:

Σ Newton (tangents) | Σ Bisection | Σ Secant | Σ Regula falsi | Σ Fixed point

Steps

- Conditions – a denominator must not be zero: $x \neq 1$
- Multiply both sides by the denominator $x - 1$: $x + 3 = x^2 - x + 2$
- Move everything to one side: $x^2 - 2x - 1 = 0$
- Compute the discriminant: $D = b^2 - 4ac = (-2)^2 - 4 \cdot 1 \cdot (-1) = 8$
- Substitute into the quadratic formula: $x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{2 \pm 2\sqrt{2}}{2}$
- The root $x = 1 - \sqrt{2}$ satisfies the conditions.
- The root $x = 1 + \sqrt{2}$ satisfies the conditions.

Graph



$y = \frac{x+3}{x-1}$

$y = \frac{2}{x-1} + x$

Intersections of the graphs:

$\left(1 - \sqrt{2}, 1 - 2\sqrt{2}\right)$ $\left(1 + \sqrt{2}, 1 + 2\sqrt{2}\right)$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of x.

5. Radical equations

The app solves an equation with the unknown under a root by isolating the root and squaring. Squaring can add roots that do not satisfy the equation – so a check always follows and the roots that fail it are discarded.

Examples

Squaring and a check – one of the roots does not satisfy the equation.

Input: `sqrt(x+5) = x - 1`

Result

$x = 4$

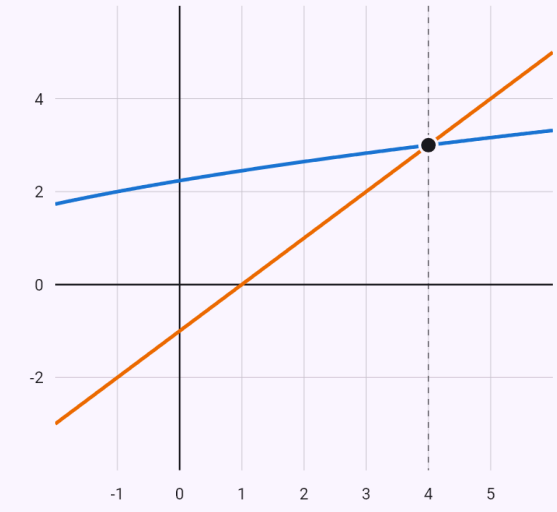
Solve by a numerical method:

Σ Newton (tangents) | Σ Bisection | Σ Secant | Σ Regula falsi | Σ Fixed point

Steps

- Square both sides (not an equivalent step, so the roots are checked): $x + 5 = (x - 1)^2$
- Move everything to one side: $x^2 - 3x - 4 = 0$
- Compute the discriminant: $D = b^2 - 4ac = (-3)^2 - 4 \cdot 1 \cdot (-4) = 25$
- Substitute into the quadratic formula: $x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{3 \pm 5}{2}$
- Check for $x = -1$: LHS = 2, RHS = -2, LHS ≠ RHS – the root does not work.
- Check for $x = 4$: LHS = 3, RHS = 3, LHS = RHS – the root works.

Graph



$y = \sqrt{x + 5}$
$y = x - 1$

Intersections of the graphs:

- $(4, 3)$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of x.

Known limitations

- Radical inequalities are not solved yet.

6. Exponential and logarithmic equations and inequalities

The app solves an exponential equation by converting to the same base, by taking logarithms or by substitution; a logarithmic one by combining the logarithms by the rules and removing the logarithm, always with the conditions. Inequalities are solved through monotonicity – for a base less than 1 the sign flips. The logarithm of a quotient whose numerator or denominator cannot be negative (a root) is split into a difference of logarithms the school way.

Equations with different bases: $a^x + b^x = c^x$ with bases from two primes ($25^x + 10^x = 4^x$) are divided by a power and turned quadratic by $t = (5/2)^x$; logarithms of different bases are converted to one ($\log_5 x = \log_2 x / \log_2 5$).

Logarithms whose bases are powers of one number ($\log_2 x + \log_4(x + 2) = 2$) are converted to one base ($\log_4 a = \log_2 a / 2$), the equation multiplied and the logarithms combined. A logarithm with the unknown in its base ($\log_x 8 = 3$) is removed by the definition ($x^3 = 8$) with the conditions $x > 0$, $x \neq 1$. An equation with the unknown in both the base and the exponent (x^(log x) = 100) is solved by taking logarithms and the substitution t = log x. Inequalities with the substitution t = log(x + 1) and systems of logarithmic and exponential inequalities are solved like single inequalities – through monotonicity.

Equations and inequalities with hyperbolic functions are rewritten by the definition ($\sinh x = (e^x - e^{-x})/2$, …) as exponential ones and solved by the substitution $t = e^x$; arsinh x = c is solved by the inverse function (x = sinh c).

Examples

The substitution t = 2ˣ.

Input: `4^x - 3*2^x - 4 = 0`

Result

$$x = 2$$

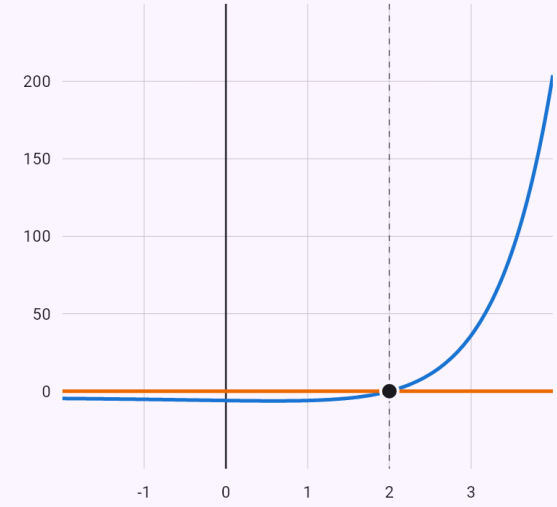
Solve by a numerical method:

Σ Newton (tangents) | Σ Bisection | Σ Secant | Σ Regula falsi | Σ Fixed point

Steps

- Substitute $t = 2^x$: $t^2 - 3t - 4 = 0$
- Compute the discriminant:
  $D = b^2 - 4ac = (-3)^2 - 4 \cdot 1 \cdot (-4) = 25$
- Substitute into the quadratic formula:
  $t_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{3 \pm 5}{2}$
- The equation $2^x = -1$ has no real solution.
- Back to the original unknown: $2^x = 4 \Rightarrow x = 2$

Graph



$y = 4^x - 3 \cdot 2^x - 4$
$y = 0$

Intersections of the graphs:

- $(2, 0)$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of x.

Combining logarithms, conditions.

Input: `log_2(x+1) + log_2(x-1) = 3`

Result

$x = 3$

Conditions: $x > 1$

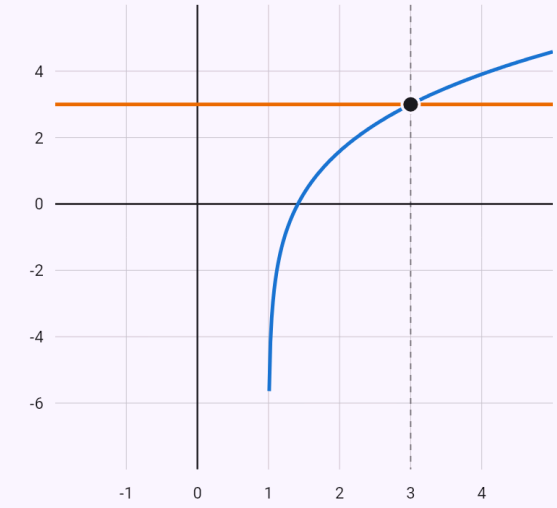
Solve by a numerical method:

Σ Newton (tangents) | Σ Bisection | Σ Secant | Σ Regula falsi | Σ Fixed point

Steps

- Domain – conditions: $x > -1,\ x > 1 \ \Rightarrow\ x > 1$
- Combine the logarithms using the rules for logarithms: $\log_2((x-1)(x+1)) = 3$
- Remove the logarithm: $(x-1)(x+1) = 2^3$
- Move everything to one side: $x^2 - 9 = 0$
- Compute the discriminant: $D = b^2 - 4ac = 0^2 - 4 \cdot 1 \cdot (-9) = 36$
- Substitute into the quadratic formula: $x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{\pm 6}{2}$
- The root $x = -3$ does not satisfy the condition $x > 1$, so it is not a solution.
- The root $x = 3$ satisfies the conditions.

Graph



$y = \log_2(x+1) + \log_2(x-1)$
$y = 3$

Intersections of the graphs:

- $(3, 3)$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of x.

Different bases – divide by 4^d, substitute $t = (5/2)^d$.

Input: `25^d + 10^d = 4^d`

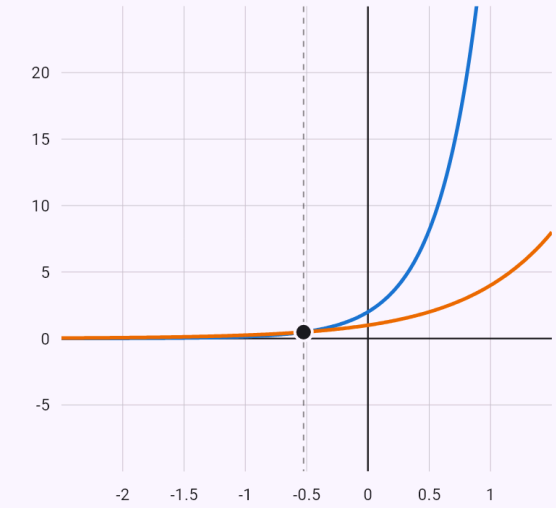
Result

$$d = \log_{\frac{5}{2}}\left(\frac{-1+\sqrt{5}}{2}\right)$$

Steps

- We divide the equation by the positive 4^d:
  $\left(\frac{5}{2}\right)^{2d} + \left(\frac{5}{2}\right)^{d} = 1$
- Substitute $t = \left(\frac{5}{2}\right)^{d}$: $t^2 + t - 1 = 0$
- Compute the discriminant:
  $D = b^2 - 4ac = 1^2 - 4 \cdot 1 \cdot (-1) = 5$
- Substitute into the quadratic formula:
  $t_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm \sqrt{5}}{2}$
- The equation $\left(\frac{5}{2}\right)^{d} = \frac{-1-\sqrt{5}}{2}$ has no real solution.
- Back to the original unknown
  $\left(\frac{5}{2}\right)^{d} = \frac{-1+\sqrt{5}}{2} \Rightarrow d = \log_{\frac{5}{2}}\left(\frac{-1+\sqrt{5}}{2}\right)$

Graph



$y = 25^d + 10^d$
$y = 4^d$

Intersections of the graphs:

- $\left(\log_{\frac{5}{2}}\left(\frac{-1+\sqrt{5}}{2}\right), 0.4829\right)$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of d.

A base less than 1 – the inequality sign flips.

Input: `(1/2)^x > 8`

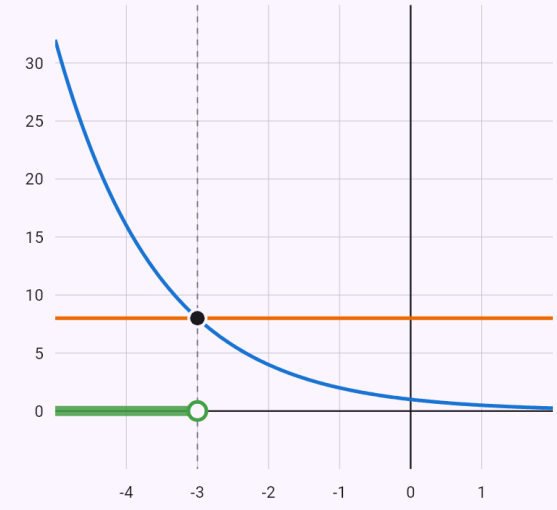
Result

$$x \in (-\infty,\, -3)$$

Steps

- Write both sides as powers with the same base: $2^{-x} > 2^3$
- The exponential function with base 2 is increasing – the exponents satisfy the inequality with the same sign: $-x > 3$
- Keep the terms with the unknown on the left, move the rest to the right: $-x > 3$
- Divide by the negative number -1, the inequality sign flips: $x < -3$
- The solution of this inequality is $x \in (-\infty,\, -3)$

Graph



$y = (\frac{1}{2})^x$

$y = 8$

Solution of the inequality – where the relation between the two curves holds

Intersections of the graphs:

- $(-3, 8)$

Hyperbolic tangent by the definition – substitution t = e²ˣ.

Input: `tanh x = 1/2`

Result

$$x = \frac{\ln 3}{2}$$

Solve by a numerical method:

Σ Newton (tangents) | Σ Bisection | Σ Secant | Σ Regula falsi | Σ Fixed point

Steps

- Substituting the definitions

  $$\tanh\alpha = \frac{e^{\alpha} - e^{-\alpha}}{e^{\alpha} + e^{-\alpha}} = \frac{e^{2\alpha} - 1}{e^{2\alpha} + 1}$$

  $$\frac{e^{2x} - 1}{e^{2x} + 1} = \frac{1}{2} \;\Rightarrow\; \frac{e^{2x} - 3}{2e^{2x} + 2} = 0 \;\Rightarrow\; e^{2x} - 3 = 0$$

- Substitute $t = e^{2x}$: $t - 3 = 0$
- Solve: $t = 3$
- Back to the original unknown: $e^{2x} = 3 \;\Rightarrow\; x = \frac{\ln 3}{2}$

The unknown in the base – by the definition; the root −1 fails the conditions.

Input: `log_x (2x+3) = 2`

Result

$$x = 3$$

Conditions: $x > 0,\ x \neq 1$

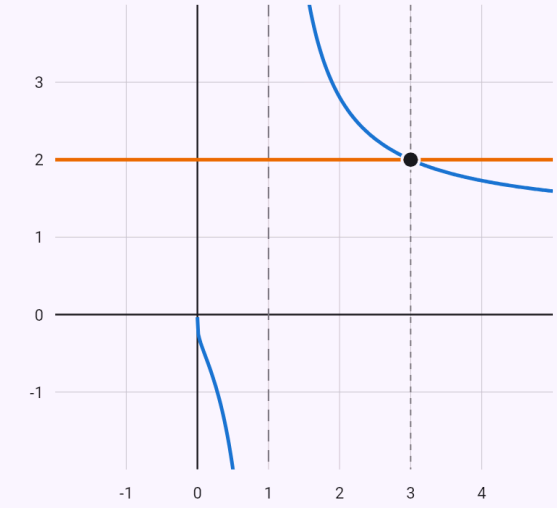
Solve by a numerical method:

Σ Newton (tangents) | Σ Bisection | Σ Secant | Σ Regula falsi | Σ Fixed point

Steps

- Domain – conditions: $x > 0,\ x \neq 1$
- Remove the logarithm: $x^2 = 2x + 3$
- Move everything to one side: $x^2 - 2x - 3 = 0$
- Compute the discriminant:
  $D = b^2 - 4ac = (-2)^2 - 4 \cdot 1 \cdot (-3) = 16$
- Substitute into the quadratic formula:
  $x_{1,2} = \dfrac{-b \pm \sqrt{D}}{2a} = \dfrac{2 \pm 4}{2}$
- The root $x = -1$ does not satisfy the condition $x > 0,\ x \neq 1$, so it is not a solution.
- The root $x = 3$ satisfies the conditions.

Graph



$y = \log_x (2x + 3)$
$y = 2$

Intersections of the graphs:

- $(3, 2)$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of x.

The unknown in the base and the exponent – take logarithms, substitute t = log x.

Input: `x^(log x) = 100`

Result

$$x_1 = 10^{-\sqrt{2}}$$

$$x_2 = 10^{\sqrt{2}}$$

Conditions: $x > 0$

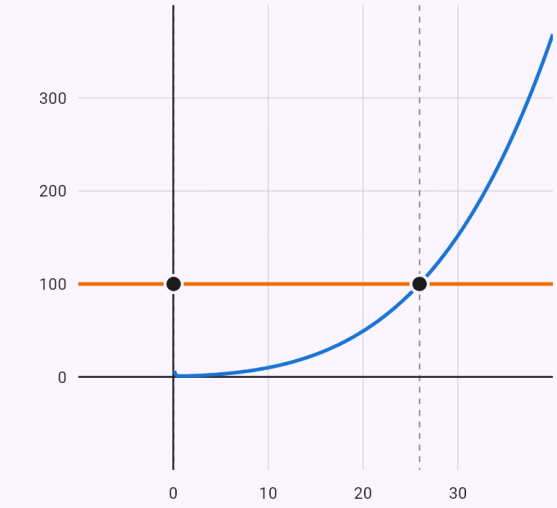
Solve by a numerical method:

Σ Newton (tangents) | Σ Bisection | Σ Secant | Σ Regula falsi | Σ Fixed point

Steps

- Take the logarithm of both sides:
  $\log\left(x^{\log x}\right) = 2 \;\Rightarrow\; \log x \cdot \log x = 2$
- Domain – conditions: $x > 0$
- Substitute $t = \log x$: $t^2 - 2 = 0$
- Compute the discriminant:
  $D = b^2 - 4ac = 0^2 - 4 \cdot 1 \cdot (-2) = 8$
- Substitute into the quadratic formula:
  $t_{1,2} = \dfrac{-b \pm \sqrt{D}}{2a} = \dfrac{\pm 2\sqrt{2}}{2}$
- Back to the original unknown:
  $\log x = -\sqrt{2} \;\Rightarrow\; x = 10^{-\sqrt{2}}$
- Back to the original unknown:
  $\log x = \sqrt{2} \;\Rightarrow\; x = 10^{\sqrt{2}}$
- The root $x = 10^{-\sqrt{2}}$ satisfies the conditions.
- The root $x = 10^{\sqrt{2}}$ satisfies the conditions.

Graph



$y = x^{\log x}$
$y = 100$

Intersections of the graphs:

- $\left(10^{-\sqrt{2}}, 100\right)$
- $\left(10^{\sqrt{2}}, 100\right)$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of x.

The logarithm of a quotient with a root – split into a difference of logarithms.

Input: `log_3(sqrt(x+1)/x) < 0`

Result

$$x \in \left(\frac{1+\sqrt{5}}{2}, \infty\right)$$

Conditions: $\sqrt{x+1} > 0,\ x > 0$

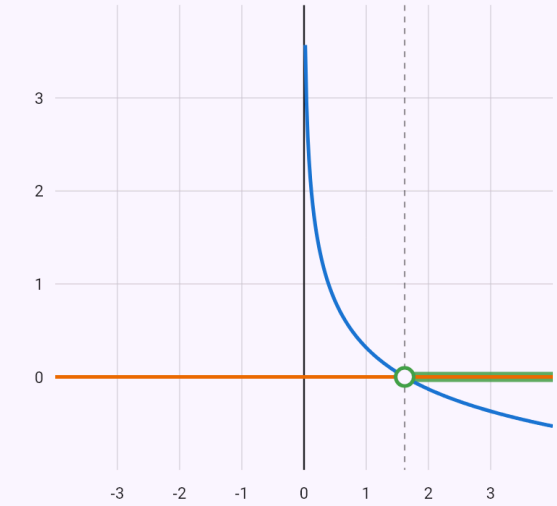
Steps

- One part of the quotient cannot be negative, so the quotient is positive exactly when both parts are – we therefore split the logarithm of the quotient into a difference of logarithms and move the subtracted one to the other side: $\log_3 \sqrt{x+1} < \log_3 x$
- Domain – conditions: $\sqrt{x+1} > 0,\ x > 0$
- The logarithm with base 3 is increasing – remove it and keep the inequality sign: $\sqrt{x+1} < x$
- Move everything to one side: $-x + \sqrt{x+1} < 0$
- Domain – conditions: $x \geq -1$
- The critical points split the number line into intervals: $x_1 = -1,\ x_2 = \frac{1+\sqrt{5}}{2}$
- Signs on the individual intervals:

	$(-\infty, -1)$	$\left(-1, \frac{1+\sqrt{5}}{2}\right)$	$\left(\frac{1+\sqrt{5}}{2}, \infty\right)$
$-x+\sqrt{x+1}$	×	+	−

- The solution of this inequality is $x \in \left(\frac{1+\sqrt{5}}{2}, \infty\right)$

Graph



$y = \log_3\left(\frac{\sqrt{x+1}}{x}\right)$

$y = 0$

Solution of the inequality – where the relation between the two curves holds

Intersections of the graphs:

- $\left(\frac{1+\sqrt{5}}{2}, 0\right)$

Known limitations

- Logarithms of different bases that aren't powers of one number ($\log_2 x + \log_3(x+1) = 2$) are solved only numerically.
- Hyperbolic equations leading to a 4th-degree equation in e^x ($\cosh^2 x - 3\sinh x = 3$) are solved only numerically; inequalities with inverse hyperbolic functions are not supported yet.

7. Absolute value

The app solves equations and inequalities with absolute values in two ways, which you choose with a chip: “By the definition” – a shortcut that writes out |a| = b or |a| < b – and “Zero points” – splitting the number line into intervals on which each absolute value has a fixed sign. Without a choice it picks the more suitable one itself.

Nested absolute values (||x| − 2| = 1) are solved the same way: by the definition step by step from the outside in, by zero points from the inside out – first the zero points of the inner absolute values, then the remaining absolute value on each interval.

An inequality |f| < |g| (just an absolute value on each side) can also be solved with the “By squaring” chip: both sides are non-negative, so it is squared and solved as a quadratic one. An expression with absolute values alone (|x − 1| + |x + 2|) is simplified by pieces – zero points, a sign chart and the expression without absolute values on each interval. An interval can follow after a comma (|x| − 2, x < 0): the expression is then simplified only on it.

Examples

One absolute value – written out by the definition.

Input: `|2x - 1| = 5`

Result

$x_1 = -2$

$x_2 = 3$

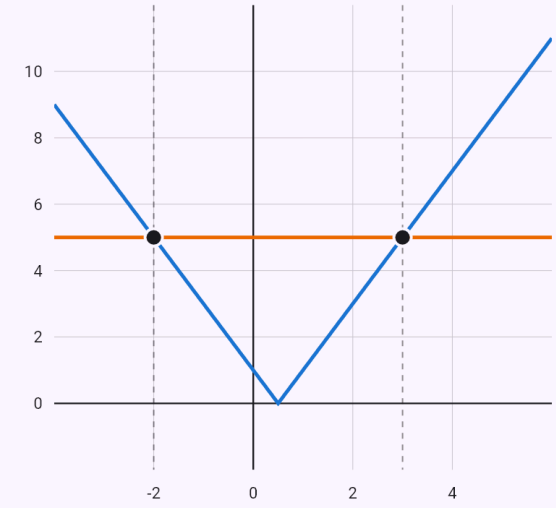
Solve by a numerical method:

Σ Newton (tangents) | Σ Bisection | Σ Secant | Σ Regula falsi | Σ Fixed point

Steps

- There is only one absolute value and a number on the other side – it is enough to use the definition.
- By the definition, $|2x - 1| = 5$ holds exactly when $2x - 1 = 5$ or $2x - 1 = -5$
- Solve: $2x - 1 = 5 \Rightarrow x = 3$
- Solve: $2x - 1 = -5 \Rightarrow x = -2$

Graph



$y = |2x - 1|$

$y = 5$

Intersections of the graphs:

● $(-2, 5)$ ● $(3, 5)$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of x.

