Mathtrace – user guide

VI. Applications: physics and engineering

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42. Applications of differential equations

The applications are in the "Applications" section of the home screen – tiles of the fields you choose in the settings (the gear, "My fields") and single applications pinned with the pin in the list or in the form; "All applications" shows them all by fields (an application can belong to several fields, an RC circuit to physics and electrical engineering, say). The differential equations use a form with 23 models: radiocarbon dating, decay and half-life, exponential growth, a decay chain, Newton's cooling (also the time of death from two measurements), chemical kinetics of the 0th–2nd order, a drug in the blood (one dose, repeated doses, an infusion), blood alcohol, savings and loans, RC, RL and RLC circuits, a mass on a spring, a fall with air resistance, a draining tank, mixing in a tank, logistic and Gompertz growth, a harvested population, the SIR epidemic, predator and prey, two competing species and a chemostat.

You choose the model, its variant and what you are looking for; the constants (^{14}C's half-life, the gravitational acceleration, …) are prefilled and can be changed. Every quantity and the result have a unit to choose – the app converts the values (the conversion is a step). The "?" by the model explains where its equation comes from.

The steps start by setting the model up (why this equation), go on solving the equation (separation of variables, an integrating factor, the characteristic equation), substitute the conditions, express the sought quantity and end by substituting the numbers. Systems (SIR, predator and prey, competition, a chemostat) are solved numerically by the Runge–Kutta method, their equilibria and stability exactly. The graph shows the quantities in time with the answer marked, for systems also the phase plane.

Examples

The age of a find with 35 % of ¹⁴C left.

Input: `model(radiocarbon; find=t[yr]; T12=5730[yr]; p=35[pct]; input=ratio)`

Result

$$t \approx 8700\,\text{yr}$$

Steps

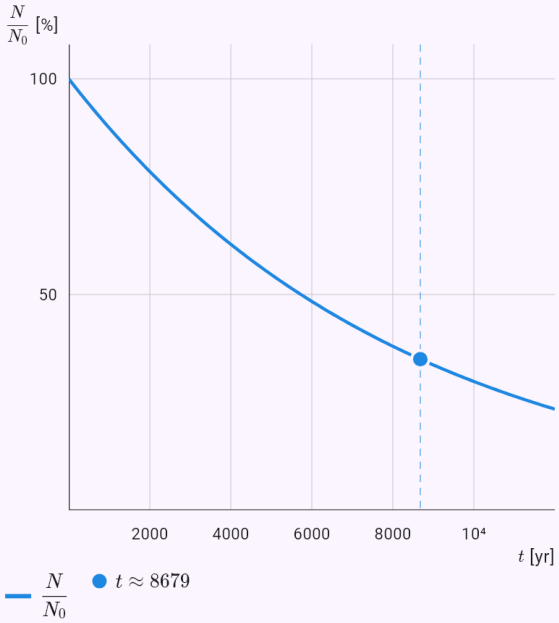
- Find t; given: $T_{1/2} = 5730\,\text{yr}, \quad \frac{N}{N_0} = 35\,\%$
- Converting the units: $\frac{N}{N_0} = 35\,\% = 0.35$
- Living organisms take in carbon from the air, so they contain the same share of the radioactive isotope ¹⁴C as the atmosphere. After death the intake stops and the ¹⁴C decays. The number of decays per unit of time is proportional to the number of nuclei left: dN/dt = −λN, λ being the decay constant. The solution is N = N₀·e^(−λt), and half of it is gone after the half-life T½ = 5730 years, so λ = ln 2 / T½. From the share of ¹⁴C left (or from the measured activity against a living sample's) we get the age. The model's equation: $\frac{dN}{dt} = -\lambda N$
- Separating the variables: $\frac{dN}{N} = -\lambda\,dt$
- Integrating both sides: $\ln|N| = -\lambda t + C$
- Exponentiating – the constant ±e^C is called C again: $N = Ce^{-\lambda t}$
- Substituting the initial condition $N(0) = N_0$: $N_0 = C$
- The solution of the initial condition (the particular solution): $N = N_0 e^{-\lambda t}$
- In a half-life the amount halves – the decay constant

  $$N(T_{1/2}) = \frac{N_0}{2} \Rightarrow \lambda = \frac{\ln 2}{T_{1/2}} = \frac{\ln 2}{5730} \approx 1.21 \cdot 10^{-4}\,\text{yr}^{-1}$$