Two absolute values – the zero-points method.

Input: `|x - 1| + |x + 2| < 5`

Result

$$x \in (-3,\ 2)$$

Steps

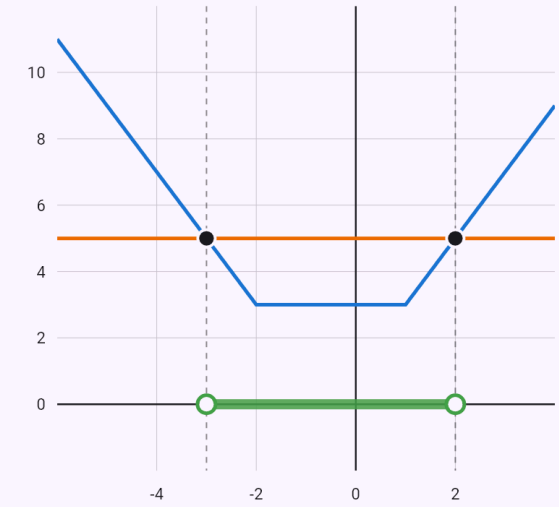
- Remove the absolute values by the zero point method: the zero points split the number line into intervals, on each of them the expressions inside keep their sign.
- Zero points of the expressions inside the absolute values: $x - 1 = 0 \ \Rightarrow \ x = 1, \quad x + 2 = 0 \ \Rightarrow \ x = -2$
- Signs on the individual intervals:

	$(-\infty,\ -2)$	$[-2,\ 1)$	$[1,\ \infty)$
$x - 1$	$-$	$-$	$+$
$x + 2$	$-$	$+$	$+$

- On the interval $(-\infty,\ -2)$, remove the absolute values by the signs: $(1 - x) + (-x - 2) < 5$
- Solve: $-2x - 1 < 5$
- Keep the terms with the unknown on the left, move the rest to the right: $-2x < 6$
- Divide by the negative number -2, the inequality sign flips: $x > -3$
- The solution of this inequality is $x \in (-3,\ \infty)$
- Intersection with the interval $(-\infty,\ -2)$: $x \in (-3,\ -2)$
- On the interval $[-2,\ 1)$, remove the absolute values by the signs: $(1 - x) + (x + 2) < 5$
- The unknown cancelled and the true statement $3 < 5$ is left – the whole interval $[-2,\ 1)$ is a solution.
- On the interval $[1,\ \infty)$, remove the absolute values by the signs: $(x - 1) + (x + 2) < 5$
- Solve: $2x + 1 < 5$
- Keep the terms with the unknown on the left, move the rest to the right: $2x < 4$
- Divide by the positive number 2, the inequality sign stays: $x < 2$

- The solution of this inequality is $x \in (-\infty,\ 2)$
- Intersection with the interval $[1,\ \infty)$: $x \in [1,\ 2)$
- The solution is the union of the results: $x \in (-3,\ 2)$

Graph



$y = |x - 1| + |x + 2|$
$y = 5$

Solution of the inequality – where the relation between the two curves holds

Intersections of the graphs:

● $(-3, 5)$ ● $(2, 5)$

A nested absolute value – by the definition from the outside in.

Input: `||x| - 2| = 1`

Result

$$x_1 = -3$$
$$x_2 = -1$$
$$x_3 = 1$$
$$x_4 = 3$$

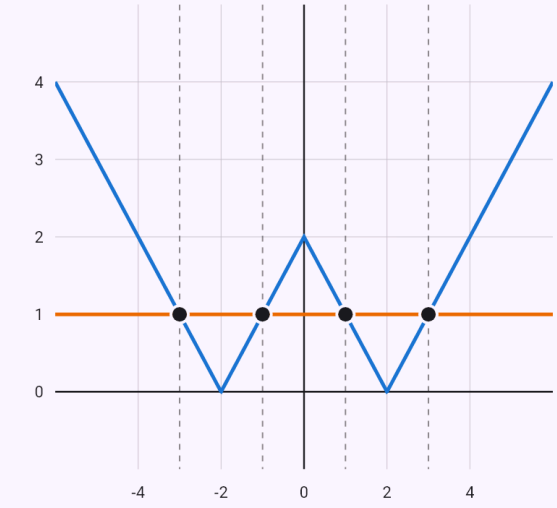
Solve by a numerical method:

Σ Newton (tangents) | Σ Bisection | Σ Secant | Σ Regula falsi | Σ Fixed point

Steps

- There is only one absolute value and a number on the other side – it is enough to use the definition.
- By the definition, $||x| - 2| = 1$ holds exactly when $|x| - 2 = 1$ or $|x| - 2 = -1$
- Isolate the absolute value: $|x| = 3$
- By the definition, $|x| = 3$ holds exactly when $x = 3$ or $x = -3$
- Isolate the absolute value: $|x| = 1$
- By the definition, $|x| = 1$ holds exactly when $x = 1$ or $x = -1$

Graph



$y = ||x| - 2|$
$y = 1$

Intersections of the graphs:

● $(-3, 1)$ ● $(-1, 1)$ ● $(1, 1)$ ● $(3, 1)$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of x.

Absolute values on both sides – squaring.

Input: `|x - 1| < |2x + 3|` By squaring

Result

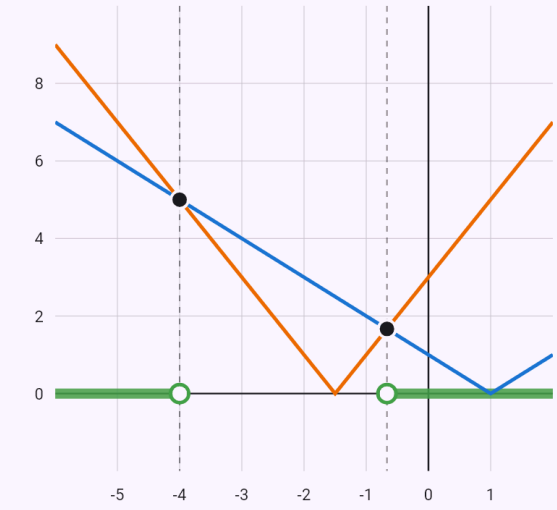
$$x \in (-\infty, -4) \cup \left(-\frac{2}{3}, \infty\right)$$

Steps

- Both sides are just absolute values – both are non-negative, so we can square the inequality (an equivalent step).
- Square both sides: $(x-1)^2 < (2x+3)^2$
- Move everything to one side: $-3x^2 - 14x - 8 < 0$
- Factor: $-(x+4)(3x+2) < 0$
- The critical points split the number line into intervals: $x_1 = -4,\ x_2 = -\frac{2}{3}$
- Signs on the individual intervals:

	$(-\infty, -4)$	$(-4, -\frac{2}{3})$	$(-\frac{2}{3}, \infty)$
-1	$-$	$-$	$-$
$x+4$	$-$	$+$	$+$
$3x+2$	$-$	$-$	$+$
$-(x+4)(3x+2)$	$-$	$+$	$-$

Graph



$y = |x - 1|$
$y = |2x + 3|$

Solution of the inequality – where the relation between the two curves holds

Intersections of the graphs:

- $(-4, 5)$
- $\left(-\frac{2}{3}, \frac{5}{3}\right)$

An expression with absolute values – by pieces on the intervals.

Input: `|x - 1| + |x + 2|`

Result

$$\begin{cases} -2x - 1, & x \in (-\infty, -2) \\ 3, & x \in [-2, 1) \\ 2x + 1, & x \in [1, \infty) \end{cases}$$

Steps

- Remove the absolute values by the zero point method: the zero points split the number line into intervals, on each of them the expressions inside keep their sign.
- Zero points of the expressions inside the absolute values: $x - 1 = 0 \Rightarrow x = 1, \quad x + 2 = 0 \Rightarrow x = -2$
- Signs on the individual intervals:

	$(-\infty, -2)$	$[-2, 1)$	$[1, \infty)$
$x - 1$	$-$	$-$	$+$
$x + 2$	$-$	$+$	$+$

- On the interval $(-\infty, -2)$, remove the absolute values by the signs: $(1 - x) + (-x - 2) = -2x - 1$
- On the interval $[-2, 1)$, remove the absolute values by the signs: $(1 - x) + (x + 2) = 3$
- On the interval $[1, \infty)$, remove the absolute values by the signs: $(x - 1) + (x + 2) = 2x + 1$

An expression with absolute values on a given interval – by pieces only there.

Input: `|x - 1| + |x + 2|, x > -1`

Result

$$\begin{cases} 3, & x \in (-1, 1) \\ 2x + 1, & x \in [1, \infty) \end{cases}$$

Steps

- We simplify the expression only for values from the interval $(-1, \infty)$ – the absolute values are removed according to the signs of the expressions inside them.
- Zero points of the expressions inside the absolute values: $x - 1 = 0 \Rightarrow x = 1, \quad x + 2 = 0 \Rightarrow x = -2$
- Signs on the individual intervals:

	$(-1, 1)$	$[1, \infty)$
$x - 1$	$-$	$+$
$x + 2$	$+$	$+$

- On the interval $(-1, 1)$, remove the absolute values by the signs: $(1 - x) + (x + 2) = 3$
- On the interval $[1, \infty)$, remove the absolute values by the signs: $(x - 1) + (x + 2) = 2x + 1$

Known limitations

- Squaring is offered only for an inequality with a single absolute value on each side.

8. Trigonometric equations and inequalities

The app brings an equation to basic equations such as sin x = a and solves them on the unit circle: it finds the angles of one turn and adds multiples of the period ($x = \pi/6 + 2k\pi$, $k \in \mathbb{Z}$). The circle is drawn right in the steps.

It prefers the known formulas – double and half angle, $\sin^2 x + \cos^2 x = 1$, sum to product –, then factoring, the substitution $t = \sin x$, dividing a homogeneous equation by $\cos^n x$ and the auxiliary angle for $a \cdot \sin x + b \cdot \cos x = c$. The result is also checked numerically over a whole period.

An unknown written as a Latin letter (x, t) is in radians, a Greek one (alpha, β, …) in degrees – as is an input with the ° sign (sin(x + 30°)). To get the roots in a given interval, add the interval after a comma or on the next line.

An inequality is solved by an arc of the unit circle, a more complex one by the zeros on one period; the result is intervals with the period.

Examples

Two branches of solutions, the angles on the unit circle.

Input: `sin x = 1/2`

Result

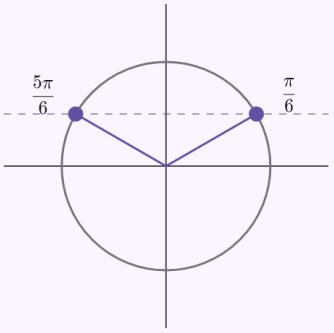
$$x_1 = \frac{\pi}{6} + 2k\pi$$

$$x_2 = \frac{5\pi}{6} + 2k\pi$$

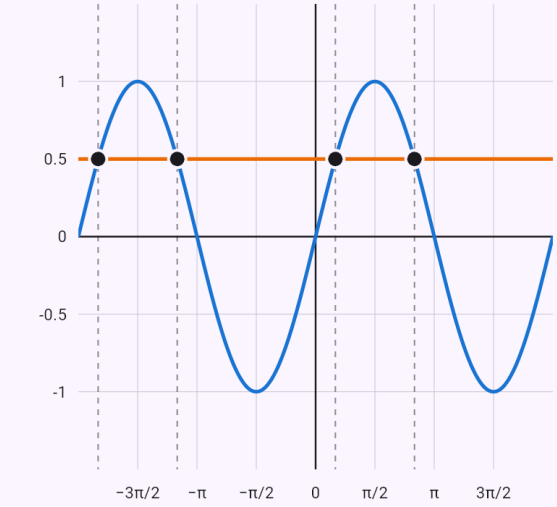
$$k \in \mathbb{Z}$$

Steps

- Basic equation $\sin x = \frac{1}{2}$: on the unit circle, the angles of one turn satisfying it are $\frac{\pi}{6}$, $\frac{5\pi}{6}$; adding multiples of the period gives all solutions: $x = \frac{\pi}{6} + 2k\pi,\ x = \frac{5\pi}{6} + 2k\pi$



Graph



$y = \sin x$

$y = \frac{1}{2}$

Intersections of the graphs:

- $\left(-\frac{11\pi}{6}, \frac{1}{2}\right)$
- $\left(-\frac{7\pi}{6}, \frac{1}{2}\right)$
- $\left(\frac{\pi}{6}, \frac{1}{2}\right)$
- $\left(\frac{5\pi}{6}, \frac{1}{2}\right)$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of x.

The substitution t = sin x and a quadratic equation.

Input: `2sin^2 x - sin x - 1 = 0`

Result

$$x_1 = \frac{\pi}{2} + 2k\pi$$

$$x_2 = \frac{7\pi}{6} + 2k\pi$$

$$x_3 = \frac{11\pi}{6} + 2k\pi$$

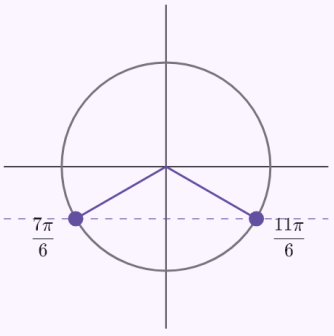
$$k \in \mathbb{Z}$$

There's no exact way – a numerical method shows where the "≈" comes from:

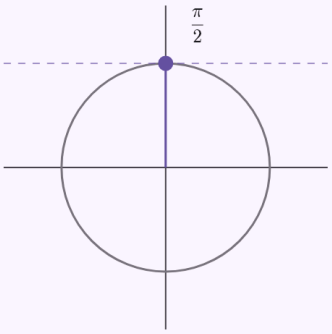
Σ Newton (tangents) | Σ Bisection | Σ Secant | Σ Regula falsi | Σ Fixed point

Steps

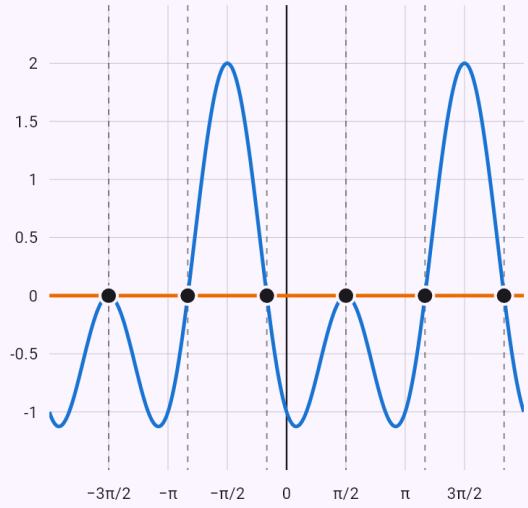
- Substitute $t = \sin x$: $2t^2 - t - 1 = 0$
- Compute the discriminant: $D = b^2 - 4ac = (-1)^2 - 4 \cdot 2 \cdot (-1) = 9$
- Substitute into the quadratic formula: $t_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{1 \pm 3}{4}$
- Basic equation $\sin x = -\frac{1}{2}$: on the unit circle, the angles of one turn satisfying it are $\frac{7\pi}{6}$, $\frac{11\pi}{6}$; adding multiples of the period gives all solutions: $x = \frac{7\pi}{6} + 2k\pi,\ x = \frac{11\pi}{6} + 2k\pi$



- Basic equation $\sin x = 1$: on the unit circle, the angles of one turn satisfying it are $\frac{\pi}{2}$; adding multiples of the period gives all solutions: $x = \frac{\pi}{2} + 2k\pi$



Graph



$y = 2\sin^2 x - \sin x - 1$
$y = 0$

Intersections of the graphs:

- $\left(-\frac{3\pi}{2}, 0\right)$
- $\left(-\frac{5\pi}{6}, 0\right)$
- $\left(-\frac{\pi}{6}, 0\right)$
- $\left(\frac{\pi}{2}, 0\right)$
- $\left(\frac{7\pi}{6}, 0\right)$
- $\left(\frac{11\pi}{6}, 0\right)$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of x.

The formula for sin 2x and factoring.

Input: `sin 2x = cos x`

Result

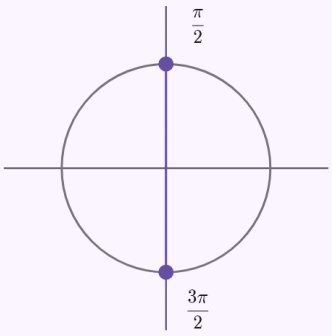
$$x_1 = \frac{\pi}{6} + 2k\pi$$
$$x_2 = \frac{\pi}{2} + k\pi$$
$$x_3 = \frac{5\pi}{6} + 2k\pi$$
$$k \in \mathbb{Z}$$

There's no exact way – a numerical method shows where the "≈" comes from:

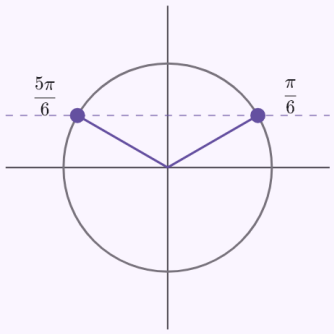
Σ Newton (tangents) Σ Bisection Σ Secant Σ Regula falsi Σ Fixed point

Steps

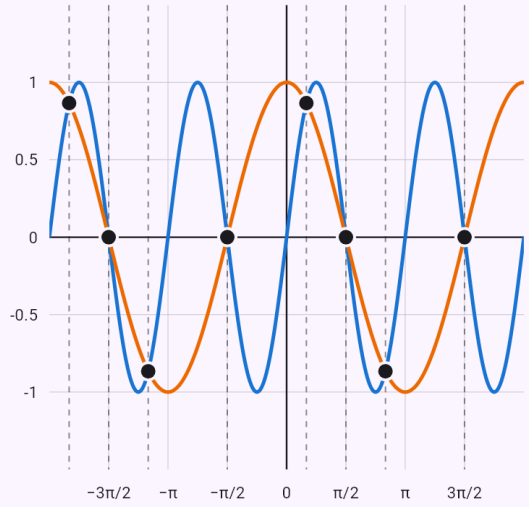
- Move everything to one side: $\sin(2x) - \cos x = 0$
- Use the formula $\sin 2x = 2\sin x \cos x$: $2\cos x \cdot \sin x - \cos x = 0$
- Factor – a product is zero exactly when one of its factors is: $(2\sin x - 1)\cos x = 0$
- Solve the equation $\cos x = 0$
- Basic equation $\cos x = 0$: on the unit circle, the angles of one turn satisfying it are $\frac{\pi}{2}$, $\frac{3\pi}{2}$; adding multiples of the period gives all solutions: $x = \frac{\pi}{2} + k\pi$



- Solve the equation $2\sin x - 1 = 0$
- Isolate the trigonometric function: $\sin x = \frac{1}{2}$
- Basic equation $\sin x = \frac{1}{2}$: on the unit circle, the angles of one turn satisfying it are $\frac{\pi}{6}$, $\frac{5\pi}{6}$; adding multiples of the period gives all solutions: $x = \frac{\pi}{6} + 2k\pi$, $x = \frac{5\pi}{6} + 2k\pi$



Graph



$y = \sin 2x$
$y = \cos x$

Intersections of the graphs:

- $\left(-\frac{11\pi}{6}, \frac{\sqrt{3}}{2}\right)$
- $\left(-\frac{3\pi}{2}, 0\right)$
- $\left(-\frac{7\pi}{6}, -\frac{\sqrt{3}}{2}\right)$
- $\left(-\frac{\pi}{2}, 0\right)$
- $\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right)$
- $\left(\frac{\pi}{2}, 0\right)$
- $\left(\frac{5\pi}{6}, -\frac{\sqrt{3}}{2}\right)$
- $\left(\frac{3\pi}{2}, 0\right)$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of x.

The auxiliary angle: sin x + cos x = √2·sin(x + π/4).

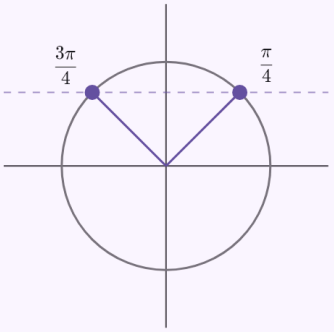
Input: `sin x + cos x = 1`

Result

$$x_1 = 2k\pi$$
$$x_2 = \frac{\pi}{2} + 2k\pi$$
$$k \in \mathbb{Z}$$

Steps

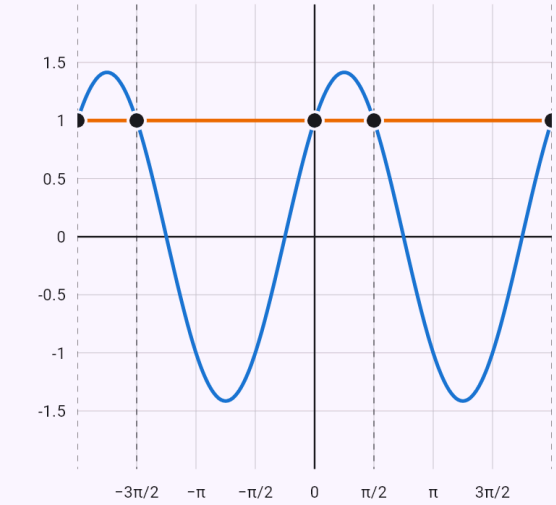
- Move everything to one side: $\cos x + \sin x - 1 = 0$
- Introduce an auxiliary angle φ:
  $\cos x + \sin x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$, where
  $\cos\varphi = \frac{\sqrt{2}}{2},\ \sin\varphi = \frac{\sqrt{2}}{2} \Rightarrow \varphi = \frac{\pi}{4}$
- Isolate the trigonometric function: $\sin\left(x + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$
- Basic equation $\sin\left(x + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$: on the unit circle, the angles of one turn satisfying it are $\frac{\pi}{4}$, $\frac{3\pi}{4}$; adding multiples of the period gives all solutions:
  $x + \frac{\pi}{4} = \frac{\pi}{4} + 2k\pi,\ x + \frac{\pi}{4} = \frac{3\pi}{4} + 2k\pi$



- Solve for the unknown

$$x + \frac{\pi}{4} = \frac{\pi}{4} + 2k\pi \Rightarrow x = 2k\pi \vee x + \frac{\pi}{4} = \frac{3\pi}{4} + 2k\pi$$

Graph



$y = \sin x + \cos x$
$y = 1$

Intersections of the graphs:

- $(-2\pi, 1)$
- $\left(-\frac{3\pi}{2}, 1\right)$
- $(0, 1)$
- $\left(\frac{\pi}{2}, 1\right)$
- $(2\pi, 1)$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of x.

A Greek letter – the result in degrees.

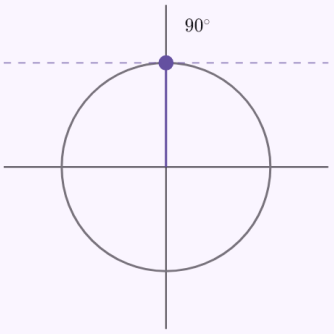
Input: `sin alpha + cos alpha = sqrt(2)`

Result

$$\alpha = 45^\circ + k \cdot 360^\circ$$
$$k \in \mathbb{Z}$$

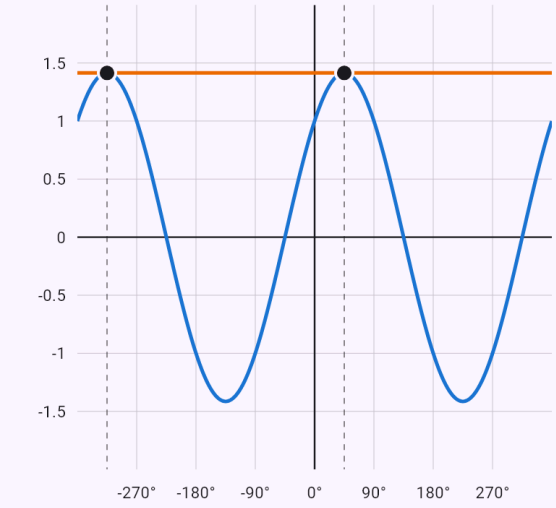
Steps

- Move everything to one side: $\cos\alpha + \sin\alpha - \sqrt{2} = 0$
- Introduce an auxiliary angle φ:
  $\cos\alpha + \sin\alpha = \sqrt{2}\sin(\alpha + 45^\circ)$, where
  $\cos\varphi = \frac{\sqrt{2}}{2},\ \sin\varphi = \frac{\sqrt{2}}{2} \Rightarrow \varphi = 45^\circ$
- Isolate the trigonometric function: $\sin(\alpha + 45^\circ) = 1$
- Basic equation $\sin(\alpha + 45^\circ) = 1$: on the unit circle, the angles of one turn satisfying it are 90°; adding multiples of the period gives all solutions:
  $\alpha + 45^\circ = 90^\circ + k \cdot 360^\circ$



- Solve for the unknown
  $\alpha + 45^\circ = 90^\circ + k \cdot 360^\circ \Rightarrow \alpha = 45^\circ + k \cdot 360^\circ$

Graph



$y = \sin\alpha + \cos\alpha$
$y = \sqrt{2}$

Intersections of the graphs:

- $(-315^\circ, \sqrt{2})$
- $(45^\circ, \sqrt{2})$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of x.

Only the roots in the given interval.

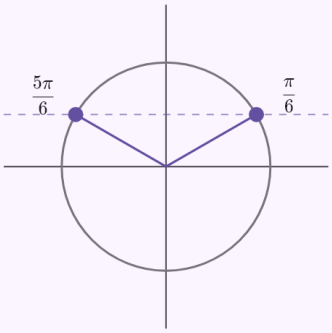
Input: `sin x = 1/2, x ∈ <0; 4pi)`

Result

$$x_1 = \frac{\pi}{6}$$
$$x_2 = \frac{5\pi}{6}$$
$$x_3 = \frac{13\pi}{6}$$
$$x_4 = \frac{17\pi}{6}$$

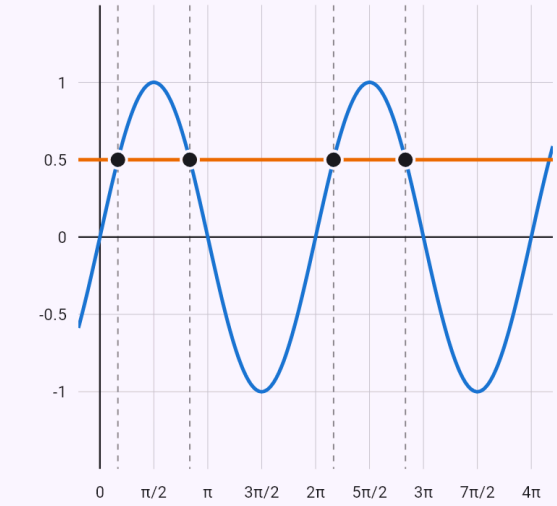
Steps

- Basic equation $\sin x = \frac{1}{2}$: on the unit circle, the angles of one turn satisfying it are $\frac{\pi}{6}$, $\frac{5\pi}{6}$; adding multiples of the period gives all solutions:
  $x = \frac{\pi}{6} + 2k\pi,\ x = \frac{5\pi}{6} + 2k\pi$



- The solutions in the interval $x \in [0,\ 4\pi)$
  $x_1 = \frac{\pi}{6},\ x_2 = \frac{5\pi}{6},\ x_3 = \frac{13\pi}{6},\ x_4 = \frac{17\pi}{6}$

Graph



$y = \sin x$

$y = \frac{1}{2}$

Intersections of the graphs:

- $\left(\frac{\pi}{6}, \frac{1}{2}\right)$
- $\left(\frac{5\pi}{6}, \frac{1}{2}\right)$
- $\left(\frac{13\pi}{6}, \frac{1}{2}\right)$
- $\left(\frac{17\pi}{6}, \frac{1}{2}\right)$

Each intersection corresponds to one solution of the equation – its first coordinate is the value of x.

An arc of the unit circle, an interval around zero.

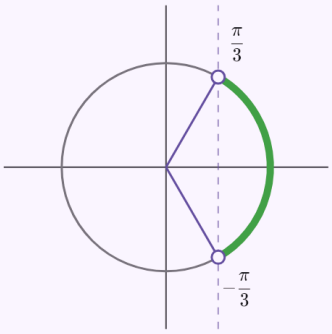
Input: `cos x > 1/2`

Result

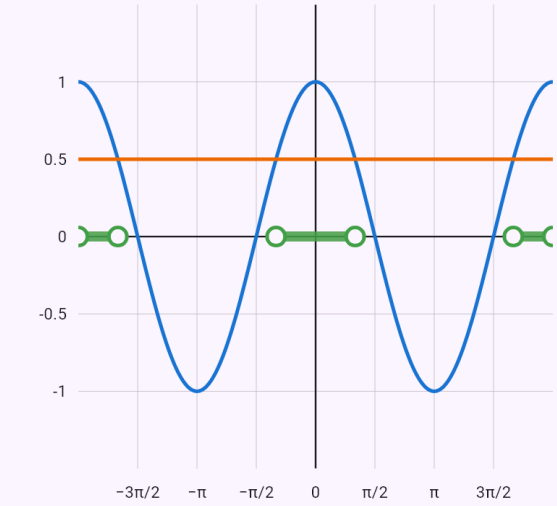
$$x \in \left(-\frac{\pi}{3} + 2k\pi,\ \frac{\pi}{3} + 2k\pi\right)$$
$$k \in \mathbb{Z}$$

Steps

- Basic inequality $\cos x > \frac{1}{2}$: on the unit circle it holds on the highlighted arc; with the period: $x \in \left(-\frac{\pi}{3} + 2k\pi,\ \frac{\pi}{3} + 2k\pi\right)$



Graph



$y = \cos x$

$y = \frac{1}{2}$

Solution of the inequality – where the relation between the two curves holds

The zeros on a period and the sign between them.

Input: `2sin^2 x - sin x - 1 > 0`

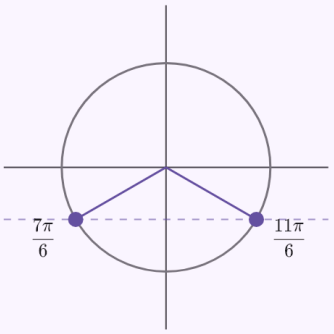
Result

$$x \in \left(\frac{7\pi}{6} + 2k\pi,\ \frac{11\pi}{6} + 2k\pi\right)$$

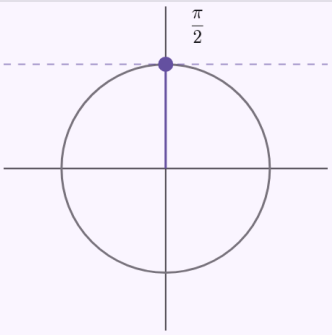
$$k \in \mathbb{Z}$$

Steps

- First solve the equation $-\sin x + 2\sin^2 x - 1 = 0$
- Substitute $t = \sin x$: $2t^2 - t - 1 = 0$
- Compute the discriminant: $D = b^2 - 4ac = (-1)^2 - 4 \cdot 2 \cdot (-1) = 9$
- Substitute into the quadratic formula: $t_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{1 \pm 3}{4}$
- Basic equation $\sin x = -\frac{1}{2}$: on the unit circle, the angles of one turn satisfying it are $\frac{7\pi}{6}$, $\frac{11\pi}{6}$; adding multiples of the period gives all solutions: $x = \frac{7\pi}{6} + 2k\pi,\ x = \frac{11\pi}{6} + 2k\pi$

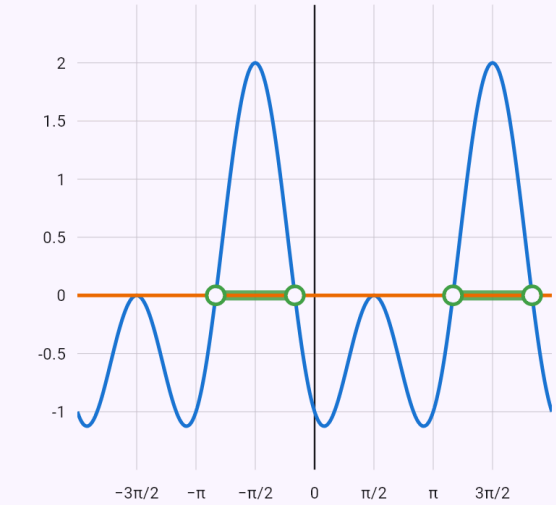


- Basic equation $\sin x = 1$: on the unit circle, the angles of one turn satisfying it are $\frac{\pi}{2}$; adding multiples of the period gives all solutions: $x = \frac{\pi}{2} + 2k\pi$



- The function has the period 2π. Its zeros (and the points where it isn't defined) $\frac{\pi}{2}$, $\frac{7\pi}{6}$, $\frac{11\pi}{6}$ split one period into intervals; by the sign on them, the inequality holds for: $x \in \left(\frac{7\pi}{6} + 2k\pi,\ \frac{11\pi}{6} + 2k\pi\right)$

Graph



$y = 2\sin^2 x - \sin x - 1$

$y = 0$

Solution of the inequality – where the relation between the two curves holds

Known limitations

- Equations with the unknown outside the trigonometric function too (x = cos x) are solved numerically only – by Newton's method with the table of iterations (see Numerical methods), without the period.
- A double trigonometric inequality (−1/2 < sin x < 1/2) and trigonometric functions inside an absolute value are not supported yet.

9. Inequalities and systems of inequalities

The app solves quadratic and rational inequalities with the zero points and a sign chart. A double inequality and a system of inequalities are solved as the intersection of the solutions of the individual inequalities, shown on a number line. A system of inequalities in two unknowns is solved as a region in the plane, shaded in the graph.

In a system of inequalities in two unknowns the app also lists the region's corners – the intersections of the boundaries with exact coordinates. A system of an equation and inequalities in one unknown is solved by solving the equation and substituting its roots into the inequalities; in two unknowns the solution is the part of the equation's curve lying in the inequalities' region – highlighted in the graph, its end points computed exactly. A system can also be written on one line, separated by semicolons ($2^x > 4$; $3^x < 81$). A system of two or more equations with inequalities is solved as a system of equations, its solutions substituted into the inequalities; with two unknowns the graph shows the curves in the inequalities' region, the failing intersections as hollow dots.

Examples

A quadratic inequality – zero points and signs.

Input: `x^2 - x - 6 > 0`

Result

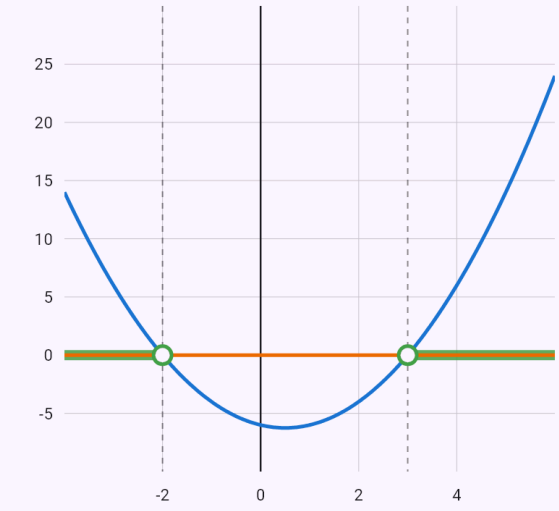
$$x \in (-\infty,\ -2) \cup (3,\ \infty)$$

Steps

- Factor: $(x-3)(x+2) > 0$
- The critical points split the number line into intervals: $x_1 = -2,\ x_2 = 3$
- Signs on the individual intervals:

	$(-\infty,\ -2)$	$(-2,\ 3)$	$(3,\ \infty)$
$x-3$	$-$	$-$	$+$
$x+2$	$-$	$+$	$+$
$(x-3)(x+2)$	$+$	$-$	$+$

Graph



$y = x^2 - x - 6$
$y = 0$

Solution of the inequality – where the relation between the two curves holds

Intersections of the graphs:

● $(-2, 0)$ ● $(3, 0)$

A rational inequality – the denominator must not be zero.

Input: `(x+1)/(x-3) <= 0`

Result

$$x \in [-1,\ 3)$$

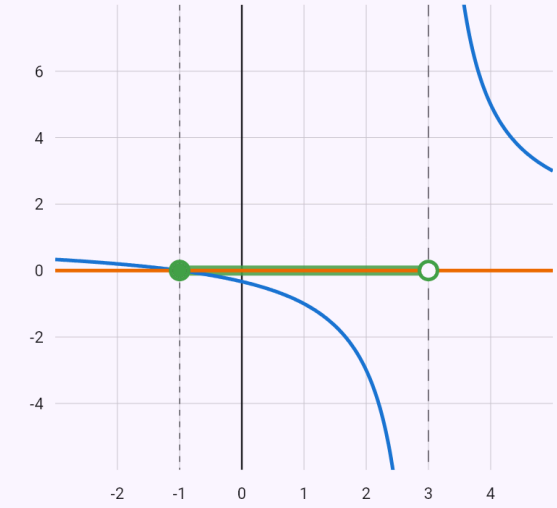
Conditions: $x \neq 3$

Steps

- Conditions – a denominator must not be zero: $x \neq 3$
- The critical points split the number line into intervals: $x_1 = -1,\ x_2 = 3$
- Signs on the individual intervals:

	$(-\infty,\ -1)$	$(-1,\ 3)$	$(3,\ \infty)$
$x - 3$	$-$	$-$	$+$
$x + 1$	$-$	$+$	$+$
$\frac{x+1}{x-3}$	$+$	$-$	$+$

Graph



$y = \dfrac{x+1}{x-3}$

$y = 0$

Solution of the inequality – where the relation between the two curves holds

Intersections of the graphs:

- $(-1, 0)$

A double inequality.

Input: `-1 <= 2x + 1 < 5`

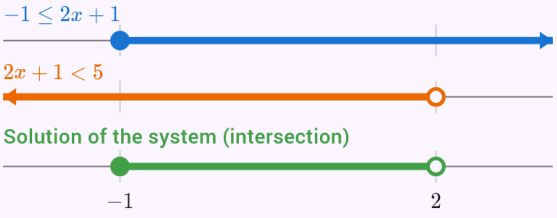
Result

$$x \in [-1,\ 2)$$

Steps

- Inequality 1: $-1 \leq 2x + 1$
- Keep the terms with the unknown on the left, move the rest to the right: $-2x \leq 2$
- Divide by the negative number -2, the inequality sign flips: $x \geq -1$
- The solution of this inequality is $x \in [-1,\ \infty)$
- Inequality 2: $2x + 1 < 5$
- Keep the terms with the unknown on the left, move the rest to the right: $2x < 4$
- Divide by the positive number 2, the inequality sign stays: $x < 2$
- The solution of this inequality is $x \in (-\infty,\ 2)$
- The system's solution is the intersection of the inequalities' solutions: $x \in [-1,\ 2)$

Graph



Each inequality has its own row on the number line, their intersection is in green. A filled dot belongs to the solution, a hollow one doesn't.

A system of inequalities in two unknowns – a region in the plane and its corners.

Input: `x + y <= 4`
`x - y >= 0`
`y >= 0`

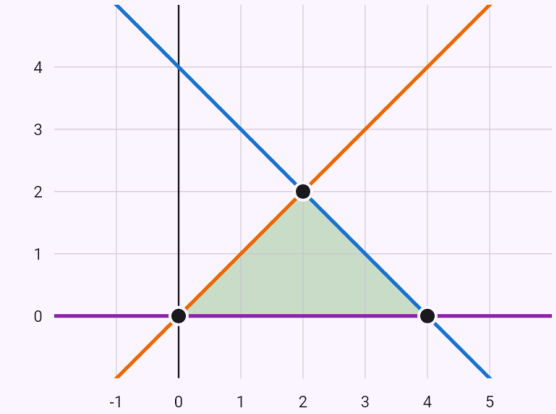
Result

$$y \le 4 - x$$
$$y \le x$$
$$y \in [0,\, \infty)$$

Steps

- Inequality 1: $x + y \le 4$
- The boundary is the curve $x + y = 4$ – drawn solid, its points belong to the solution.
- Substitute the point $[0;\ 0]$: $0 \le 4$ holds – the solution is the part of the plane containing the point.
- Inequality 2: $x - y \ge 0$
- The boundary is the curve $x - y = 0$ – drawn solid, its points belong to the solution.
- Substitute the point $[1;\ 0]$: $1 \ge 0$ holds – the solution is the part of the plane containing the point.
- Inequality 3: $y \ge 0$
- The boundary is the curve $y = 0$ – drawn solid, its points belong to the solution.
- Substitute the point $[0;\ 1]$: $1 \ge 0$ holds – the solution is the part of the plane containing the point.
- The system's solution is the common part of these regions – shaded in the graph.
- The region's corners (where the boundaries meet): $[0;\ 0]$, $[2;\ 2]$, $[4;\ 0]$

Graph



$x + y \le 4 \ \Leftrightarrow\ y \le 4 - x$
$x - y \ge 0 \ \Leftrightarrow\ y \le x$
$y \ge 0 \ \Leftrightarrow\ y \in [0,\, \infty)$

The solution is the shaded region. A solid boundary belongs to it, a dashed one doesn't.

The region's corners:

- $(0, 0)$
- $(2, 2)$
- $(4, 0)$

An inequality in the integers – the integers of the interval are listed.

Input: `x^2 < 10`

Result

$$x \in \{-3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3\}$$

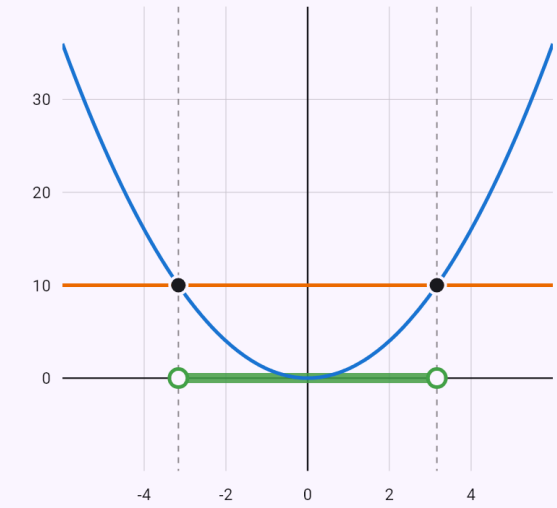
Steps

- Move everything to one side: $x^2 - 10 < 0$
- The critical points split the number line into intervals: $x_1 = -\sqrt{10},\ x_2 = \sqrt{10}$
- Signs on the individual intervals:

	$(-\infty,\ -\sqrt{10})$	$(-\sqrt{10},\ \sqrt{10})$	$(\sqrt{10},\ \infty)$
$x^2 - 10$	$+$	$-$	$+$

- In the real numbers $x \in \left(-\sqrt{10},\ \sqrt{10}\right)$; in $\mathbb{Z}$: $x \in \{-3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3\}$

Graph



$y = x^2$
$y = 10$

Solution of the inequality – where the relation between the two curves holds

Intersections of the graphs:

- $\left(-\sqrt{10}, 10\right)$
- $\left(\sqrt{10}, 10\right)$

An equation and an inequality – the equation's roots substituted into the inequality.

Input: `x^2 - x - 6 = 0`
`x > 0`

Result

$$x = 3$$

Steps

- First we solve the equation $x^2 - x - 6 = 0$, then substitute its roots into the inequalities.
- Compute the discriminant: $D = b^2 - 4ac = (-1)^2 - 4 \cdot 1 \cdot (-6) = 25$
- Substitute into the quadratic formula: $x_{1,2} = \dfrac{-b \pm \sqrt{D}}{2a} = \dfrac{1 \pm 5}{2}$
- The root $x = -2$ does not satisfy the condition $x > 0$, so it is not a solution.
- The root $x = 3$ satisfies the conditions.

A circle and a half-plane – the solution is an arc, its end points don't belong to it.

Input: `x^2 + y^2 = 4`

```
y > x
```

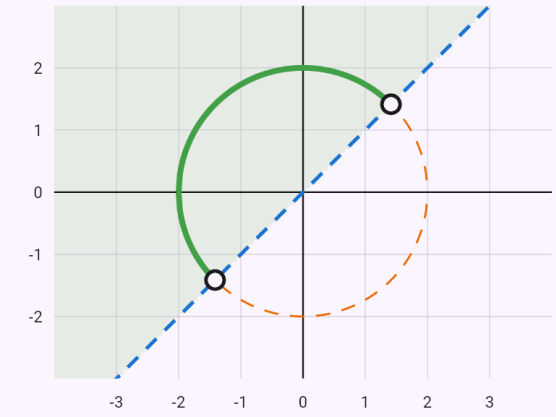
Result

$$x^2 + y^2 = 4$$
$$y > x$$

Steps

- The solution is the points of the curve $x^2 + y^2 = 4$ lying in the region given by the inequalities. First we find this region:
- The boundary is the curve $y = x$ – drawn dashed, its points don't belong to the solution.
- Substitute the point $[1;\ 0]$: $0 > 1$ doesn't hold – the solution is the part of the plane not containing the point.
- The end points of the curve's part: $\left[-\sqrt{2};\ -\sqrt{2}\right]$, $\left[\sqrt{2};\ \sqrt{2}\right]$ – on a dashed boundary, not part of the solution.

Graph



$y > x$

$x^2 + y^2 = 4 \ \Leftrightarrow\ y = \pm\sqrt{4 - x^2}$

The solution is the part of the curve highlighted in green – the part lying in the shaded region. A filled end point belongs to the solution, a hollow one doesn't.

End points:

$\circ\ \left(-\sqrt{2}, -\sqrt{2}\right)$ $\circ\ \left(\sqrt{2}, \sqrt{2}\right)$

A circle and a line with an inequality – the intersection below the x-axis fails.

Input:

```
x^2 + y^2 = 25
x - y = 1
y > 0
```

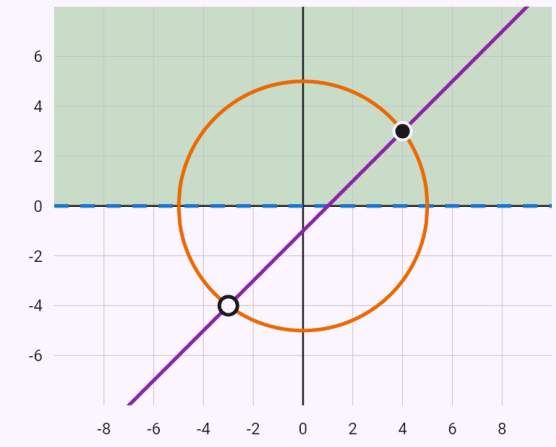
Result

$$x = 4$$
$$y = 3$$

Steps

- First we solve the system of equations, then substitute its solutions into the inequalities.
- The solution $x = -3,\ y = -4$ does not satisfy the inequality $y > 0$, so it is not a solution of the system.
- The solution $x = 4,\ y = 3$ satisfies the inequalities.