- Expressing the sought quantity: $t = -\frac{1}{\lambda}\ln\frac{N}{N_0} = \frac{T_{1/2}}{\ln 2}\ln\frac{N_0}{N}$
- Substituting: $t = \frac{5730}{\ln 2}\ln\frac{1}{0.35} \approx 8679\,\text{yr}$

- Laboratories give the "radiocarbon age" with Libby's half-life of 5568 years and convert it by a calibration curve (the atmosphere's ¹⁴C hasn't always been the same) – the true age may therefore differ somewhat.

Graph



The model's quantities in time, computed from the solution; the answer is marked.

The time of death: the body was 30 °C when found, 28 °C an hour later, the room 20 °C.

Input: `model(cooling; find=t[h]; T0=30[degC]; Ta=20[degC]; t1=1[h]; T1=28[degC];`

`Tt=37[degC]; k=measured)`

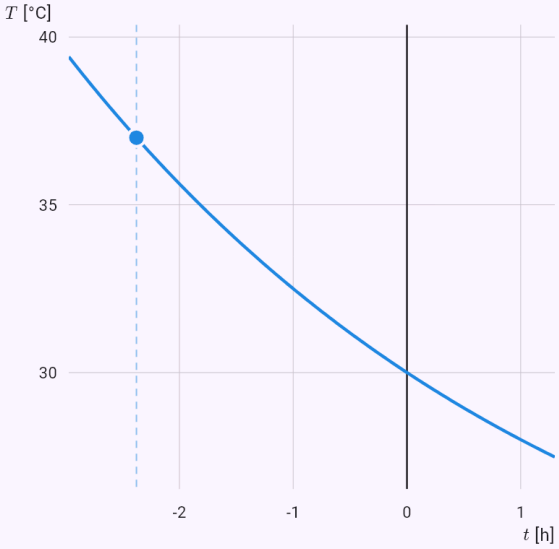
Result

$$t \approx -2.38\,\text{h}$$

Steps

- Find t; given
  $T_0 = 30\,^\circ\text{C}, \quad T_a = 20\,^\circ\text{C}, \quad t_1 = 1\,\text{h}, \quad T_1 = 28\,^\circ\text{C},$
- A body gives off heat the faster, the bigger the difference of temperatures: dT/dt = −k(T − T_a), T_a being the ambient temperature and k depending on the body (size, material, air flow). The solution is T = T_a + (T_0 − T_a)·e^(−kt) − the difference falls exponentially. k can be found from two measurements. That's how forensic scientists estimate the time of death: they measure the body's temperature when found and a bit later and calculate when it was 37 °C. The model's equation: $\frac{dT}{dt} = -k(T - T_a)$
- Separating the variables: $\frac{dT}{T - T_a} = -k\,dt$
- Integrating both sides: $\ln|T - T_a| = -kt + C$
- Exponentiating − the constant ±e^C is called C again: $T = T_a + Ce^{-kt}$
- Substituting the initial condition $T(0) = T_0$: $T_0 = T_a + C \Rightarrow C = T_0 - T_a$
- The solution of the initial condition (the particular solution): $T = T_a + (T_0 - T_a)e^{-kt}$
- k from the second measurement
  $T(t_1) = T_1 \Rightarrow k = -\frac{1}{t_1}\ln\frac{T_1 - T_a}{T_0 - T_a} = -\frac{1}{1}\ln\frac{28 - 20}{30 - 20} \approx$
- Expressing the sought quantity
  $T_a + (T_0 - T_a)e^{-kt} = T \Rightarrow t = -\frac{1}{k}\ln\frac{T - T_a}{T_0 - T_a}$
- Substituting: $t = -\frac{1}{0.2231}\ln\frac{37 - 20}{30 - 20} \approx -2.378\,\text{h}$
- A negative time means before the first measurement (for the time of death: that many hours before the body was found).