Graph



$y > 0 \ \Leftrightarrow\ y \in (0,\ \infty)$

$x^2 + y^2 = 25 \ \Leftrightarrow\ y = \pm\sqrt{25 - x^2}$

$x - y = 1 \ \Leftrightarrow\ y = x - 1$

The solutions are the curves' intersections in the shaded region (a full dot); a hollow dot fails the inequalities.

Intersections of the graphs:

$\circ\ (-3, -4)$ $\bullet\ (4, 3)$

10. Systems of equations

The app solves a system of linear equations in two unknowns by elimination (addition) or substitution, and larger systems by Gaussian elimination – as equations or on the matrix (the augmented matrix and row operations). A square system can also be solved by Cramer's rule (determinants). You choose the method with a chip. A system of two equations in two unknowns also gets a graph in which the solutions are the intersections of the lines or curves.

The numbers of equations and unknowns don't have to match. Gaussian elimination tells when a system has no solution (an equation like 0 = 1 comes out) and when it has infinitely many: the unknowns the equations don't determine become parameters t, s, … (chosen so the solution has no fractions), and the solution is also written in vector form. In matrix form the Rouché–Capelli (Frobenius) theorem decides the number of solutions – it compares the rank of the matrix, the rank of the augmented matrix and the number of unknowns. For a homogeneous system (all right-hand sides 0) the app shows a basis of the solution space.

Unknowns can have an index: x_1, x_1 or x1. A system can also be given by matrices – the augmented matrix with the right-hand sides after a vertical bar, or a matrix A, a vector of right-hand sides b and the equation Ax = b; the unknowns are then x_1, x_2, … and the default method is Gaussian elimination on the matrix.

A system may contain one or two parameters – letters that aren't unknowns (the unknowns are x, y, z, u, v, w, t and x_1, x_2, …). The app then discusses solvability by the parameters' values: with the chip “Gauss – matrix” (the default for three or more equations) it reduces the augmented matrix; where it would multiply by an expression in the parameter that can be zero, it assumes it isn't, and finishes its zeros separately with the value put in – the Frobenius theorem decides in each case. With the chip “Cramer's rule” it computes the determinants as expressions in the parameter: formulas for D ≠ 0, the zeros of D put in. With two parameters there are cases like “b = 4, a ≠ 2” or “a = b”.

A nonlinear system (say a circle and a line) is solved and drawn, so far without the working.

How to enter it

`x_1 + x_2 + x_3 = 6`	Unknowns with an index – x_1, x_1 and x1.
`[1 1 1 | 6; 2 -1 1 | 3; 1 2 -1 | 2]`	The augmented matrix – the right-hand sides after a vertical bar.

`A = [1 2; 4 5]`
`b = [3; 6]`
`Ax = b`
	The matrix of the system, the vector of right-hand sides and the equation Ax = b.

Examples

The elimination (addition) method.

Input: `2x + 3y = 7`
`x - y = 1` Addition

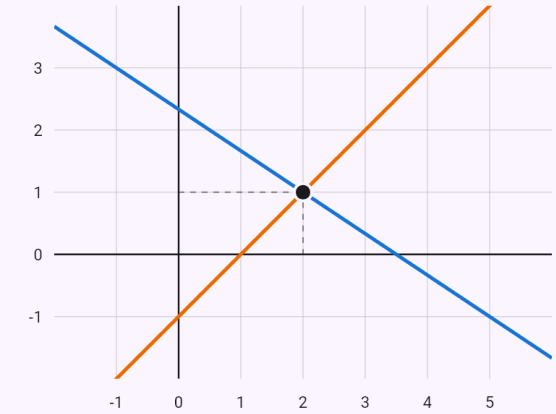
Result

$x = 2$
$y = 1$

Steps

- Multiply the first equation by 1 and the second by -2 so that x cancels when they are added
  $\begin{cases} 2x + 3y = 7 \\ -2x + 2y = -2 \end{cases}$
- Add the equations, x cancels: $5y = 5$
- Solve: $5y = 5 \;\Rightarrow\; y = 1$
- Substitute into equation 2: $x - 1 = 1 \;\Rightarrow\; x = 2$

Graph



$2x + 3y = 7 \;\Leftrightarrow\; y = \frac{7}{3} - \frac{2x}{3}$
$x - y = 1 \;\Leftrightarrow\; y = x - 1$

Intersections of the graphs:

- $(2, 1)$

Each intersection corresponds to one solution of the system – its coordinates are the values of x and y.

Three equations – Gaussian elimination.

Input: `x + y + z = 6`
`2x - y + z = 3`

`x + 2y - z = 2` Gaussian elimination

Result

$$x = 1$$
$$y = 2$$
$$z = 3$$

Steps

- Eliminate x from the equations below

$$\begin{cases} x + y + z = 6 \\ -3y - z = -9 \\ y - 2z = -4 \end{cases}$$

- Eliminate y from the equations below

$$\begin{cases} x + y + z = 6 \\ -3y - z = -9 \\ -z = -3 \end{cases}$$

- Solve: $-z = -3 \Rightarrow z = 3$
- Substitute into equation 2: $-3y - 3 = -9 \Rightarrow y = 2$
- Substitute into equation 1: $x + 5 = 6 \Rightarrow x = 1$

Cramer's rule – the determinants D, D_x, D_y.

Input: `2x + 3y = 7`

`x - y = 1` Cramer's rule

Result

$$x = 2$$
$$y = 1$$

Steps

- Compute the determinant

$$D = \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = 2 \cdot (-1) - 3 \cdot 1 = -5$$

- Compute the determinant

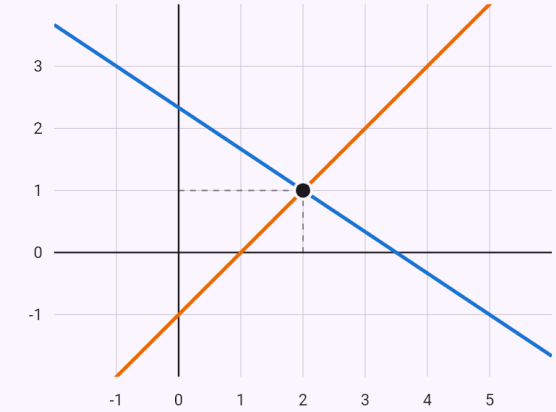
$$D_x = \begin{vmatrix} 7 & 3 \\ 1 & -1 \end{vmatrix} = 7 \cdot (-1) - 3 \cdot 1 = -10$$

- Compute the determinant

$$D_y = \begin{vmatrix} 2 & 7 \\ 1 & 1 \end{vmatrix} = 2 \cdot 1 - 7 \cdot 1 = -5$$

- By Cramer's rule: $x = \frac{D_x}{D} = \frac{-10}{-5} = 2$
- By Cramer's rule: $y = \frac{D_y}{D} = \frac{-5}{-5} = 1$

Graph



$2x + 3y = 7 \Leftrightarrow y = \frac{7}{3} - \frac{2x}{3}$

$x - y = 1 \Leftrightarrow y = x - 1$

Intersections of the graphs:

- $(2, 1)$

Each intersection corresponds to one solution of the system – its coordinates are the values of x and y.

Infinitely many solutions – the Frobenius theorem, a parameter and the vector form.

Input: `x + 2y + z = 3`

`2x + 4y + 3z = 7`

`3x + 6y + 4z = 10` Gauss – matrix

Result

There are infinitely many solutions.

$x = 2 - 2t$

$y = t$

$z = 1$

$t \in \mathbb{R}$

Steps

- The augmented matrix of the system
$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 4 & 3 & 7 \\ 3 & 6 & 4 & 10 \end{array}\right)$$
- Row operations $R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1$
$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$
- Row operations $R_3 \to R_3 - R_2$
$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$
- The equation 0 = 0 always holds – we leave it out
$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

- Rouché–Capelli (Frobenius) theorem:
$h(A) = h(A|b) = 2 < n = 3$ – the ranks are equal but less than the number of unknowns, so the system has infinitely many solutions.
- Write the echelon (triangular) matrix back as equations
$$\begin{cases} x + 2y + z = 3 \\ z = 1 \end{cases}$$
- The unknowns the equations don't determine are chosen as parameters: $y = t,\ t \in \mathbb{R}$
- Solve: $z = 1$
- Substitute into equation 1:
$x + 2t + 1 = 3 \ \Rightarrow \ x = 2 - 2t$
- The solution in vector form
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

A system with no solution – an equation that never holds comes out.

Input: `x + y + z = 2`

`x - y + z = 0`

`2x + 2z = 3`

Result

There is no solution.

Steps

- Eliminate x from the equations below
$$\begin{cases} x + y + z = 2 \\ -y = -1 \\ -2y = -1 \end{cases}$$
- Eliminate y from the equations below
$$\begin{cases} x + y + z = 2 \\ -y = -1 \\ 0 = 1 \end{cases}$$
- We got the equation $0 = 1$, which has no solution – so the system has no solution.

A homogeneous system – a basis of the solution space.

Input: `x + 2y - z = 0`

`2x + 4y - 2z = 0`

`x - y + z = 0` Gauss – matrix

Result

There are infinitely many solutions.

$$x = -t$$
$$y = 2t$$
$$z = 3t$$
$$t \in \mathbb{R}$$

Steps

- The system is homogeneous (all right-hand sides are 0) – it always has the trivial solution (all zeros); it has a non-trivial one exactly when the rank of its matrix is less than the number of unknowns.
- The augmented matrix of the system
$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 4 & -2 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right)$$
- Row operations $R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - R_1$
$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & 2 & 0 \end{array}\right)$$
- Swap equations 2 and 3
$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$
- The equation 0 = 0 always holds – we leave it out
$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -3 & 2 & 0 \end{array}\right)$$

- Rouché–Capelli (Frobenius) theorem:
$h(A) = h(A|b) = 2 < n = 3$ – the ranks are equal but less than the number of unknowns, so the system has infinitely many solutions.
- Write the echelon (triangular) matrix back as equations
$$\begin{cases} x + 2y - z = 0 \\ -3y + 2z = 0 \end{cases}$$
- The unknowns the equations don't determine are chosen as parameters: $z = 3t,\ t \in \mathbb{R}$
- Substitute into equation 2: $-3y + 6t = 0 \ \Rightarrow\ y = 2t$
- Substitute into equation 1: $x + t = 0 \ \Rightarrow\ x = -t$
- The solution in vector form
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$
- A basis of the solution space (dimension 1)
$$\left\{ \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \right\}$$

A system given by its augmented matrix.

Input: `[1 1 1 | 6; 2 -1 1 | 3; 1 2 -1 | 2]`

Result

$$x_1 = 1$$
$$x_2 = 2$$
$$x_3 = 3$$

Steps

- The augmented matrix of the system
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & -1 & 1 & 3 \\ 1 & 2 & -1 & 2 \end{array}\right)$$
- Row operations $R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - R_1$
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & -1 & -9 \\ 0 & 1 & -2 & -4 \end{array}\right)$$
- Row operations $R_3 \to \frac{1}{7}(3R_3 + R_2)$
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & -1 & -9 \\ 0 & 0 & -1 & -3 \end{array}\right)$$

- Rouché–Capelli (Frobenius) theorem: $h(A) = h(A|b) = 3 = n$ – the rank of the matrix equals the rank of the augmented matrix and the number of unknowns, so the system has exactly one solution.
- Write the echelon (triangular) matrix back as equations
$$\begin{cases} x_1 + x_2 + x_3 = 6 \\ -3x_2 - x_3 = -9 \\ -x_3 = -3 \end{cases}$$
- Solve: $-x_3 = -3 \ \Rightarrow\ x_3 = 3$
- Substitute into equation 2: $-3x_2 - 3 = -9 \ \Rightarrow\ x_2 = 2$
- Substitute into equation 1: $x_1 + 5 = 6 \ \Rightarrow\ x_1 = 1$

A system with a parameter – the augmented matrix, the cases a = 1 and a = −2.

Input: `ax + y + z = 1`

`x + ay + z = 1`

`x + y + az = 1` Gauss – matrix

Result

$$\begin{cases} x = \dfrac{1}{a+2},\ y = \dfrac{1}{a+2},\ z = \dfrac{1}{a+2}, \\ x = 1 - t - s,\ y = t,\ z = s,\ t, s \in \\ \emptyset, \end{cases}$$

Steps

- The unknowns are $x,\ y,\ z$, the parameter a is any real number. The solution depends on its value, so we go through the cases (a discussion of the solution).
- The augmented matrix of the system
$$\left(\begin{array}{ccc|c} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \end{array}\right)$$
- Swap equations 1 and 2
$$\left(\begin{array}{ccc|c} 1 & a & 1 & 1 \\ a & 1 & 1 & 1 \\ 1 & 1 & a & 1 \end{array}\right)$$
- Row operations $R_2 \to R_2 - aR_1,\ R_3 \to R_3 - R_1$
$$\left(\begin{array}{ccc|c} 1 & a & 1 & 1 \\ 0 & (1-a)(a+1) & 1-a & 1-a \\ 0 & 1-a & a-1 & 0 \end{array}\right)$$
- We go on with the entry $1 - a$, which can be zero – so we assume $a \neq 1$; the case $a = 1$ is dealt with separately.
- Swap equations 2 and 3
$$\left(\begin{array}{ccc|c} 1 & a & 1 & 1 \\ 0 & 1-a & a-1 & 0 \\ 0 & (1-a)(a+1) & 1-a & 1-a \end{array}\right)$$
- Row operations $R_3 \to \dfrac{1}{a-1}(R_3 - (a+1)R_2)$
$$\left(\begin{array}{ccc|c} 1 & a & 1 & 1 \\ 0 & 1-a & a-1 & 0 \\ 0 & 0 & -a-2 & -1 \end{array}\right)$$
- We go on with the entry $-a - 2$, which can be zero – so we assume $a \neq -2$; the case $a = -2$ is dealt with separately.
- For $a \neq -2,\ a \neq 1$: $h(A) = h(A|b) = 3 = n$ – the system has exactly one solution:
$x = \dfrac{1}{a+2},\ y = \dfrac{1}{a+2},\ z = \dfrac{1}{a+2}$
- For $a = 1$: $h(A) = h(A|b) = 1 < n = 3$ – the system has infinitely many solutions:
$x = 1 - t - s,\ y = t,\ z = s,\ t, s \in \mathbb{R}$. The matrix with the value put in
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$
- For $a = -2$: $h(A) = 2 \neq h(A|b) = 3$ – the system has no solution. The matrix with the value put in
$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

Two parameters – a discussion by a and b.

Input: `x + y + z = 1`

`x + 2y + az = 2`

`x + 4y + a^2z = b`

Result

$$\begin{cases} x = \dfrac{b-4}{a-1},\ y = \dfrac{a-b+2}{a-2},\ z = \dfrac{}{(a} \\ \emptyset, \\ x = 0,\ y = 1-t,\ z = t,\ t \in \mathbb{R}, \\ x = -t,\ y = 1,\ z = t,\ t \in \mathbb{R}, \end{cases}$$

Steps

- The unknowns are $x,\ y,\ z$, the parameters $a,\ b$ are any real numbers. The solution depends on their values, so we go through the cases (a discussion of the solution).
- The augmented matrix of the system
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & a & 2 \\ 1 & 4 & a^2 & b \end{array}\right)$$
- Row operations $R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & a-1 & 1 \\ 0 & 3 & (a-1)(a+1) & b-1 \end{array}\right)$$
- Row operations $R_3 \to R_3 - 3R_2$
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & a-1 & 1 \\ 0 & 0 & (a-2)(a-1) & b-4 \end{array}\right)$$
- We go on with the entry $(a-2)(a-1)$, which can be zero – so we assume $a \neq 1,\ a \neq 2$; the case $a = 2,\ a = 1$ is dealt with separately.
- For $a \neq 1,\ a \neq 2$: $h(A) = h(A|b) = 3 = n$ – the system has exactly one solution
$x = \dfrac{b-4}{a-1},\ y = \dfrac{a-b+2}{a-2},\ z = \dfrac{b-4}{(a-2)(a-1)}$
- For $a = 2,\ b \neq 4$: $h(A) = 2 \neq h(A|b) = 3$ – the system has no solution. The matrix with the value put in
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & b-4 \end{array}\right)$$
- For $a = 2,\ b = 4$: $h(A) = h(A|b) = 2 < n = 3$ – the system has infinitely many solutions:
$x = 0,\ y = 1-t,\ z = t,\ t \in \mathbb{R}$. The matrix with the value put in
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$
- For $a = 1,\ b \neq 4$: $h(A) = 2 \neq h(A|b) = 3$ – the system has no solution. The matrix with the value put in
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & b-4 \end{array}\right)$$
- For $a = 1,\ b = 4$: $h(A) = h(A|b) = 2 < n = 3$ – the system has infinitely many solutions:
$x = -t,\ y = 1,\ z = t,\ t \in \mathbb{R}$. The matrix with the value put in
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

A circle and a line.

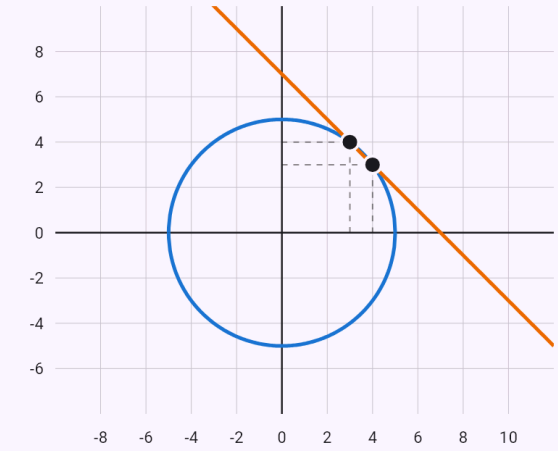
Input: `x^2 + y^2 = 25`

```
x + y = 7
```

Result

$$x_1 = 3,\ y_1 = 4$$
$$x_2 = 4,\ y_2 = 3$$

Graph



$x^2 + y^2 = 25 \ \Leftrightarrow \ y = \pm\sqrt{25 - x^2}$
$x + y = 7 \ \Leftrightarrow \ y = 7 - x$

Intersections of the graphs:

● $(3, 4)$ ● $(4, 3)$

Each intersection corresponds to one solution of the system – its coordinates are the values of x and y.

Known limitations

- Nonlinear systems come without the working for now.

11. Equations and inequalities with a parameter

A letter in an equation besides the unknown is a parameter – any real number the solution depends on. The app rearranges the equation into the form A·x = B (a quadratic one into A·x² + B·x + C = 0) and discusses the solution by the parameter's values: where the coefficient of the unknown is zero, where the discriminant is positive, zero or negative, where the conditions exclude a root and for which values the problem makes no sense. The result sums up the cases.

The app takes the unknown in the order x, y, z, u, v, w, t; the other letters are parameters. Choose another unknown with the “Unknown” chip or write it after a comma. The graph shows all solutions as a curve (a region for an inequality) in the parameter–unknown plane; pick the parameter's value with the slider and read the solutions off the vertical line.

After a comma you can also ask about the roots: “x^2 + px + 4 = 0, x > 0” – for which p every root is positive (the chip “The condition holds for” switches to at least one root), “…, x = 3” – for which p 3 is a root. A linear equation’s root is put in the condition, a quadratic one’s by Vieta’s formulas. A system of two linear equations with a parameter is solved by determinants (Cramer’s rule), with the cases of one, no and infinitely many solutions. A quadratic or a rational inequality is discussed by the order of its zeros. The graph’s slider stops exactly at the parameter’s critical values, which are also listed under the graph. A system with a parameter of more equations, with two parameters or by the matrix method is in the chapter Systems of equations.

In an indefinite integral, other letters are constants – the app adds which must not be zero (∫ dx/(a − bx)³ for b ≠ 0). In a limit and a definite integral letters are parameters too – see the chapters Limits and Definite integral.

How to enter it

`2ax + a = 4x`	Every other letter is a parameter; the unknown is x (or y, z, …).
`(q+3)x = 2x + q, unknown q`	You can name the unknown or the parameter after a comma (“a is a parameter”, “unknown q”).
`x^2 + px + 4 = 0, x > 0`	A question about the roots after a comma: for which p every root (or at least one) has the property.

Examples

A linear equation – no solution for a = 2.

Input: `2ax + a = 4x`

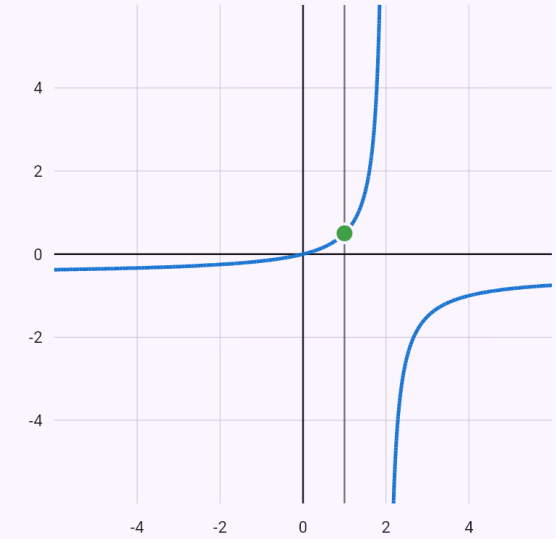
Result

$$\begin{cases} x = \dfrac{a}{2(2-a)}, & a \neq 2 \\ x \in \emptyset, & a = 2 \end{cases}$$

Steps

- The unknown is x, the parameter a is any real number. The solution depends on its value, so we go through the individual cases (discussion of the solution).
- Rearrange into the basic form: $2(a-2)x = -a$
- For $a \neq 2$, $2(a-2)$ is non-zero, so we can divide by it: $x = \dfrac{a}{2(2-a)}$
- For $a = 2$, the coefficient of the unknown is zero: $0 \cdot x = -2$, so $x \in \emptyset$

Graph



$2ax + a = 4x$

Every point [a; x] of the curve is a solution. For a chosen value of a, the solutions lie on the vertical line – move it with the slider.

a = 2

$a = 1: \quad x \approx 0.5$

A quadratic equation – discussed by the discriminant.

Input: `x^2 + 6x + p = 0`

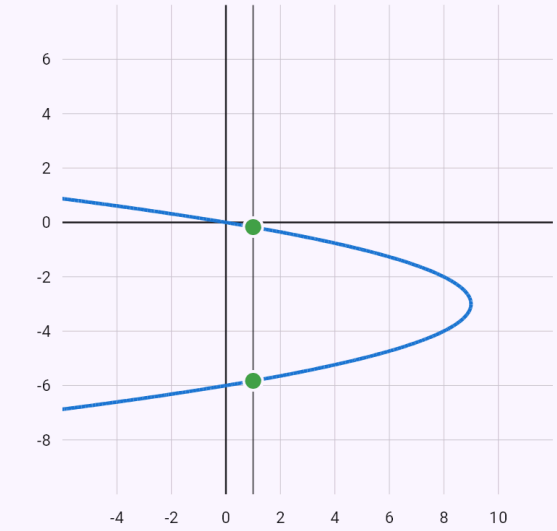
Result

$$\begin{cases} x_{1,2} = -3 \pm \sqrt{9-p}, & p < 9 \\ x = -3, & p = 9 \\ x \in \emptyset, & p > 9 \end{cases}$$

Steps

- The unknown is x, the parameter p is any real number. The solution depends on its value, so we go through the individual cases (discussion of the solution).
- Rearrange into the basic form: $x^2 + 6x + p = 0$
- The discriminant depending on the parameter: $D = 6^2 - 4 \cdot 1 \cdot p = 36 - 4p = 4(9-p)$
- For $p < 9$, D > 0 and the equation has two different roots: $x_{1,2} = -3 \pm \sqrt{9-p}$
- For $p > 9$, D < 0 and the equation has no real solution.
- For $p = 9$, D = 0 and the equation has one double root: $x = -3$

Graph



$x^2 + 6x + p = 0$

Every point [p; x] of the curve is a solution. For a chosen value of p, the solutions lie on the vertical line – move it with the slider.

p = 9

$p = 1: \quad x \approx -5.83;\ -0.17$

An inequality – the coefficient's sign decides the direction.

Input: `2x + 2p - 5 < 2px + 7`

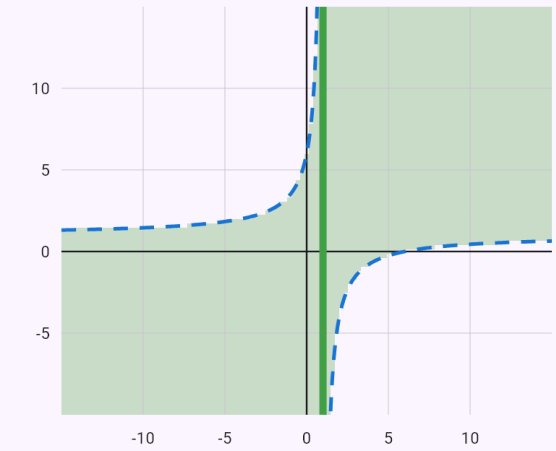
Result

$$\begin{cases} x < \dfrac{p-6}{p-1}, & p < 1 \\ x \in \mathbb{R}, & p = 1 \\ x > \dfrac{p-6}{p-1}, & p > 1 \end{cases}$$

Steps

- The unknown is x, the parameter p is any real number. The solution depends on its value, so we go through the individual cases (discussion of the solution).
- Rearrange into the basic form: $2(1-p)x < 2(6-p)$
- For $p < 1$, $2(1-p)$ is positive – dividing keeps the inequality sign: $x < \dfrac{p-6}{p-1}$
- For $p > 1$, $2(1-p)$ is negative – dividing reverses the inequality sign: $x > \dfrac{p-6}{p-1}$
- For $p = 1$, the coefficient of the unknown is zero: $0 \cdot x < 10$, so $x \in \mathbb{R}$

Graph



$2x + 2p - 5 < 2px + 7 \;\Leftrightarrow\; x = \dfrac{p-6}{p-1}$

The shaded region contains all solutions [p; x]. For a chosen value of p, the solution is the highlighted part of the vertical line – move it with the slider.

p = 1

$p = 1$

Both roots greater than 1 – the discriminant and Vieta's formulas.

Input: `x^2 - 2px + p + 2 = 0, x > 1`

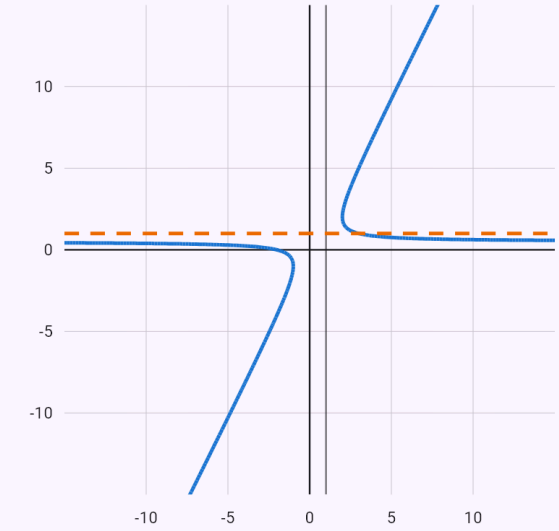
Result

$$2 \leq p < 3$$

Steps

- The unknown is x, the parameter p is any real number. The solution depends on its value, so we go through the individual cases (discussion of the solution).
- Rearrange into the basic form: $x^2 - 2px + p + 2 = 0$
- The discriminant depending on the parameter $D = (-2p)^2 - 4 \cdot 1 \cdot (p+2) = 4p^2 - 4p - 8 = 4(p - 2$
- For $p \in (-\infty, -1) \cup (2, \infty)$, D > 0 and the equation has two different roots: $x_{1,2} = p \pm \sqrt{p^2 - p - 2}$
- For $-1 < p < 2$, D < 0 and the equation has no real solution.
- For $p = -1$, D = 0 and the equation has one double root: $x = -1$
- For $p = 2$, D = 0 and the equation has one double root: $x = 2$
- We look for the values of the parameter for which every root of the equation satisfies $x > 1$ (and there is a root).
- By Vieta's formulas: $x_1 + x_2 = 2p, \quad x_1 \cdot x_2 = p + 2$
- The equation must have real roots: $D \geq 0$, so $p \in (-\infty, -1] \cup [2, \infty)$
- The condition on the sum: $(x_1 - 1) + (x_2 - 1) = 2(p - 1) > 0$, so $p > 1$
- The condition on the product: $(x_1 - 1)(x_2 - 1) = 3 - p > 0$, so $p < 3$
- The condition holds for $2 \leq p < 3$

Graph



$x^2 - 2px + p + 2 = 0$
$x = 1$

Every point [p; x] of the curve is a solution. For a chosen value of p, the solutions lie on the vertical line – move it with the slider.

p = −1 | p = 2 | p = 3

$p = 1: \quad x \in \emptyset$

A system with a parameter – determinants; the lines meet, are parallel or the same.

Input: `ax + y = 1`

`x + ay = 1`

Result

$$\begin{cases} x = \dfrac{1}{a+1},\ y = \dfrac{1}{a+1}, & a \neq -1, \\ \emptyset, & a = -1 \\ y = 1 - x,\ x \in \mathbb{R}, & a = 1 \end{cases}$$

Steps

- The unknowns are $x,\ y$, the parameter a is any real number. The solution depends on its value, so we go through the cases (a discussion of the solution).
- Rearrange into the basic form
  $ax + y = 1$
  $x + ay = 1$
- We solve the system by determinants (Cramer's rule)
  $D = \begin{vmatrix} a & 1 \\ 1 & a \end{vmatrix} = (a-1)(a+1)$
  $D_x = \begin{vmatrix} 1 & 1 \\ 1 & a \end{vmatrix} = a - 1$
  $D_y = \begin{vmatrix} a & 1 \\ 1 & 1 \end{vmatrix} = a - 1$
- For $a \neq -1,\ a \neq 1$, D ≠ 0 – the system has a single solution (the lines meet): $x = \dfrac{1}{a+1},\ y = \dfrac{1}{a+1}$
- For $a = -1$, D = 0 and the system has no solution (the lines are parallel).
  $-x + y = 1$
  $x - y = 1$
- For $a = 1$, D = 0 and both equations of describe the same line – infinitely many solutions: $y = 1 - x,\ x \in \mathbb{R}$
  $x + y = 1$
  $x + y = 1$

A rational inequality – by the order of the zero p and the point 1.

Input: `(x - p)/(x - 1) > 0`

Result

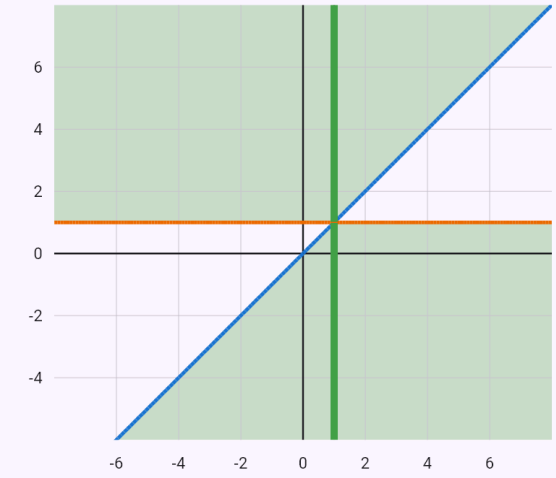
$$\begin{cases} x \in (-\infty,\ p) \cup (1,\ \infty), & p < 1 \\ x \in (-\infty,\ 1) \cup (1,\ \infty), & p = 1 \\ x \in (-\infty,\ 1) \cup (p,\ \infty), & p > 1 \end{cases}$$

Conditions: $x \neq 1$

Steps

- The unknown is x, the parameter p is any real number. The solution depends on its value, so we go through the individual cases (discussion of the solution).
- Conditions – a denominator must not be zero: $x \neq 1$
- Rearrange into the basic form: $\dfrac{x-p}{x-1} > 0$
- The numerator's zeros: p; the denominator is zero for 1
- For $p < 1$ – by the order of the zeros and the signs on the intervals: $x \in (-\infty,\ p) \cup (1,\ \infty)$
- For $p > 1$ – by the order of the zeros and the signs on the intervals: $x \in (-\infty,\ 1) \cup (p,\ \infty)$
- For $p = 1$, we solve $\dfrac{x-1}{x-1} > 0$: $x \in (-\infty,\ 1) \cup (1,\ \infty)$

Graph



$\dfrac{x-p}{x-1} > 0$
$x - 1 = 0$

The shaded region contains all solutions [p; x]. For a chosen value of p, the solution is the highlighted part of the vertical line – move it with the slider.

p = 1

$p = 1$

Known limitations

- The full discussion by intervals with one parameter only, with more parameters the general cases.
- Questions about the roots only for linear and quadratic equations; systems with a parameter only two linear equations; a parameter in a root or a logarithm not yet.

12. Solving a formula for a letter

A formula from mathematics, physics or chemistry (m = ρV, S = πr², v = s/t) is rearranged so that the chosen letter stands alone on one side. The other letters are given quantities – the app works with them like with known numbers. The steps are equivalent steps as at school: moving terms to the other side, multiplying by a denominator, dividing, factoring out, taking a root, reciprocals.

The letter to solve for is chosen with the “Solve for” chip; without a choice the app solves for the first letter on the formula's side. The result is the formula with its conditions – what it divides by must not be zero (V = m/ρ for ρ ≠ 0). The quantities in a formula are positive, so a root has just its positive value: r = √(S/π).

An equation in x and y is solved for y (the line 2x + y = x + 2y is y = x); the other equations in x and y have “Solve for y” and “Solve for x” among the chips of the derivative, the integral and the graph. Here a root gives both signs: x² + y² = 25 is y = ±√(25 − x²).

Equations with a parameter (2ax + a = 4x, the unknown x) are not rearranged as a formula but solved with the discussion – see the previous chapter.

How to enter it

`m = ρV`	A formula – a quantity alone on one side, capital and small letters (E, I, V are quantities).
`1/R = 1/R_1 + 1/R_2`	Indices with an underscore (R_1, v_0), Greek letters (ρ, rho, \rho), π is the constant.
`2x + y = x + 2y`	An equation in x and y – solved for y.

Examples

Density – dividing by a quantity with the condition ρ ≠ 0.

Input: `m = ρV`

Result

$$V = \frac{m}{\rho}$$

Conditions: $\rho \neq 0$

Steps

- We solve for V. The other letters ($m,\ \rho$) are treated as known numbers, and the equation is changed by equivalent steps until V stands alone on one side.
- Swap the sides of the equation: $\rho V = m$
- Divide both sides by ρ, which must not be zero ($\rho \neq 0$): $V = \frac{m}{\rho}$

A trapezoid's area – multiplying by two, dividing by the height, moving a over.

Input: `S = (a + c)/2 · v`

Result

$$c = \frac{2S - av}{v}$$

Conditions: $v \neq 0$

Steps

- We solve for c. The other letters ($S,\ a,\ v$) are treated as known numbers, and the equation is changed by equivalent steps until c stands alone on one side.
- Swap the sides of the equation: $\frac{v(a+c)}{2} = S$
- Multiply both sides by 2: $v(a+c) = 2S$
- Divide both sides by v, which must not be zero ($v \neq 0$): $a + c = \frac{2S}{v}$
- Subtract a from both sides: $c = \frac{2S}{v} - a$
- Bring to a common denominator: $c = \frac{2S - av}{v}$

A circle's area – taking the root, the radius is positive.

Input: `S = πr^2`

Result

$$r = \sqrt{\frac{S}{\pi}}$$

Steps

- We solve for r. The other letters (S) are treated as known numbers, and the equation is changed by equivalent steps until r stands alone on one side.
- Swap the sides of the equation: $\pi r^2 = S$
- Divide both sides by π: $r^2 = \frac{S}{\pi}$
- Take the root of both sides. r is a quantity, and quantities are positive – only the positive root holds: $r = \sqrt{\frac{S}{\pi}}$

Resistors in parallel – a common denominator and reciprocals.

Input: `1/R = 1/R_1 + 1/R_2`

Result

$$R_1 = \frac{RR_2}{R_2 - R}$$

Conditions: $R \neq 0,\ R_1 \neq 0,\ R_2 \neq 0$

Steps

- We solve for R_1. The other letters ($R,\ R_2$) are treated as known numbers, and the equation is changed by equivalent steps until R_1 stands alone on one side.
- Conditions – a denominator must not be zero: $R \neq 0,\ R_1 \neq 0,\ R_2 \neq 0$
- Swap the sides of the equation: $\frac{1}{R_2} + \frac{1}{R_1} = \frac{1}{R}$
- Subtract $\frac{1}{R_2}$ from both sides: $\frac{1}{R_1} = \frac{1}{R} - \frac{1}{R_2}$
- Bring to a common denominator: $\frac{1}{R_1} = \frac{R_2 - R}{RR_2}$
- If two expressions are equal, so are their reciprocals: $R_1 = \frac{RR_2}{R_2 - R}$

A line – the terms with y to the left, the others to the right.

Input: `2x + y = x + 2y`

Result

$$y = x$$

Steps

- We solve for y. The other letters (x) are treated as known numbers, and the equation is changed by equivalent steps until y stands alone on one side.
- Move the terms with y to the left side and the others to the right (with the opposite sign), then add them up

$$y - 2y = x - 2x$$
$$-y = -x$$

- Multiply both sides by (-1): $y = x$

A circle solved for y – the root with ±.

Input: `x^2 + y^2 = 25`

Result

$$y = \pm\sqrt{25 - x^2}$$

Steps

- We solve for y. The other letters (x) are treated as known numbers, and the equation is changed by equivalent steps until y stands alone on one side.
- Subtract x^2 from both sides: $y^2 = 25 - x^2$
- Take the square root of both sides; a number and its opposite have the same square, hence ±:

$$y = \pm\sqrt{25 - x^2}$$

Known limitations

- The letter only in powers, fractions and roots; not yet in functions (sin, log, an exponent) or squared together with itself (s = v_0t + ½at² for t).

13. Simplifying expressions

The app transforms an expression without an equals sign according to the chosen chip: it simplifies it (a rational expression over a common denominator and cancelled, with conditions), expands it, factors it (by taking out a common factor, by formulas and by synthetic division) or splits it into partial fractions – as on paper, with equations for the unknown coefficients. Factoring is over the integers; if the expression factors further over the reals (by irrational roots), the next step shows it. A root of a monomial is taken out partly – with an even root as an absolute value where the base may be negative. Complex numbers work too – a quotient by multiplying with the complex conjugate.

Numeric expressions with fractions are worked out step by step as at school: the innermost brackets first (round, square or curly), a compound fraction's numerator and denominator separately, then powers, multiplication and division (by a fraction as multiplication by its reciprocal, also with a colon or ÷) and the addition over a common denominator last. Type a mixed number with a space – 2 1/3 (from a photo also 2⅓) – and the app first turns it into a fraction; write a product of two numbers always with a dot. A result above 1 is shown as a mixed number too.

Write a polynomial divided by a polynomial with a colon: (2x^3 + 3x^2 - x + 5) : (x^2 + 1). The app does long division as on paper – divides the leading terms, multiplies the divisor by the result and subtracts – and shows the quotient, the remainder and a check; for a divisor x − a it also offers Horner's scheme. A fraction p/q is simplified as before; choose the division with the "Polynomial division" chip.

The binomial theorem: the k-th term of an expansion ("5th term of (2x - 1/x)^8") and the term without x ("term without x in (x + 1/x)^10") from the general term. Permutations P(5) and with repetition P′(2, 3, 1) are worked out as factorials.

Combinatorics can also be asked in a short sentence: "How many ways can we choose 3 of 10?" (with "order matters", "with repetition"), "In how many ways can 5 people be arranged?" (also around a round table), "How many three-digit even numbers can be made from the digits 0-5 without repetition?" and "How many words can be formed from the letters of the word BANANA?". Two classic probability problems too: the birthday problem (at least two of n people share a birthday) and the cloakroom attendant who loses the tickets (nobody gets their own coat – or letters in envelopes).

Examples

A numeric expression with fractions – multiplication first, then the common denominator.

Input: `2/3 - 1/6*2` Simplify

Result

$$\frac{1}{3}$$

Steps

- Multiplication and division first: $\frac{2}{3} - \frac{1}{3} = \frac{1}{3}$

Nested brackets from the innermost.

Input: `{2 - [1 - (1/2 - 1/3)]} * 6`

Result

$$7$$

Steps

- First the innermost bracket: $\frac{1}{2} - \frac{1}{3} = \frac{3}{6} - \frac{2}{6} = \frac{1}{6}$
- Substituted: $\left\{2 - \left[1 - \frac{1}{6}\right]\right\} \cdot 6$
- First the innermost bracket: $1 - \frac{1}{6} = \frac{6}{6} - \frac{1}{6} = \frac{5}{6}$
- Substituted: $\left\{2 - \frac{5}{6}\right\} \cdot 6$
- First the innermost bracket: $2 - \frac{5}{6} = \frac{12}{6} - \frac{5}{6} = \frac{7}{6}$
- Substituted: $\frac{7}{6} \cdot 6$
- We multiply – numerator by numerator, denominator by denominator – and cancel: $\frac{7}{6} \cdot 6 = \frac{7 \cdot 6}{6} = \frac{42}{6} = 7$

A compound fraction – the numerator and the denominator separately, then times the reciprocal.

Input: `(1/2 + 1/3)/(2 - 1/4)`

Result

$$\frac{10}{21}$$

Steps

- First the numerator and the denominator of the compound fraction separately: $\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$
- First the numerator and the denominator of the compound fraction separately: $2-\frac{1}{4}=\frac{8}{4}-\frac{1}{4}=\frac{7}{4}$
- Substituted: $\frac{\frac{5}{6}}{\frac{7}{4}}$
- A compound fraction: the numerator times the reciprocal of the denominator

  $\frac{\frac{5}{6}}{\frac{7}{4}}=\frac{5}{6}\cdot\frac{4}{7}=\frac{5\cdot 4}{6\cdot 7}=\frac{20}{42}=\frac{10}{21}$

Mixed numbers – into fractions and the result as a mixed number too.

Input: `2 1/3 + 1 1/2`

Result

$$\frac{23}{6}=3\frac{5}{6}$$

Steps

- We turn the mixed numbers into fractions – the whole part times the denominator plus the numerator

  $2\frac{1}{3}=\frac{2\cdot 3+1}{3}=\frac{7}{3},\quad 1\frac{1}{2}=\frac{1\cdot 2+1}{2}=\frac{3}{2}$
- Substituted: $\frac{7}{3}+\frac{3}{2}$
- Bring to a common denominator: $\frac{14}{6}+\frac{9}{6}=\frac{23}{6}$
- The result as a mixed number: $\frac{23}{6}=3\frac{5}{6}$

Simplifying a rational expression.

Input: `(x^2 - 1)/(x^2 + 2x + 1)` Simplify

Result

$$\frac{x-1}{x+1}$$

Conditions: $x \neq -1$

Steps

- Conditions – a denominator must not be zero: $x \neq -1$
- Factor the numerator and denominator: $\frac{(x-1)(x+1)}{(x+1)^2}$
- Cancel the common factor $x+1$: $\frac{x-1}{x+1}$

Expanding.

Input: `(x + 2)^3` Expand

Result

$$x^3 + 6x^2 + 12x + 8$$

Steps

- Multiply out the brackets and collect like terms: $x^3 + 6x^2 + 12x + 8$

Factoring by synthetic division.

Input: `x^3 - 6x^2 + 11x - 6` Factor

Result

$$(x-3)(x-2)(x-1)$$

Steps

- Rational roots can only be divisors of the constant term -6: $\pm 1,\ \pm 2,\ \pm 3,\ \pm 6$
- Horner's scheme confirms the root $x = 1$ and divides the polynomial by it:

	1	−6	11	−6
1	1	−5	6	0

$P(x) = (x-1)(x^2 - 5x + 6)$

- Substituting shows these candidates aren't roots: $P(-1) = -24 \neq 0$
- Horner's scheme confirms the root $x = 2$ and divides the polynomial by it:

	1	−5	6
2	1	−3	0

$P(x) = (x-1)(x-2)(x-3)$

Partial fractions.

Input: `(3x + 5)/(x^2 + x - 2)` Partial fractions

Result

$$\frac{8}{3(x-1)} + \frac{1}{3(x+2)}$$

Conditions: $x \neq -2,\ x \neq 1$

Steps

- Conditions – a denominator must not be zero: $x \neq -2,\ x \neq 1$
- Factor the denominator: $(x-1)(x+2)$
- Look for a decomposition of the form: $\frac{A}{x-1} + \frac{B}{x+2}$
- Multiply both sides by the denominator $(x-1)(x+2)$: $3x + 5 = A(x+2) + B(x-1)$
- Substitute $x = 1$: $8 = 3A \ \Rightarrow\ A = \frac{8}{3}$
- Substitute $x = -2$: $-1 = -3B \ \Rightarrow\ B = \frac{1}{3}$

Taking a root out partly, with an absolute value.

Input: `sqrt(8a^5 b^2)` Simplify

Result

$$2a^2|b|\sqrt{2a}$$

Steps

- Take what we can out of the root – powers under it come out in front (with an even root as an absolute value where the base may be negative): $2a^2|b|\sqrt{2a}$

Dividing complex numbers.

Input: `(3+4i)/(1-2i)` Algebraic form

Result

$$-1 + 2i$$

Steps

- Multiply the fraction by the denominator's complex conjugate, $1 + 2i$

$$\frac{(3+4i)(1+2i)}{(1-2i)(1+2i)} = \frac{-5+10i}{5} = -1+2i$$

An equation with factorials – conditions, cancelling and checking the roots.

Input: `(x+1)!/(x-1)! = 30`

Result

$$x = 5$$

Steps

- Conditions: factorials and combination numbers exist only for whole non-negative numbers – every argument (for a combination number n, k and n − k) must be a whole number ≥ 0:
  $x + 1 \geq 0,\ x - 1 \geq 0 \ \Rightarrow\ x \in \mathbb{Z},\ x \geq 1$
- The factorials written by the smallest one, (x + 1)! = (x − 1)! · x · (x + 1), and cancelled; a combination number as a product: $\frac{(x+1)!}{(x-1)!} = 30 \ \Rightarrow\ x^2 + x = 30$
- An equation without factorials is left, everything on one side: $x^2 + x - 30 = 0$
- The equation's roots are $x = -6,\ x = 5$; the conditions are met by $x = 5$

Long division – the quotient, the remainder and a check.