Graph



The model's quantities in time, computed from the solution; the answer is marked.

A 500 mg drug every 8 hours – the steady level between doses.

Input: `model(drug; find=steady[mg_l]; D=500[mg]; V=40[l]; T12=6[h]; tau=8[h];`

`mode=repeated)`

Result

$$c_{max} \approx 20.7\,\mathrm{mg/l}$$
$$c_{min} \approx 8.22\,\mathrm{mg/l}$$

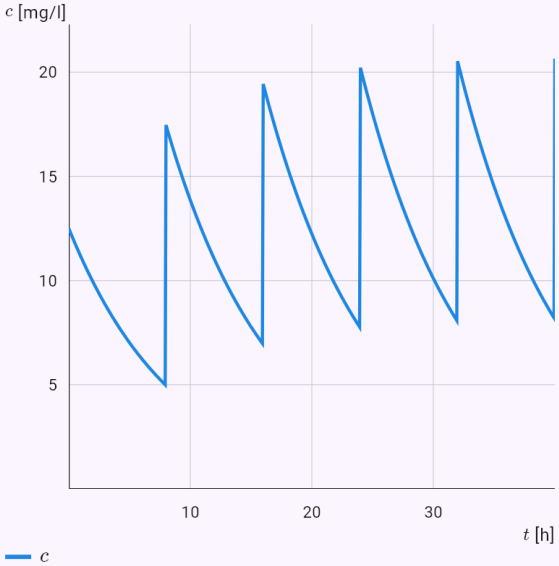
Steps

- Find c_{max}; given $D = 500\,\mathrm{mg}, \quad V = 40\,\mathrm{l}, \quad T_{1/2} = 6\,\mathrm{h}, \quad \tau = 8\,\mathrm{h}$
- A drug in the blood spreads in the volume of distribution V; the kidneys and the liver eliminate it at a rate proportional to the concentration: dc/dt = −kc, k = ln 2 / T½ (the biological half-life). After one dose D, c = (D/V)·e^(−kt). With repeated doses every τ the remainders add up (a geometric series) and the level settles between c_min and c_max. An infusion at a rate R gives dc/dt = R/V − kc, the concentration approaching the steady value R/(kV). The model's equation: $\frac{dc}{dt} = -kc$
- Initial concentration after the dose: $c_0 = \frac{D}{V} = \frac{500}{40} \approx 12.5\,\mathrm{mg/l}$
- Elimination constant from the biological half-life: $k = \frac{\ln 2}{T_{1/2}} = \frac{\ln 2}{6} \approx 0.1155\,\mathrm{h^{-1}}$
- Separating the variables: $\frac{dc}{c} = -k\,dt$
- Integrating both sides: $\ln|c| = -kt + C$
- Exponentiating – the constant ±e^C is called C again: $c = Ce^{-kt}$
- Substituting the initial condition $c(0) = c_0$: $c_0 = C$
- The solution of the initial condition (the particular solution): $c = c_0 e^{-kt}$
- Share of the drug left by the next dose: $q = \frac{c(\tau)}{c_0} = e^{-k\tau} = e^{-0.1155 \cdot 8} \approx 0.3969$
- Expressing the sought quantity

$$c_{max} = c_0\left(1 + q + q^2 + \ldots\right) = \frac{c_0}{1-q}, \quad c_{min} = q\,c_{max}$$

- Substituting: $c_{max} = \frac{12.5}{1 - 0.3969} \approx 20.72\,\mathrm{mg/l}$
- Substituting: $c_{min} = 0.3969 \cdot 20.72 \approx 8.225\,\mathrm{mg/l}$
- Accumulation factor (how many times the steady level exceeds one dose's): $R = \frac{1}{1-q} \approx 1.658$

Graph



How the quantities change in time.

The monthly repayment of a loan of 1 000 000 CZK for 20 years at 5 %.

Input: `model(interest; find=a[CZK_mo]; B0=1000000[CZK]; r=5[pct_yr]; t=20[yr]; B=0[CZK];`

kind=loan)

Result

$$a \approx 6{,}592\,\mathrm{CZK/mo}$$

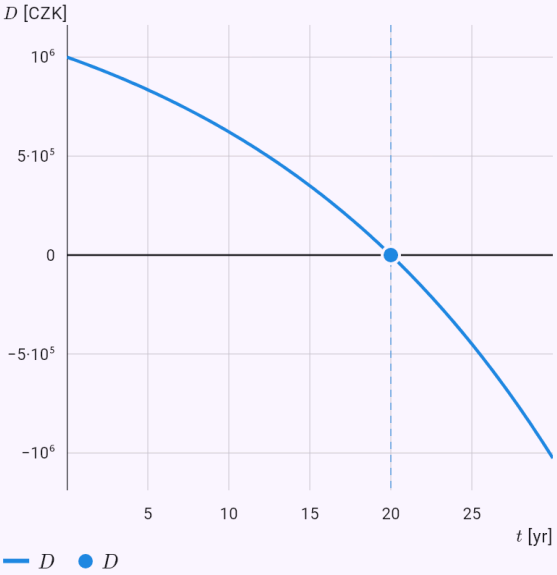
Steps

- Find a; given $D_0 = 1{,}000{,}000\,\mathrm{CZK},\quad r = 5\,\%\ \mathrm{p.a.},\quad t = 20\,\mathrm{yr},\quad D$
- Converting the units: $r = 5\,\%\ \mathrm{p.a.} = 0.05\,\mathrm{yr}^{-1}$
- With continuous compounding, interest accrues at every moment in proportion to the balance: dB/dt = rB, r being the annual rate. Regular deposits a raise the balance (dB/dt = rB + a), a loan's repayments lower the debt (dD/dt = rD − a). The equation is solved by separating the variables, and the solution gives the balance, the time to repay or the payment (the continuous counterpart of an annuity). The model's equation: $\frac{dD}{dt} = rD - a$
- Separating the variables: $\frac{dD}{D - \frac{a}{r}} = r\,dt$
- Integrating both sides: $\ln\left|D - \frac{a}{r}\right| = rt + C$
- Exponentiating – the constant ±e^C is called C again: $D = \frac{a}{r} + Ce^{rt}$
- Substituting the initial condition $D(0) = D_0$: $C = D_0 - \frac{a}{r}$
- The solution of the initial condition (the particular solution): $D = \frac{a}{r} + \left(D_0 - \frac{a}{r}\right)e^{rt}$
- Continuous compounding is a model – banks compound monthly or yearly; the results differ only a little.
- Expressing the sought quantity: $a = \frac{r(D_0 e^{rt} - D)}{e^{rt} - 1}$
- Substituting

$$a = \frac{0.05(1\,000\,000e^{0.05\cdot 20} - 0)}{e^{0.05\cdot 20} - 1} \approx 79{,}100\,\mathrm{CZK/yr}$$

- Paid in total: $a\,t = 1{,}582{,}000\,\mathrm{CZK}$
- Converting the units: $a \approx 79{,}100\,\mathrm{CZK/yr} = 6{,}592\,\mathrm{CZK/mo}$

Graph



The model's quantities in time, computed from the solution; the answer is marked.

How long a capacitor takes to charge to half the source voltage.