Input: `(2x^3 + 3x^2 - x + 5) : (x^2 + 1)`

Result

$$\left(2x^3 + 3x^2 - x + 5\right) : \left(x^2 + 1\right)$$
$$= 2x + 3 - \frac{3x - 2}{x^2 + 1}$$

Steps

- Step 1 – divide the leading terms $2x^3 : x^2 = 2x$, multiply the divisor by the result $2x \cdot \left(x^2 + 1\right) = 2x^3 + 2x$ and subtract, leaving $3x^2 - 3x + 5$
- Step 2 – divide the leading terms $3x^2 : x^2 = 3$, multiply the divisor by the result $3 \cdot \left(x^2 + 1\right) = 3x^2 + 3$ and subtract, leaving $-3x + 2$
- Long division $\left(2x^3 + 3x^2 - x + 5\right) : \left(x^2 + 1\right)$

$$\begin{array}{rrrr} 2x^3 & +3x^2 & -x & +5 \\ -2x^3 & & -2x & \\ \hline & 3x^2 & -3x & +5 \\ & -3x^2 & & -3 \\ \hline & & -3x & +2 \end{array}$$

- The quotient is $2x + 3$, the remainder $-3x + 2$
- Check – the quotient times the divisor plus the remainder gives the dividend:
  $(2x + 3)\left(x^2 + 1\right) + (-3x + 2) = 2x^3 + 3x^2 - x + 5$

Division by x − 1 with Horner's scheme.

Input: `(x^3 - 2x + 1) : (x - 1)` Horner's scheme

Result

$$\left(x^3 - 2x + 1\right) : (x - 1)$$
$$= x^2 + x - 1$$

Steps

- Horner's scheme for $x = 1$ – the dividend's coefficients on top, their running sums below; the last number is the remainder

	1	0	-2	1
1	1	1	-1	$\boxed{0}$

- The quotient is $x^2 + x - 1$, the remainder 0
- Check – the quotient times the divisor plus the remainder gives the dividend: $\left(x^2 + x - 1\right)(x - 1) = x^3 - 2x + 1$

An equation with a combination number C(n, 2) – also V(n, k), C', V' with repetition.

Input: `C(n, 2) = 45`

Result

$$n = 10$$

Steps

- Conditions: factorials and combination numbers exist only for whole non-negative numbers – every argument (for a combination number n, k and n − k) must be a whole number ≥ 0: $n \geq 0,\ n - 2 \geq 0 \ \Rightarrow\ n \in \mathbb{Z},\ n \geq 2$
- The factorials written by the smallest one, (x + 1)! = (x − 1)! · x · (x + 1), and cancelled; a combination number as a product: $\binom{n}{2} = 45 \ \Rightarrow\ \frac{n(n-1)}{2} = 45$
- An equation without factorials is left, everything on one side: $n^2 - n - 90 = 0$
- The equation's roots are $n = -9,\ n = 10$; the conditions are met by $n = 10$

Choosing k of n in a sentence – combinations.

Input: `Kolika způsoby lze vybrat 3 z 10?`

Result

120

Steps

- We choose 3 of 10, the order doesn't matter and nothing repeats – the number of combinations:
- $$C(10,3) = \binom{10}{3} = \frac{10!}{3! \cdot 7!} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$$

Even three-digit numbers – two cases by the last digit.

Input: `Kolik je trojmístných sudých čísel z číslic 0, 1, 2, 3, 4, 5, číslice se nesmí opakovat?`

Result

52

Steps

- The numbers have 3 digits taken from $0,\ 1,\ 2,\ 3,\ 4,\ 5$. Zero can't come first. Digits may not repeat. A number is even exactly when its last digit is even.
- The last digit is 0 – all the other digits are left for the other places: $5 \cdot 4 \cdot 1 = 20$
- The last digit is a suitable one other than 0 – then neither zero nor it can come first: $4 \cdot 4 \cdot 2 = 32$
- We add both cases: $20 + 32 = 52$

Words from the letters of a word – permutations with repetition.

Input: `Kolik slov lze sestavit z písmen slova MATEMATIKA?`

Result

151200

Steps

- The word has 10 letters; we count how many times each occurs: $\mathrm{M}:2,\ \mathrm{A}:3,\ \mathrm{T}:2,\ \mathrm{E}:1,\ \mathrm{I}:1,\ \mathrm{K}:1$
- The words are permutations with repetition – swapping equal letters gives the same word:
  $$P'(2,3,2,1,1,1) = \frac{10!}{2! \cdot 3! \cdot 2!} = \frac{3628800}{24} = 151200$$

The birthday problem through the complementary event.

Input: `Jaká je pravděpodobnost, že ve skupině 23 lidí slaví alespoň dva narozeniny ve stejný den?`

Result

$$P(A) \approx 0.5073$$

Steps

- The complementary event A′ is easier: all 23 people have birthdays on different days (365 days, each equally likely).
- The probability that all birthdays differ
  $$P(A') = \frac{V(365,23)}{365^{23}} = \frac{365 \cdot 364 \cdot \ldots \cdot 343}{365^{23}} \approx 0.4927$$
- The probability asked for is the complement to one:
  $P(A) = 1 - P(A') \approx 0.5073 = 50.73\,\%$

The lost cloakroom tickets – derangements.

Input: `Jaká je pravděpodobnost, že nikdo z 5 lidí odcházejících z divadla nedostane svůj kabát, když šatnářka ztratí lístky?`

Result

$$P = \frac{11}{30} \approx 0.3667$$

Steps

- The ways in which nobody gets their own are the derangements – their number by inclusion and exclusion
  $$D_5 = 5!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right) = 44$$
- There are n! ways in all, the probability is the ratio:
  $$P = \frac{D_5}{5!} = \frac{44}{120} = \frac{11}{30} \approx 0.3667$$
- For more people the probability hardly changes:
  $\frac{D_n}{n!} \to \frac{1}{e} \approx 0.3679$

The term without x of a binomial expansion.

Input: `term without x in (x + 1/x)^10`

Result

$$T_6 = 252$$

Steps

- The general $(k+1)$-th term of the binomial expansion:
  $T_{k+1} = \binom{10}{k} x^{10-k} \cdot \left(\frac{1}{x}\right)^k$
- The term without x has the exponent of x zero:
  $10 - 2k = 0 \;\Rightarrow\; k = 5$
- The term asked for, number 6:
  $T_6 = \binom{10}{5} x^5 \cdot \left(\frac{1}{x}\right)^5 = 252$

14. Divisibility and primes

The app factors a whole number on its own into primes – it divides by the smallest primes one after another and writes 360 = 2³·3²·5; for a prime it shows that no prime up to its square root divides it. You can also write the test as isprime(97).

The greatest common divisor gcd(60, 90) and the least common multiple lcm(18, 30) come from the prime factorizations, or with the “Euclidean algorithm” chip by division with a remainder. The extended Euclidean algorithm with its table gives Bézout’s identity bezout(240, 46): gcd(240, 46) = 240·u + 46·v.

A number's divisors divisors(36) come in pairs 1·36, 2·18, … up to the square root, their count checked by the factorization. The divisibility rules divisibility(1236) form a table for 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 and 25 (the last digit, the digit sum, the alternating sum…); one divisor is written divisibility(1236, 7), as the sentence “is 1236 divisible by 7” or 7 | 1236 – a divisor without a school rule is divided with a remainder. The primes up to 200 come from the sieve of Eratosthenes (“primes up to 50”). A linear equation ax + by = c in the integers (“3x + 5y = 1, x, y ∈ ℤ” or the ℤ chip) is solved by Bézout's identity – all the solutions $x = x_0 + 5k$, $y = y_0 - 3k$.

Modular arithmetic: the remainder 17 mod 5, the inverse 3^(-1) mod 11 (from the extended Euclidean algorithm, with a check) and the power 7^13 mod 13 by repeated squaring. RSA is built on them – rsa(p = 11, q = 13, e = 7, m = 9) computes n, φ(n), the private key d, then encrypts the message and decrypts it again; e and m may be left out.

How to enter it

`360, factor(360), isprime(97)`	The prime factorization, a primality test.
`gcd(60, 90), lcm(18, 30)`	The greatest common divisor and the least common multiple.
`divisors(36), divisibility(1236), divisibility(1236, 7)`	A number's divisors and the divisibility rules – all, or one divisor (also the sentence “is 1236 divisible by 7”, 7 | 1236).
`primes up to 50`	The primes up to n (at most 200) by the sieve of Eratosthenes; also primes(50).
`3x + 5y = 1, x, y ∈ ℤ`	A linear equation in the integers (Diophantine); also diophantine(3, 5, 1).

`bezout(240, 46)`	Bézout's identity.
`17 mod 5, 3^(-1) mod 11, 7^13 mod 13`	The remainder, the inverse and the power modulo n.
`rsa(p = 11, q = 13, e = 7, m = 9)`	RSA with small numbers.

Examples

The prime factorization by successive division.

Input: `360`

Result

$$360 = 2^3 \cdot 3^2 \cdot 5$$

Steps

- Divide 360 by the smallest primes, one after another, until 1 is left

360	2
180	2
90	2
45	3
15	3
5	5
1	

- The prime factorization: $360 = 2^3 \cdot 3^2 \cdot 5$

A prime – no divisor up to the square root.

Input: `97`

Result

The number is a prime.

Steps

- It is enough to try the primes up to $\sqrt{97} \approx 9.85$: $2,\ 3,\ 5,\ 7$ – none of them divides 97, so it is a prime.

The gcd from the prime factorizations.

Input: `gcd(60, 90)`

Result

$$\gcd(60, 90) = 30$$

Steps

- The prime factorization: $60 = 2^2 \cdot 3 \cdot 5$
- The prime factorization: $90 = 2 \cdot 3^2 \cdot 5$
- The greatest common divisor is the product of the common prime factors with the smallest exponents: $\gcd(60, 90) = 2 \cdot 3 \cdot 5 = 30$

The lcm via the gcd by the Euclidean algorithm.

Input: `lcm(60, 90)` Euclidean algorithm

Result

$$\operatorname{lcm}(60, 90) = 180$$

Steps

- The Euclidean algorithm – divide the larger number by the smaller one with a remainder, then the divisor by the remainder, until the remainder is 0; the last non-zero remainder is the greatest common divisor

$$\begin{aligned} 90 &= 1 \cdot 60 + 30 \\ 60 &= 2 \cdot 30 + 0 \end{aligned}$$

$$\Rightarrow \gcd(60, 90) = 30$$

- The least common multiple is the product of the numbers divided by their greatest common divisor

$$\operatorname{lcm}(60, 90) = \frac{60 \cdot 90}{\gcd(60, 90)} = \frac{5400}{30} = 180$$

The divisors in pairs, their count from the factorization.

Input: `dělitelé(36)`

Result

$$\{1, 2, 3, 4, 6, 9, 12, 18, 36\}$$

Steps

- We look for the divisors in pairs d · 36/d – it is enough to try d up to 6, the pair's other divisor follows

  $36 = 1 \cdot 36$

  $36 = 2 \cdot 18$

  $36 = 3 \cdot 12$

  $36 = 4 \cdot 9$

  $36 = 6 \cdot 6$

- A check of the number of divisors from the prime factorization – the exponents increased by 1 multiplied: $36 = 2^2 \cdot 3^2,\ (2+1) \cdot (2+1) = 9$

The table of divisibility rules.

Input: `dělitelnost(1236)`

Result

$$2, 3, 4, 6, 12 \mid 1236$$

Steps

- Divisibility rules: 2, 5, 10 – the last digit; 4, 25 – the last two digits; 8 – the last three digits; 3, 9 – the digit sum; 6 = 2 · 3 and 12 = 3 · 4 – both at once; 7 – cut off the last digit and subtract its double from the rest; 11 – the alternating digit sum

2	6	$2 \mid 1236$
3	$1+2+3+6=12$	$3 \mid 1236$
4	36	$4 \mid 1236$
5	6	$5 \nmid 1236$
6	$2 \mid 1236,\ 3 \mid 1236$	$6 \mid 1236$
7	$123 - 2 \cdot 6 = 111,\ 11 - 2 \cdot 1 = 9$	$7 \nmid 1236$
8	236	$8 \nmid 1236$
9	$1+2+3+6=12$	$9 \nmid 1236$
10	6	$10 \nmid 1236$
11	$6-3+2-1=4$	$11 \nmid 1236$
12	$3 \mid 1236,\ 4 \mid 1236$	$12 \mid 1236$
25	36	$25 \nmid 1236$

Divisibility by seven – cutting off the last digit.

Input: `dělitelnost(1309, 7)`

Result

$$7 \mid 1309$$

Steps

- Divisibility by 7: cut off the last digit, subtract its double from the rest and repeat until at most two digits are left – that number is divisible by 7 just when the original one is (for large numbers a division is often quicker): $130 - 2 \cdot 9 = 112,\ 11 - 2 \cdot 2 = 7$

The sieve of Eratosthenes.

Input: `prvočísla do 50`

Result

$$2, 3, 5, 7, 11, 13, 17, 19,$$
$$23, 29, 31, 37, 41, 43, 47$$

Steps

- 2 is a prime – we strike out all its multiples; the smaller ones are struck already, so we start at $2^2 = 4$: $4, 6, 8, 10, \ldots, 50$
- 3 is a prime – we strike out all its multiples; the smaller ones are struck already, so we start at $3^2 = 9$: $9, 12, 15, 18, \ldots, 48$
- 5 is a prime – we strike out all its multiples; the smaller ones are struck already, so we start at $5^2 = 25$: $25, 30, 35, 40, 45, 50$
- 7 is a prime – we strike out all its multiples; the smaller ones are struck already, so we start at $7^2 = 49$: 49
- The sieve of Eratosthenes: striking the multiples of the primes up to the square root of 50 is enough. The numbers left are the primes – 15 of them

1	**2**	**3**	4	**5**	6	**7**	8	9	10
11	12	**13**	14	15	16	**17**	18	**19**	20
21	22	**23**	24	25	26	27	28	**29**	30
31	32	33	34	35	36	**37**	38	39	40
41	42	**43**	44	45	46	**47**	48	49	50

Bézout's identity by the extended Euclidean algorithm.

Input: `bezout(240, 46)`

Result

$$\gcd(240, 46) = 2 = 240 \cdot (-9) + 46 \cdot$$

Steps

- The extended Euclidean algorithm – each remainder r is written as r = 240·u + 46·v; a new row is the row before last minus q times the last one

r	q	u	v
240		1	0
46	5	0	1
10	4	1	−5
6	1	−4	21
4	1	5	−26
2	2	−9	47
0			

- The last non-zero row of the table gives Bézout's identity: $\gcd(240, 46) = 2 = 240 \cdot (-9) + 46 \cdot 47$

A Diophantine equation by Bézout's identity.

Input: `3x + 5y = 1, x, y ∈ ℤ`

Result

$$x = 2 + 5k,\ y = -1 - 3k,\ k \in \mathbb{Z}$$

Steps

- We solve $3x + 5y = 1$ in the integers. It has a solution just when the greatest common divisor of 3 and 5 divides the right side.
- The extended Euclidean algorithm – each remainder r is written as r = 5·u + 3·v; a new row is the row before last minus q times the last one

r	q	u	v
5		1	0
3	1	0	1
2	1	1	−1
1	2	−1	2
0			

- The last non-zero row of the table gives Bézout's identity: $1 = 3 \cdot 2 + 5 \cdot (-1)$
- Bézout's identity multiplied by 1 gives one solution: $1 = 3 \cdot 2 + 5 \cdot (-1)$, so $x_0 = 2,\ y_0 = -1$
- All the solutions: add multiples of 5 to x and at the same time multiples of -3 to y (the left side doesn't change); shifted to the least non-negative $x_0 = 2,\ y_0 = -1$: $x = 2 + 5k,\ y = -1 - 3k,\ k \in \mathbb{Z}$
- Check: $3 \cdot 2 + 5 \cdot (-1) = 1$

A power modulo n by repeated squaring.

Input: `7^13 mod 13`

Result

$$7^{13} \mod 13 = 7$$

Steps

- Write the exponent as a sum of powers of 2:
  $13 = 2^3 + 2^2 + 1$
- Square repeatedly, taking the remainders modulo 13
  $7^1 \equiv 7$
  $7^2 \equiv 7^2 = 49 \equiv 10$
  $7^4 \equiv 10^2 = 100 \equiv 9$
  $7^8 \equiv 9^2 = 81 \equiv 3$
- Multiply the powers needed, taking the remainders modulo 13 as you go
  $7^{13} = 7^8 \cdot 7^4 \cdot 7^1$
  $\equiv 3 \cdot 9 \cdot 7$
  $3 \cdot 9 = 27 \equiv 1$
  $1 \cdot 7 = 7$

RSA – the keys, encryption and decryption.

Input: `rsa(p = 11, q = 13, e = 7, m = 9)`

Result

$$n = 143,\ e = 7,\ d = 103$$
$$c = 48$$
$$m = 9$$

Steps

- The modulus: $n = p \cdot q = 11 \cdot 13 = 143$
- Euler's totient: $\varphi(n) = (p-1)(q-1) = 10 \cdot 12 = 120$
- The Euclidean algorithm – divide the larger number by the smaller one with a remainder, then the divisor by the remainder, until the remainder is 0; the last non-zero remainder is the greatest common divisor

$$120 = 17 \cdot 7 + 1$$
$$7 = 7 \cdot 1 + 0$$
$$\Rightarrow \gcd(120, 7) = 1$$

- The public exponent e must be coprime to φ(n): $\gcd(7, 120) = 1$ – it is.
- The extended Euclidean algorithm – each remainder r is written as r = 120·u + 7·v; a new row is the row before last minus q times the last one

r	q	u	v
120		1	0
7	17	0	1
1	7	1	−17
0			

- The private exponent d is the inverse of e modulo φ(n): $d = e^{-1} \bmod \varphi(n) = 7^{-1} \bmod 120 = 103$
- The public key $(n, e) = (143, 7)$, the private key $(n, d) = (143, 103)$
- Encrypt the message m: $c = m^e \bmod n = 9^7 \bmod 143$
- Write the exponent as a sum of powers of 2: $7 = 2^2 + 2^1 + 1$
- Square repeatedly, taking the remainders modulo 143
  $9^1 \equiv 9$
  $9^2 \equiv 9^2 = 81$
  $9^4 \equiv 81^2 = 6561 \equiv 126$
- Multiply the powers needed, taking the remainders modulo 143 as you go
  $9^7 = 9^4 \cdot 9^2 \cdot 9^1$
  $\equiv 126 \cdot 81 \cdot 9$
  $126 \cdot 81 = 10206 \equiv 53$
  $53 \cdot 9 = 477 \equiv 48$
- Decrypt – the original message comes back: $m = c^d \bmod n = 48^{103} \bmod 143$
- Write the exponent as a sum of powers of 2: $103 = 2^6 + 2^5 + 2^2 + 2^1 + 1$
- Square repeatedly, taking the remainders modulo 143
  $48^1 \equiv 48$
  $48^2 \equiv 48^2 = 2304 \equiv 16$
  $48^4 \equiv 16^2 = 256 \equiv 113$
  $48^8 \equiv 113^2 = 12769 \equiv 42$
  $48^{16} \equiv 42^2 = 1764 \equiv 48$
  $48^{32} \equiv 48^2 = 2304 \equiv 16$
  $48^{64} \equiv 16^2 = 256 \equiv 113$
- Multiply the powers needed, taking the remainders modulo 143 as you go
  $48^{103} = 48^{64} \cdot 48^{32} \cdot 48^4 \cdot 48^2 \cdot 48^1$
  $\equiv 113 \cdot 16 \cdot 113 \cdot 16 \cdot 48$
  $113 \cdot 16 = 1808 \equiv 92$
  $92 \cdot 113 = 10396 \equiv 100$
  $100 \cdot 16 = 1600 \equiv 27$
  $27 \cdot 48 = 1296 \equiv 9$

Known limitations

- Prime factorization works for numbers up to 10^{18}; a larger prime is only confirmed by a test.
- RSA is a school example with small numbers (n = p·q up to about a trillion), not real encryption.

15. Angles in degrees, minutes and seconds

An angle alone is converted to all its forms: degrees, minutes and seconds (5.5° = 5°30′), decimal degrees, minutes, seconds and radians – 1° = 60′, 1′ = 60″, 180° = π rad. Just one form comes with “= ?′”, “= ?°”, “= ?″” or “= ? rad”. Radians to degrees: “π/6 rad”.

Angles are added and subtracted as at school – degrees, minutes and seconds separately; 60′ is carried as 1°, and when there are not enough minutes, one degree is broken into 60′. They are multiplied and divided by a number: 100° : 3 is divided step by step, the remainder of the degrees turned into minutes and that of the minutes into seconds.

Minutes and seconds are written with ′ and ″ (on the app's keyboard) or with an apostrophe and quotes: 25°30'15". Degrees with a function (sin 30°) belong to the chapter Trigonometric equations.

How to enter it

`25°30’15”, 25°30'15", 5,5°`	An angle – degrees, minutes, seconds, also decimal degrees.
`25°30’ + 18°45’, 3 · 25°30’, 100° : 3`	A sum, a difference, a multiple and a quotient by a number.
`5,5° = ?’, 30° = ? rad, π/6 rad`	One form of the conversion, radians to degrees.

Examples

Decimal degrees in all the forms.

Input: `5,5°`

Result

$$5^\circ 30'$$
$$= 330'$$
$$= 19800''$$
$$= \frac{11\pi}{360}\,\mathrm{rad} \approx 0.09599\,\mathrm{rad}$$

Steps

- The decimal part of the degree goes to minutes (1° = 60'), the decimal part of the minute to seconds (1' = 60"): $5.5^\circ = 5^\circ + 0.5 \cdot 60' = 5^\circ 30'$
- To minutes: the degrees times 60 (1° = 60') plus the minutes: $5^\circ 30' = 5 \cdot 60' + 30' = 330'$
- To seconds: the minutes times 60 (1' = 60"): $330' = 330 \cdot 60'' = 19800''$
- To radians: 180° = π rad, so the degrees times π/180 $5.5^\circ = 5.5 \cdot \frac{\pi}{180}\,\mathrm{rad} = \frac{11\pi}{360}\,\mathrm{rad} \approx 0.09599\,\mathrm{rad}$

A sum – 60' carried as 1°.

Input: `25°30′ + 18°45′`

Result

$$44^\circ 15'$$

Steps

- Add the degrees, minutes and seconds separately: $25^\circ 30' + 18^\circ 45' = 43^\circ 75'$
- Every 60" makes 1' and every 60' makes 1° – carried: $43^\circ 75' = 44^\circ 15'$

A difference – one degree broken into 60′.

Input: `25°15′ - 18°45′`

Result

$$6^\circ 30'$$

Steps

- There are fewer minutes (seconds) than subtracted – one degree (minute) is broken into 60′ (60″): $25^\circ 15' = 24^\circ 75'$
- Subtract the degrees, minutes and seconds separately: $24^\circ 75' - 18^\circ 45' = 6^\circ 30'$

Division by a number, the remainder in minutes.

Input: `100° : 3`

Result

$$33^\circ 20'$$

Steps

- Divide step by step: the degrees, the remainder to minutes added; the minutes, the remainder to seconds; the seconds

$$\begin{aligned} & 100^\circ : 3 = 33^\circ,\ 1^\circ = 60' \\ & 60' : 3 = 20' \\ \Rightarrow\ & 100^\circ : 3 = 33^\circ 20' \end{aligned}$$

Radians to degrees.

Input: `π/6 rad`

Result

$$\begin{aligned} & 30^\circ \\ & = 1800' \\ & = 108000'' \end{aligned}$$

Steps

- From radians to degrees: π rad = 180°, so times 180°/π: $\frac{\pi}{6}\,\mathrm{rad} = \frac{\pi}{6} \cdot \frac{180^\circ}{\pi} = 30^\circ$
- To minutes: the degrees times 60 (1° = 60′) plus the minutes: $30^\circ = 30 \cdot 60' = 1800'$
- To seconds: the minutes times 60 (1′ = 60″): $1800' = 1800 \cdot 60'' = 108000''$

Known limitations

- Equations with angles in minutes (x + 25°30′ = 90°) not yet.

16. Percentages, ratios and the rule of three

Percentages are entered briefly, without a word problem: a part “20 % of 480”, the base “15 % = 45”, the rate “96 of 480 = ? %”, an increase and a decrease “480 + 15 %”, “480 − 20 %” and the original value before a change “x − 20 % = 384”. The app works through 1 % as taught at school today (1 % = 480 : 100 = 4.8); the “Formula” chip switches to part = base · rate with the rate as a decimal (480 · 0.2).

A ratio “12 : 18” is reduced to 2 : 3, a whole is split in a ratio (“300 in the ratio 2 : 3 : 5” – the parts added, one part, the shares) and a proportion “3 : 4 = x : 20” is solved by the products of the outer and inner terms. A ratio of fractions (“2/3 : 4/9”) is first multiplied by the least common multiple of the denominators; inside a longer expression the colon divides.

The rule of three is written as at school – two rows with three dots (or …, →): “10 kg … 250 Kč, 12 kg … x”, the units may be left out. A direct proportion by default, the “Inverse” chip (or the word “inverse” after a comma) for an inverse one. Simple interest: interest(principal, rate %, years, tax %) – the tax is optional. Compound interest: compound interest(principal, rate %, years, tax %, frequency) – the tax and the frequency (annually, semiannually, quarterly, monthly, continuously) are optional; more tasks (savings, loans, interest for days) are in Applications, the Economics chapter.

How to enter it

`20 % of 480, 15 % = 45, 96 of 480 = ? %, 480 − 20 %, x − 20 % = 384`	Percentages – a part, the base, the rate, a change, the original value.
`12 : 18, 300 in the ratio 2 : 3 : 5, 3 : 4 = x : 20`	A ratio reduced, split in a ratio, a proportion.
`10 kg … 250 Kč, 12 kg … x`	The rule of three – a direct and an inverse proportion.
`interest(10000, 3 %, 2, 15 %)`	Simple interest with a tax.
`compound interest(10000, 3 %, 5, 15 %, monthly)`	Compound interest with a tax and the crediting frequency.

Examples

A part through 1 %.

Input: `20 % z 480`

Result

96

Steps

- First 1 % – the base (the whole, 100 %) divided by 100:
  $1\,\% = 480 : 100 = 4.8$
- The percentages asked for are so many times 1 %:
  $20\,\% = 20 \cdot 4.8 = 96$

A decrease by the formula.

Input: `480 - 20 %` Formula part = base · rate

Result

384

Steps

- Decreased value = base · (1 − rate), $20\,\% = 0.2$:
  $480 \cdot (1 - 0.2) = 480 \cdot 0.8 = 384$

The original price before a discount.

Input: `x - 20 % = 384`

Result

$x = 480$

Steps

- The original value is 100 %; after the decrease fewer percent are left – the known value:
  $100\,\% - 20\,\% = 80\,\%, \quad 80\,\% = 384$
- 1 % is the known value divided by its number of percent:
  $1\,\% = 384 : 80 = 4.8$
- The base is 100 %, a hundred times 1 %:
  $100\,\% = 100 \cdot 4.8 = 480$

A split in a ratio.

Input: `300 v poměru 2 : 3 : 5`

Result

$$60,\ 90,\ 150$$

Steps

- The whole is split into as many parts as the terms add up to: $2 + 3 + 5 = 10$
- One part is: $300 : 10 = 30$
- Each share has as many parts as its term says: $2 \cdot 30 = 60,\ 3 \cdot 30 = 90,\ 5 \cdot 30 = 150$; check $60 + 90 + 150 = 300$

A ratio of fractions.

Input: `2/3 : 4/9`

Result

$$3 : 2 = \frac{3}{2}$$

Steps

- We clear the fractions – every term of the ratio times the least common multiple of the denominators, 9
  $\frac{2}{3} : \frac{4}{9} = \left(\frac{2}{3} \cdot 9\right) : \left(\frac{4}{9} \cdot 9\right) = 6 : 4$
- The ratio reduced – all terms divided by their greatest common divisor 2: $6 : 4 = (6 : 2) : (4 : 2) = 3 : 2$

An inverse proportion.

Input: `4 dělníci … 6 dní, 3 dělníci … x` Inverse

Result

$$x = 8\,\text{dní}$$

Steps

- The rule of three. An inverse proportion – so many times more of one, so many times less of the other, the arrows opposite

  $$\begin{array}{ccc} 4\,\text{dělníci} & \ldots & 6\,\text{dní} \\ 3\,\text{dělníci} & \ldots & x \\ \uparrow & & \downarrow \end{array}$$

- By the arrows a proportion – the second ratio turned over: $\frac{x}{6} = \frac{4}{3} \Rightarrow x = 6 \cdot \frac{4}{3} = 8$

Simple interest with a 15 % tax.

Input: `úrok(10000, 3 %, 2, 15 %)`

Result

$$10000 + 510 = 10510$$

Steps

- Interest = principal · p/100 · time in years: $10000 \cdot \frac{3}{100} \cdot 2 = 600$
- The tax $15\,\%$ is paid off the interest, the net interest is left: $600 \cdot \frac{15}{100} = 90, \quad 600 - 90 = 510$
- At the end, the principal with the interest: $10000 + 510 = 10510$

Compound interest with a 15 % tax, credited monthly.

Input: `compound interest(10000, 3 %, 5, 15 %, monthly)`

Result

$$K_n \approx 11{,}358.31$$

Steps

- Given: $K_0 = 10{,}000, \quad p = 3\,\%, \quad n = 5, \quad d = 15\,\%$
- Converting the units: $p = 3\,\% = 0.03, \quad d = 15\,\% = 0.15$
- With simple interest the interest is always on the original deposit: K_n = K_0(1 + pn). With compound interest it's credited and earns interest too: K_n = K_0(1 + p/m)^(mn), m being the credits a year (annual, semiannual, quarterly, monthly). Continuous compounding is the limit m → ∞: K_n = K_0e^(pn). A tax on interest d lowers the rate to p(1 − d). The effective annual rate compares the frequencies. Discounting is the other way round – how much to deposit today. With inflation π the real value is K_n/(1 + π)^n. Mixed interest (how banks credit interest once a year) compounds the whole years and adds simple interest for the rest of the last one: K_n = K_0(1 + p)^k·(1 + p·f), n = k + f. The model's equation: $K_n = K_0\left(1 + \frac{p'}{m}\right)^{mn}$
- Interest credits a year: $m = 12$
- The rate after the tax on interest: $p' = p(1 - d) = 3\,\% \cdot (1 - 0.15) = 2.55\,\%$
- Effective annual rate: $p_{ef} = \left(1 + \frac{p'}{m}\right)^{m} - 1 \approx 2.58\,\%$
- Substituting: $K_n = 10{,}000 \cdot \left(1 + \frac{0.0255}{12}\right)^{60} \approx 11{,}358.31$
- Interest in total: $K_n - K_0 \approx 1{,}358.31$

Known limitations

- No word problems yet – the problem has to be written briefly; compound interest with regular deposits and more financial tasks are in the forms in Applications.

17. Unit conversions

A conversion is written as “5 m² = ? cm²” (or “5 m2 to cm2”). The app writes the conversion relation and multiplies by it (to a smaller unit) or divides (to a larger one); for decimal units it adds how many places the decimal point moves. For areas and volumes it explains why the number is a power: 1 m = 100 cm, so 1 m² = 100² cm² = 10 000 cm².

Length mm to km, area mm² to km² and are and hectare, volume mm³ to m³ and ml, cl, dl, l, hl (1 l = 1 dm³), mass mg to t, time s, min, h, d – also a compound “1 h 30 min” and the result in hours and minutes –, speed km/h and m/s (1 m/s = 3.6 km/h) and temperature °C, K, °F by their formulas.

How to enter it

`5 m² = ? cm², 3.5 l to m³, 1 h 30 min = ? min`	A value with its unit, “= ?” (or “to”) and the target unit.

Examples

An area – the number is a square.

Input: `5 m² = ? cm²`

Result

$$5\,\mathrm{m}^2 = 50\,000\,\mathrm{cm}^2$$

Steps

- For areas and volumes both (all three) dimensions are converted – the lengths' number is raised to the power
  $1\,\mathrm{m} = 100\,\mathrm{cm} \;\Rightarrow\; 1\,\mathrm{m}^2 = 100^2\,\mathrm{cm}^2 = 10\,000\,\mathrm{cm}^2$
- To a smaller unit – more of them, so multiply:
  $5\,\mathrm{m}^2 = 5 \cdot 10\,000\,\mathrm{cm}^2 = 50\,000\,\mathrm{cm}^2$
- Multiplying by a one with zeros (4 here) moves the decimal point right by as many places as there are zeros; missing places become zeros.

Litres to cubic metres.

Input: `3,5 l = ? m³`

Result

$$3.5\,\mathrm{l} = 0.0035\,\mathrm{m}^3$$

Steps

- The conversion relation: $1\,\mathrm{m}^3 = 1000\,\mathrm{l}$
- To a larger unit – fewer of them, so divide:
  $3.5\,\mathrm{l} = 3.5 : 1000\,\mathrm{m}^3 = 0.0035\,\mathrm{m}^3$
- Dividing by a one with zeros (3 here) moves the decimal point left by as many places as there are zeros; missing places become zeros.

Minutes to hours, also in hours and minutes.

Input: `150 min = ? h`

Result

$$150\,\mathrm{min} = 2.5\,\mathrm{h}$$
$$= 2\,\mathrm{h}\ 30\,\mathrm{min}$$

Steps

- The conversion relation: $1\,\mathrm{h} = 60\,\mathrm{min}$
- To a larger unit – fewer of them, so divide:
  $150\,\mathrm{min} = 150 : 60\,\mathrm{h} = 2.5\,\mathrm{h}$
- In hours and minutes: $2\,\mathrm{h}\ 30\,\mathrm{min}$

Speed – divided by 3.6.

Input: `72 km/h = ? m/s`

Result

$$72\,\mathrm{km/h} = 20\,\mathrm{m/s}$$

Steps

- The conversion relation: $1\,\mathrm{m/s} = 3.6\,\mathrm{km/h}$
- To a larger unit – fewer of them, so divide:
  $72\,\mathrm{km/h} = 72 : 3.6\,\mathrm{m/s} = 20\,\mathrm{m/s}$

A temperature by its formula.

Input: `20 °C = ? K`

Result

$$20\,^{\circ}\mathrm{C} = 293.15\,\mathrm{K}$$

Steps

- Temperatures aren't converted by a multiple but by a formula (the scales have different zeros):
  $T = t + 273.15 = 20 + 273.15 = 293.15\,\mathrm{K}$

18. Logic and sets

A proposition with the connectives ¬, ∧, ∨, ⇒, ⇔ is evaluated by a truth table – a column for every connective, ones for true – and the app says whether it is a tautology, a contradiction or merely satisfiable; for a satisfiable one it adds the disjunctive and conjunctive normal forms and a simplified form.

With the "Negation" chip the app negates the proposition step by step by De Morgan's laws and the negations of an implication and an equivalence. A proposition with quantifiers (∀x ∈ ℝ: x² ≥ 0) is negated directly – the universal quantifier becomes existential, an inequality the opposite one. For an implication it also offers the contrapositive and the converse.

Sets given by their elements ({1, 2, 3}) or as intervals ((1; 5), ⟨3; 7)) and their union, intersection, difference, complement in the universe U and symmetric difference are computed operation by operation. Listed sets get a Venn diagram with the elements in its regions, intervals the result on the number line.

Without elements (A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C)) the app checks an equality of set expressions by a membership table and Venn diagrams of both sides; an expression alone, (A \ B) ∪ (B \ C), is shaded in a Venn diagram. It also decides whether 3 ∈ A or A ⊆ B holds.

How to enter it

`¬p, p ∧ q, p ∨ q, p ⇒ q, p ⇔ q`	The connectives from the app's keyboard (the ∧∪ tab) or from a photo (\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow).
`p => q, p <=> q`	An implication and an equivalence as arrows from an ordinary keyboard. &, | and ! are not read as connectives – they are the absolute value and the factorial.
`∀x ∈ ℝ: x^2 ≥ 0, ∃x > 0: x^2 < x`	Quantifiers with a domain (∈ ℝ) or a condition (> 0), the proposition after a colon.
`negace(p ∧ q), obmena(p ⇒ q), obracena(p ⇒ q)`	The task as a word – the negation, the contrapositive and the converse (also negace, obmena, obracena).
`A = {1, 2, 3}, B = (1; 5), U = {1, 2, ..., 10}`	A set by its elements (separated by commas, by semicolons with a decimal comma), an interval, the

	universe U; each definition on its own line or after a comma.
`A ∪ B, A ∩ B, A \ B, A', A △ B`	The union, intersection, difference (also \), complement A', symmetric difference; the intersection goes before the others.
`3 ∈ A, A ⊆ B`	An element of a set and a subset – true or false.

Examples

A tautology – an implication is equivalent to its contrapositive.

Input: `(p ⇒ q) ⇔ (¬q ⇒ ¬p)`

Result

The proposition is a tautology – true for every valuation.

Steps

- The truth table (1 – true, 0 – false), a column for every connective

p	q	$p \Rightarrow q$	$\neg q$	$\neg p$	$\neg q \Rightarrow \neg p$	$(p \Rightarrow q) \Leftrightarrow (\neg q$
1	1	1	0	0	1	1
1	0	0	1	0	0	1
0	1	1	0	1	1	1
0	0	1	1	1	1	1

- The last column has only ones – the proposition is true for every valuation, a tautology.

A satisfiable proposition with its normal forms.

Input: `(p ∨ q) ⇒ r`

Result

The proposition is satisfiable – true for some valuations only. A simplified equivalent form:
$(\neg p \wedge \neg q) \vee r$

Steps

- The truth table (1 – true, 0 – false), a column for every connective

p	q	r	$p \vee q$	$(p \vee q) \Rightarrow r$
1	1	1	1	1
1	1	0	1	0
1	0	1	1	1
1	0	0	1	0
0	1	1	1	1
0	1	0	1	0
0	0	1	0	1
0	0	0	0	1

- The last column has ones and zeros – the proposition is satisfiable but not a tautology.
- A simplified form: $(\neg p \wedge \neg q) \vee r$; the full disjunctive normal form – the disjunction of the rows where it is true
  $(p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge$
- The conjunctive normal form – from the rows where it is false
  $(\neg p \vee \neg q \vee r) \wedge (\neg p \vee q \vee r) \wedge (p \vee \neg q \vee r)$

A negation by De Morgan's laws step by step.

Input: `p ∧ (q ⇒ r)` Negation

Result

$\neg p \vee (q \wedge \neg r)$

Steps

- The negation: $\neg(p \wedge (q \Rightarrow r))$
- De Morgan's law, $\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$:
  $\neg p \vee \neg(q \Rightarrow r)$
- The negation of an implication, $\neg(A \Rightarrow B) \Leftrightarrow A \wedge \neg B$:
  $\neg p \vee (q \wedge \neg r)$

The negation of a quantified proposition.

Input: `∀x ∈ ℝ: x^2 + 1 > 0`

Result

$\exists x \in \mathbb{R}\colon\ x^2 + 1 \leq 0$

Steps

- The negation: $\neg\left(\forall x \in \mathbb{R}\colon\ x^2 + 1 > 0\right)$
- Negating "for all … holds" gives "there is one for which it does not hold": $\exists x \in \mathbb{R}\colon\ \neg\left(x^2 + 1 > 0\right)$
- The negation of an equality or inequality is the opposite relation ($<$ to $\geq$, $=$ to $\neq$): $\exists x \in \mathbb{R}\colon\ x^2 + 1 \leq 0$

The contrapositive.

Input: `p ⇒ ¬q` Contrapositive

Result

$q \Rightarrow \neg p$

Steps

- The contrapositive of $A \Rightarrow B$ is $\neg B \Rightarrow \neg A$ – equivalent to it: $\neg\neg q \Rightarrow \neg p$, the negations simplified $q \Rightarrow \neg p$

The complement of a union with a Venn diagram.

Input: `A = {1, 2, 3, 4}, B = {3, 4, 5}, U = {1, 2, ..., 6}, (A ∪ B)'`

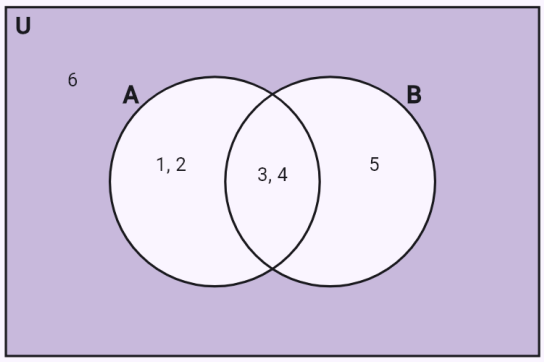
Result

$(A \cup B)' = \{6\}$

Steps

- Given: $A = \{1, 2, 3, 4\}$
- Given: $B = \{3, 4, 5\}$
- Given: $U = \{1, 2, 3, 4, 5, 6\}$
- The union – the elements in at least one of the sets: $A \cup B = \{1, 2, 3, 4, 5\}$
- The complement – the elements of the universe not in the set: $(A \cup B)' = \{6\}$

Graph



The intersection of intervals on the number line.

Input: `A = (1; 5), B = ⟨3; 7), A ∩ B`

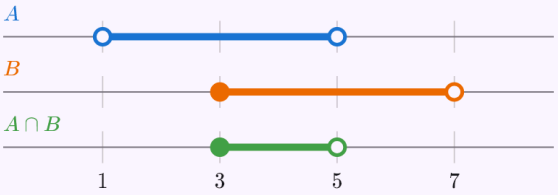
Result

$$A \cap B = [3,\ 5)$$

Steps

- Given: $A = (1,\ 5)$
- Given: $B = [3,\ 7)$
- The intersection – the elements in both sets: $A \cap B = [3,\ 5)$

Graph



Each given set has its own row on the axis, the result is in green. A filled dot belongs to the set, a hollow one does not.

The distributive law by a membership table.

Input: `A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C)`

Result

The equality holds – both sides are the same set.

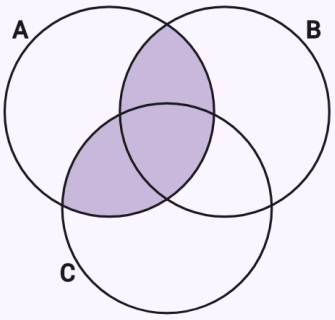
Steps

- The membership table – for every way an element can belong to the sets (1 – it belongs), both sides compared

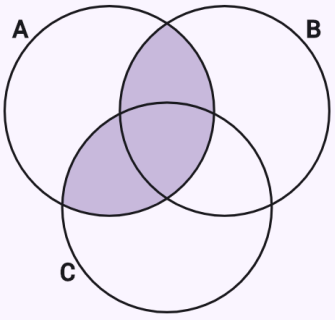
A	B	C	$A \cap (B \cup C)$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1
1	1	0	1	1
1	0	1	1	1
1	0	0	0	0
0	1	1	0	0
0	1	0	0	0
0	0	1	0	0
0	0	0	0	0

Graph

$A \cap (B \cup C)$



$(A \cap B) \cup (A \cap C)$



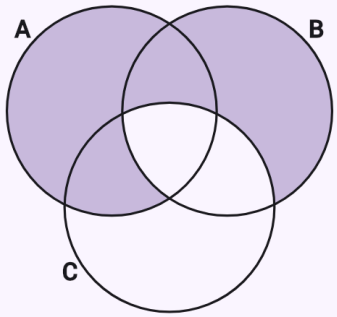
A set expression in a Venn diagram.

Input: `(A \ B) ∪ (B \ C)`

Result

$$(A \setminus B) \cup (B \setminus C)$$

Graph



Known limitations

- A truth table for 5 variables at most (32 rows), a Venn diagram for 4 sets at most.
- Sets given by a property ({x ∈ ℝ: x² < 4}) are not read yet – write them as intervals.

19. Complex numbers

Write a complex number with the imaginary unit i: 1 - i√3 (or 1 - i*sqrt(3)), in the trigonometric form 2(cos(pi/3) + i sin(pi/3)) or cis(pi/3), in the exponential form 3e^(i pi/4); angles may be in degrees too (cos 60° + i sin 60°). Numbers can be named, each on its own line (z_1 = 1 + i), with an expression in their names on the last line (z_1/z_2, z^8).

For a single number the app computes the modulus $|z| = \sqrt{(a^2 + b^2)}$ and the argument φ – from the table values of cosine and sine, otherwise by the acute angle from $\tan \alpha = |b/a|$ and the quadrant (exactly with arctan and approximately in radians and degrees) – and writes the trigonometric and exponential forms. A number in the trigonometric form is converted to the algebraic one. The argument lies in $\langle 0; 2\pi)$; the chip “Principal argument” switches it to $(-\pi; \pi\rangle$. The graph shows the number in the complex plane as a vector with the argument’s arc.

A power such as (1 + i)^10 is computed by de Moivre’s theorem with the argument reduced by multiples of 2π; a product and a quotient by the moduli and arguments. The chips choose “Trigonometric form” (for a power “De Moivre’s theorem”) or “Algebraic form”; dividing algebraic numbers by the conjugate is in the chapter Simplifying expressions. A root root(3, 8i) and a binomial equation $z^3 = 8i$ give all n roots by $z_k = \sqrt[n]{|a|}(\cos((\varphi + 2k\pi)/n) + i \sin((\varphi + 2k\pi)/n))$; in the graph they are the vertices of a regular n-gon. An equation in the unknown z is solved over the complex numbers straight away; the chip “Solve in:” switches it to $\mathbb{R}$. A quadratic equation with complex coefficients ($z^2 + i = 2z$) is solved by completing the square, $(z - 1)^2 = 1 - i$, the complex square root found as a + bi by comparing the real and imaginary parts.

An equation with the conjugate $\bar{z}$ (the $\bar{z}$ key types conj(z), a photo gives \bar{z}), with |z|, Re z and Im z is solved by substituting z = x + yi: the app writes both sides as a + bi, compares the real and the imaginary parts and solves the resulting system of two equations in x and y – one equation is solved for a variable and substituted into the other, and if it factors (y(2x + 1) = 0), the solution splits into cases. The solutions are shown as points of the complex plane. If the imaginary parts agree by themselves ($|z - 1| = 2$, $\mathrm{Re}\, z \geq 1$, $|z - i| < |z + 1|$), the result is a set of points: the app squares the moduli, completes the squares and describes the circle (disc, outside of a circle), line, half-plane or perpendicular bisector. Several conditions, each on its own line or separated by commas ($|z| \leq 2$, $\mathrm{Re}\, z > 0$), and a double inequality ($1 < |z - 1| \leq 2$) are drawn as their intersection – a shaded region; a solid boundary belongs to the set, a dashed one does not.

How to enter it

`1 - i*sqrt(3)`	The algebraic form with the imaginary unit i.

`2(cos(pi/3) + i sin(pi/3)), cis(pi/3)`	The trigonometric form, cis φ for short.
`3e^(i pi/4)`	The exponential form.
`z_1 = 1 + i ↵ z_2 = 2 - 2i ↵ z_1/z_2`	Named numbers, each on its own line, and an expression in them.
`root(3, 8i), z^3 = 8i`	All n-th roots, a binomial equation.
`z + 2conj(z) = 3 - i, Re(z), Im(z)`	The conjugate, the real and the imaginary part.
`|z - i| < 3, |z| <= 2 ↵ Re(z) > 0`	A set of points in the complex plane, several conditions line by line.

Examples

The modulus, the argument from table values, the trigonometric and exponential forms.

Input: `1 - i*sqrt(3)`

Result

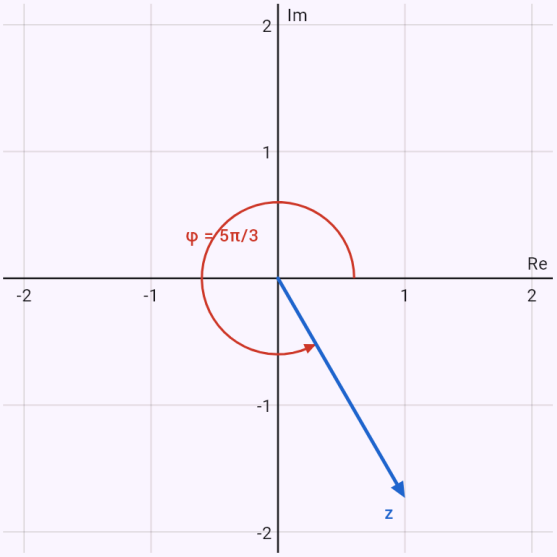
$$|z| = 2,\ \varphi = \frac{5\pi}{3}$$

$$z = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right) = 2\,e^{i\frac{5\pi}{3}}$$

Steps

- The modulus |z| = √(a² + b²):
$$\left|1 - \sqrt{3}i\right| = \sqrt{1^2 + \left(-\sqrt{3}\right)^2} = \sqrt{4} = 2$$
- The argument φ from cos φ = a/|z| and sin φ = b/|z| – values from the table
$$\cos\varphi = \frac{1}{2},\ \sin\varphi = \frac{-\sqrt{3}}{2} = -\frac{\sqrt{3}}{2} \Rightarrow \varphi = \frac{5\pi}{3}$$
- The trigonometric (polar) form z = |z|(cos φ + i sin φ):
$$z = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$$
- The exponential form $z = |z|\,e^{i\varphi}$: $z = 2\,e^{i\frac{5\pi}{3}}$

Graph



The complex plane – the given numbers blue, the result red, the argument φ as an arc from the positive real axis.

An argument not in the table – the acute angle and the quadrant.

Input: `-3 + 4i`

Result

$$|z| = 5,\ \varphi = \pi - \arctan\frac{4}{3}$$

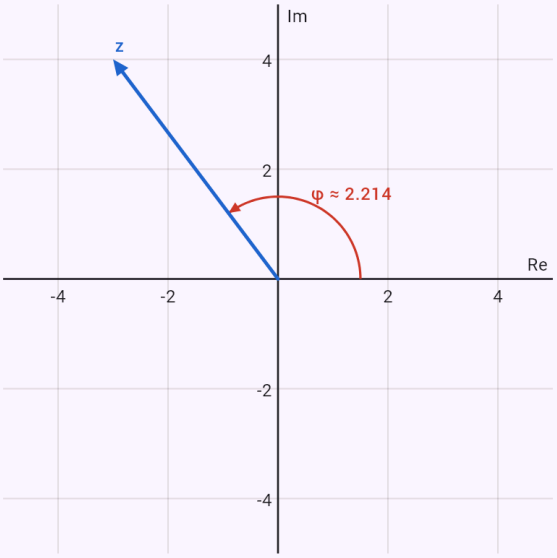
$$\varphi \approx 2.214\ (126^\circ 52')$$

$$z = 5(\cos\varphi + i\sin\varphi) = 5\,e^{i\varphi}$$

Steps

- The modulus |z| = √(a² + b²):
  $|-3+4i| = \sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5$
- The acute angle α from tan α = |b/a|. The number lies in quadrant 2, so the argument is
  $\tan\alpha = \left|\frac{4}{-3}\right| = \frac{4}{3},\ \alpha = \arctan\frac{4}{3} \approx 0.927\ (53^\circ 8')$
  $\varphi = \pi - \alpha = \pi - \arctan\frac{4}{3} \approx 2.214\ (126^\circ 52')$
- The trigonometric (polar) form z = |z|(cos φ + i sin φ):
  $z = 5(\cos\varphi + i\sin\varphi),\ \varphi = \pi - \arctan\frac{4}{3}$
- The exponential form $z = |z|\,e^{i\varphi}$: $z = 5\,e^{i\varphi}$

Graph



The complex plane – the given numbers blue, the result red, the argument φ as an arc from the positive real axis.

From the trigonometric form to the algebraic one.

Input: `2(cos(pi/3) + i sin(pi/3))`

Result

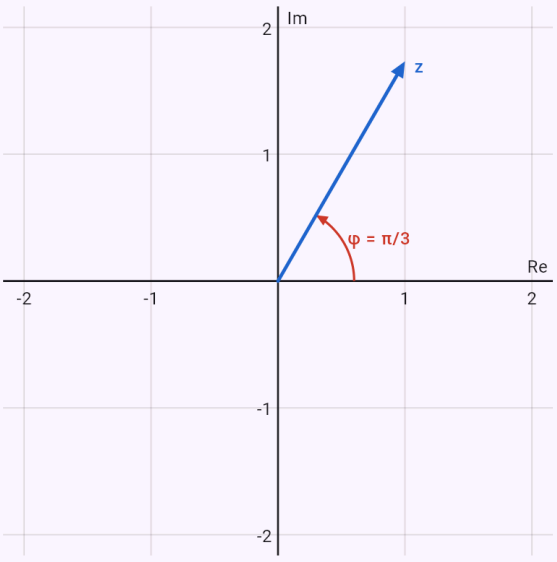
$$z = 1 + \sqrt{3}i$$

$$z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 2\,e^{i\frac{\pi}{3}}$$

Steps

- The number is given in the trigonometric (exponential) form – its modulus and argument read off
  $z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right):\ |z| = 2,\ \varphi = \frac{\pi}{3}$
- The algebraic form – the values of cosine and sine substituted, multiplied out
  $z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 1 + \sqrt{3}i$
- The exponential form $z = |z|\,e^{i\varphi}$: $z = 2\,e^{i\frac{\pi}{3}}$

Graph



The complex plane – the given numbers blue, the result red, the argument φ as an arc from the positive real axis.

De Moivre's theorem.

Input: `(1 + i)^10` De Moivre's theorem

Result

$$(1+i)^{10} = 32\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$(1+i)^{10} = 32i$$

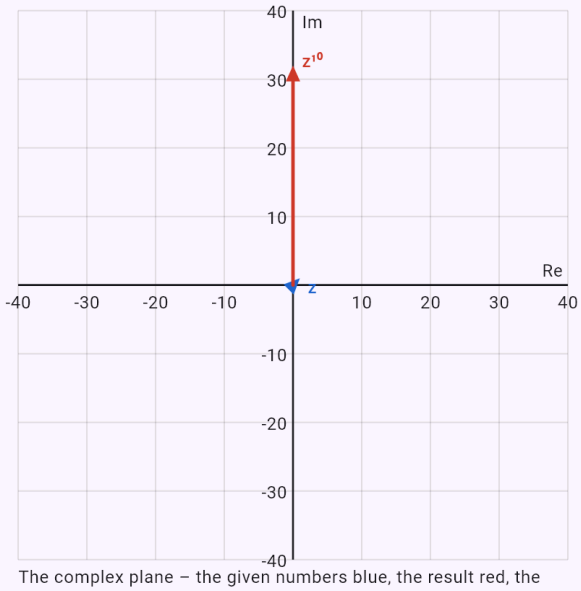
Steps

- The modulus |z| = √(a² + b²): $|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$
- The argument φ from cos φ = a/|z| and sin φ = b/|z| – values from the table

  $\cos\varphi = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2},\ \sin\varphi = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \Rightarrow \varphi = \frac{\pi}{4}$
- De Moivre's theorem $z^n = |z|^n(\cos n\varphi + i \sin n\varphi)$

  $(1+i)^{10} = \left(\sqrt{2}\right)^{10}\left(\cos\left(10\cdot\frac{\pi}{4}\right) + i\sin\left(10\cdot\frac{\pi}{4}\right)\right) =$
- The argument brought into $\langle 0; 2\pi)$ by a multiple of the full angle: $\frac{5\pi}{2} = \frac{\pi}{2} + 2\pi$
- The algebraic form – the values of cosine and sine substituted, multiplied out:
  $32\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 32(0+i) = 32i$

Graph



The complex plane – the given numbers blue, the result red, the argument φ as an arc from the positive real axis.

A quotient in the trigonometric form.

Input: `z_1 = 1 + i`

`z_2 = 2 - 2i`

z_1/z_2 Trigonometric form

Result

$$\frac{z_1}{z_2} = \frac{1}{2}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$\frac{z_1}{z_2} = \frac{1}{2}i$$

Steps

- The modulus |z| = √(a² + b²): $|z_1| = \sqrt{1^2 + 1^2} = \sqrt{2}$
- The argument φ from cos φ = a/|z| and sin φ = b/|z| – values from the table

  $\cos\varphi_1 = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2},\ \sin\varphi_1 = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \Rightarrow \varphi_1 = \frac{\pi}{4}$
- The modulus |z| = √(a² + b²):

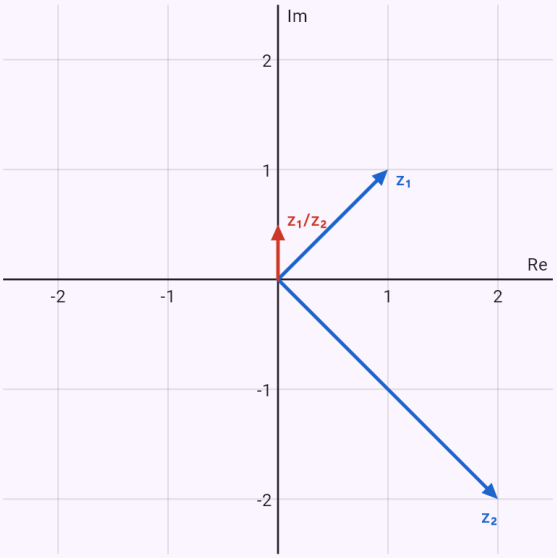
  $|z_2| = \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$
- The argument φ from cos φ = a/|z| and sin φ = b/|z| – values from the table

  $\cos\varphi_2 = \frac{2}{2\sqrt{2}} = \frac{\sqrt{2}}{2},\ \sin\varphi_2 = \frac{-2}{2\sqrt{2}} = -\frac{\sqrt{2}}{2} \Rightarrow \varphi_2 =$
- Dividing, the moduli divide and the arguments subtract

  $\frac{z_1}{z_2} = \frac{\sqrt{2}}{2\sqrt{2}}\left(\cos\left(\frac{\pi}{4} - \frac{7\pi}{4}\right) + i\sin\left(\frac{\pi}{4} - \frac{7\pi}{4}\right)\right) = \frac{1}{2}($
- The argument brought into $\langle 0; 2\pi)$ by a multiple of the full angle: $-\frac{3\pi}{2} = \frac{\pi}{2} - 2\pi$
- The algebraic form – the values of cosine and sine substituted, multiplied out

$$\frac{1}{2}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = \frac{1}{2}(0 + i) = \frac{1}{2}i$$

Graph



The complex plane – the given numbers blue, the result red, the argument φ as an arc from the positive real axis.

A binomial equation – three roots at the vertices of a triangle.

Input: `z^3 = 8i`

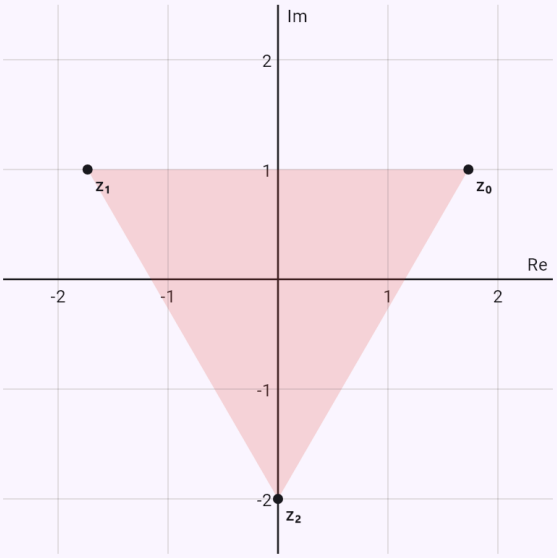
Result

$$z_0 = \sqrt{3} + i$$
$$z_1 = -\sqrt{3} + i$$
$$z_2 = -2i$$

Steps

- A binomial equation $z^n = a$ – its roots are all the n-th roots of a: $z^3 = 8i$
- The modulus $|z| = \sqrt{a^2 + b^2}$:
  $|8i| = \sqrt{0^2 + 8^2} = \sqrt{64} = 8$
- The number lies on an axis – its argument straight away:
  $\varphi = \frac{\pi}{2}$
- All 3 roots by the formula $z_k = \sqrt[n]{|a|}\,(\cos((\varphi + 2k\pi)/n) + i \sin((\varphi + 2k\pi)/n))$
  $\sqrt[3]{8} = 2,\ z_k = 2\left(\cos\frac{\frac{\pi}{2} + 2k\pi}{3} + i\sin\frac{\frac{\pi}{2} + 2k\pi}{3}\right),\ k =$
- For k = 0: $z_0 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \sqrt{3} + i$
- For k = 1: $z_1 = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = -\sqrt{3} + i$
- For k = 2: $z_2 = 2\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = -2i$
- In the complex plane the roots are the vertices of a regular 3-gon inscribed in the circle about 0 of radius 2

Graph



The complex plane – the given numbers blue, the result red, the argument φ as an arc from the positive real axis.

An equation with the conjugate – the real and imaginary parts compared.

Input: `z + 2conj(z) = 3 - i`

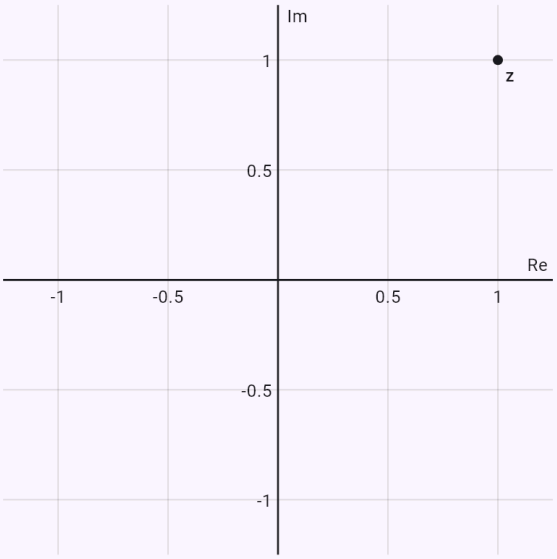
Result

$$z = 1 + i$$

Steps

- Substitute z = x + yi with x and y real: $z = x + yi,\ \bar{z} = x - yi$
- Write both sides in the form a + bi: $3x - yi = 3 - i$
- Two complex numbers are equal when their real parts and their imaginary parts are equal: $3x = 3$ and $-y = -1$
- Substitute $y = 1$ into the other equation: $3x = 3$
- Move everything to one side: $3x - 3 = 0$
- Keep the terms with the unknown on the left, move the rest to the right: $3x = 3$
- Divide by the coefficient 3: $x = 1$

Graph



The complex (Gauss) plane – the solutions as points: the real part on the horizontal axis, the imaginary part on the vertical one.

The equation z² = z̄ – factoring and cases, four points.

Input: `z^2 = conj(z)`

Result

$$z_1 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$
$$z_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$
$$z_3 = 0$$
$$z_4 = 1$$

Solve by a numerical method:

Σ Newton (tangents) | Σ Bisection | Σ Secant | Σ Regula falsi | Σ Fixed point

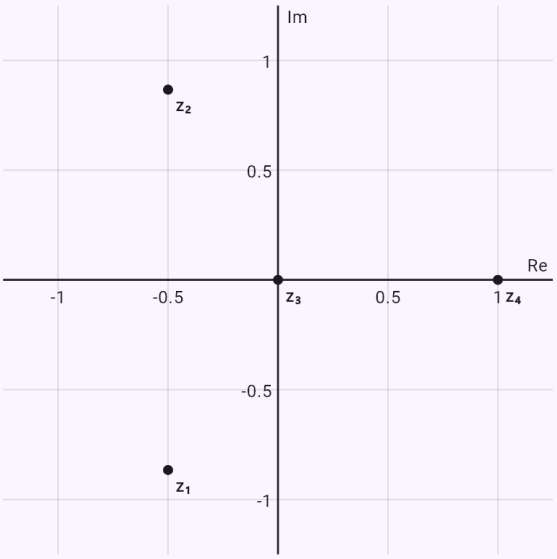
Steps

- Substitute z = x + yi with x and y real: $z = x + yi,\ \bar{z} = x - yi$
- Write both sides in the form a + bi: $x^2 - y^2 + 2xyi = x - yi$
- Two complex numbers are equal when their real parts and their imaginary parts are equal: $x^2 - y^2 = x$ and $2xy = -y$
- Factor the equation – a product is zero when one of its factors is zero

$$2xy + y = 0 \ \Rightarrow\ y(2x+1) = 0 \ \Rightarrow\ x = -\frac{1}{2} \ \vee\ y = 0$$

- Substitute $x = -\frac{1}{2}$ into the other equation: $\frac{1}{4} - y^2 = -\frac{1}{2}$
- Move everything to one side: $\frac{3}{4} - y^2 = 0$
- Compute the discriminant: $D = b^2 - 4ac = 0^2 - 4 \cdot (-1) \cdot \left(\frac{3}{4}\right) = 3$
- Substitute into the quadratic formula: $y_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{\pm\sqrt{3}}{-2}$
- Substitute $y = 0$ into the other equation: $x^2 = x$
- Move everything to one side: $x^2 - x = 0$
- Compute the discriminant: $D = b^2 - 4ac = (-1)^2 - 4 \cdot 1 \cdot 0 = 1$
- Substitute into the quadratic formula: $x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{1 \pm 1}{2}$

Graph



The complex (Gauss) plane – the solutions as points: the real part on the horizontal axis, the imaginary part on the vertical one.

The intersection of a disc and a half-plane – a shaded region, the dashed boundary not included.

Input: `|z| <= 2`

`Re(z) > 0`

Result

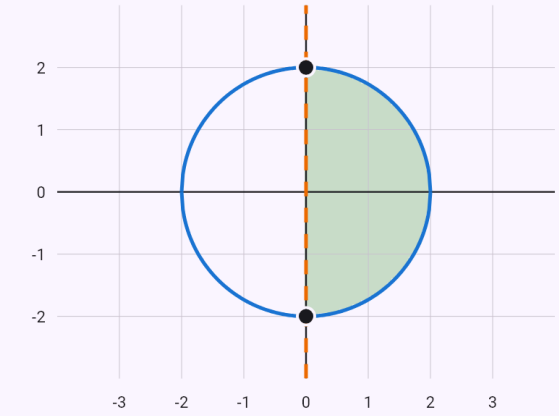
$$x^2 + y^2 \leq 4$$
$$x > 0$$

Steps

- Substitute z = x + yi with x and y real: $z = x + yi,\ |z| = \sqrt{x^2 + y^2},\ \operatorname{Re} z = x$
- Condition 1 for x and y: $\sqrt{x^2 + y^2} \leq 2$
- Both sides are non-negative, so square them: $x^2 + y^2 \leq 4$
- It is the disc with centre 0 and radius 2, the boundary circle included.
- Condition 2 for x and y: $x > 0$
- It is the half-plane to the right of the line $x = 0$, the boundary line not included.
- Inequality 1: $x^2 + y^2 \leq 4$
- The boundary is the curve $x^2 + y^2 = 4$ – drawn solid, its points belong to the solution.
- Substitute the point $[0;\ 0]$: $0 \leq 4$ holds – the solution is the part of the plane containing the point.
- Inequality 2: $x > 0$
- The boundary is the curve $x = 0$ – drawn dashed, its points don't belong to the solution.
- Substitute the point $[1;\ 0]$: $1 > 0$ holds – the solution is the part of the plane containing the point.
- The system's solution is the common part of these regions – shaded in the graph.

- The region's corners (where the boundaries meet): $[0;\ -2],\ [0;\ 2]$

Graph



$|z| \leq 2 \ \Leftrightarrow\ x^2 + y^2 \leq 4$
$\operatorname{Re}(z) > 0 \ \Leftrightarrow\ x > 0$

The complex (Gauss) plane – x = Re z on the horizontal axis, y = Im z on the vertical one. The set of points is the shaded region. A solid boundary belongs to it, a dashed one does not.

The points equally far from i and from −1 – a perpendicular bisector.

Input: `|z - i| = |z + 1|`

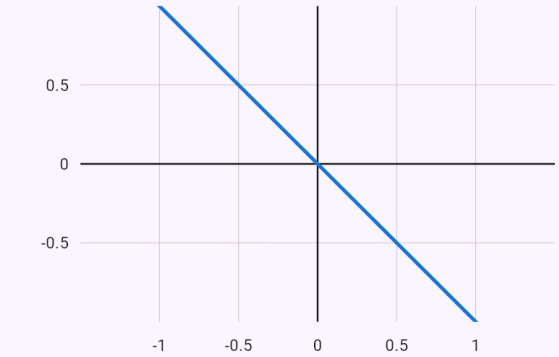
Result

$$y = -x$$

Steps

- Substitute z = x + yi with x and y real: $z = x + yi$
- The condition for x and y: $\sqrt{x^2 + (y-1)^2} = \sqrt{(x+1)^2 + y^2}$
- Both sides are non-negative, so square them: $x^2 + (y-1)^2 = (x+1)^2 + y^2$
- These are the points equally far from i and -1 – the perpendicular bisector of the segment between i and -1 : $y = -x$

Graph



$|z - i| = |z + 1|$

The complex (Gauss) plane – x = Re z on the horizontal axis, y = Im z on the vertical one. The set of points is the line drawn.

Known limitations

- Roots whose argument is not a table value are given in the trigonometric form and approximately in the algebraic one.
- A set of points is described when it is a circle (disc), a line (half-plane) or a perpendicular bisector; other conics (Im $z^2 = 2$) are only drawn and a sum of moduli ($|z - 1| + |z + 1| = 4$, an ellipse) is not supported yet.
- A system in x and y that cannot be solved by expressing one unknown is solved without steps.

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