Input: `model(rc; find=t[ms]; R=10[kohm]; C=100[uF]; U=12[V]; u=6[V]; mode=charge)`

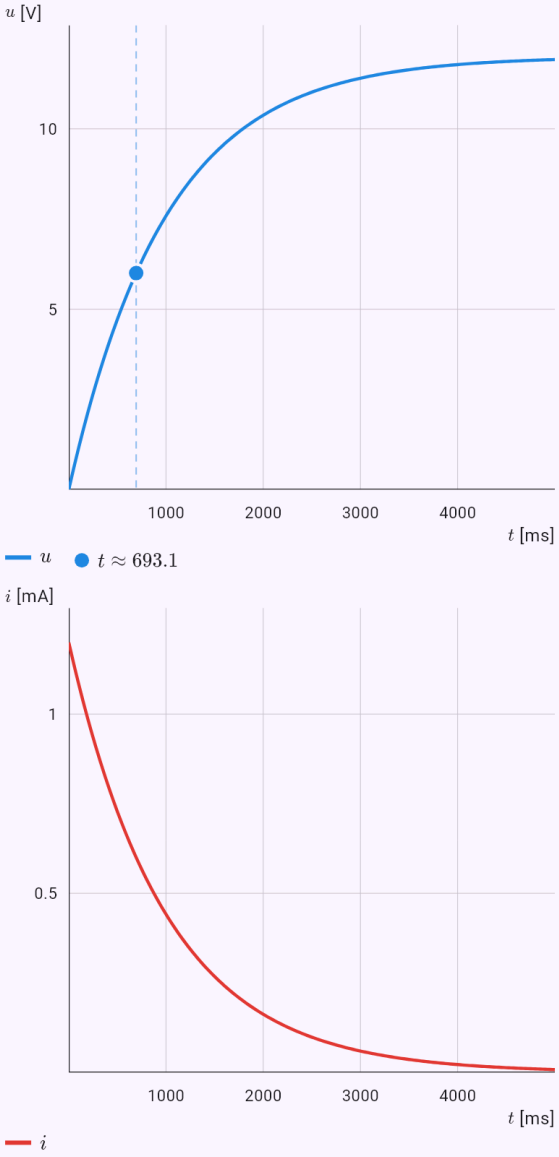
Result

$t \approx 693\,\text{ms}$

Steps

- Find t; given $R = 10\,\text{k}\Omega, \quad C = 100\,\mu\text{F}, \quad U = 12\,\text{V}, \quad u = 6\,\text{V}$
- Charging a capacitor through a resistor, Kirchhoff's law gives U = Ri + u, and the current charges the capacitor: i = C·du/dt. Together RC·du/dt + u = U. The voltage approaches the source's: u = U(1 − e^(−t/τ)), τ = RC being the time constant; the current i = (U/R)·e^(−t/τ) decreases. When discharging, u = U₀·e^(−t/τ). The model's equation: $RC\frac{du}{dt} + u = U$
- Time constant: $\tau = RC = 10 \cdot 100 \approx 1000\,\text{ms}$
- Separating the variables: $\frac{du}{u - U} = -\frac{dt}{\tau}$
- Integrating both sides: $\ln|u - U| = -\frac{t}{\tau} + C$
- Exponentiating – the constant ±e^C is called C again: $u = U + Ce^{-\frac{t}{\tau}}$
- Substituting the initial condition $u(0) = 0$: $0 = U + C \Rightarrow C = -U$
- The solution of the initial condition (the particular solution): $u = U\left(1 - e^{-\frac{t}{\tau}}\right)$
- Expressing the sought quantity: $t = -\tau \ln\left(1 - \frac{u}{U}\right)$
- Substituting: $t = -1000 \ln\left(1 - \frac{6}{12}\right) \approx 693.1\,\text{ms}$

Graph



The model's quantities in time, computed from the solution; the answer is marked.

A damped spring – the displacement at 1 s.

Input: `model(spring; find=x[cm]; m=0.5[kg]; k=20[N_m]; b=1[kg_s]; x0=5[cm]; v0=0[m_s];`

t=1[s])

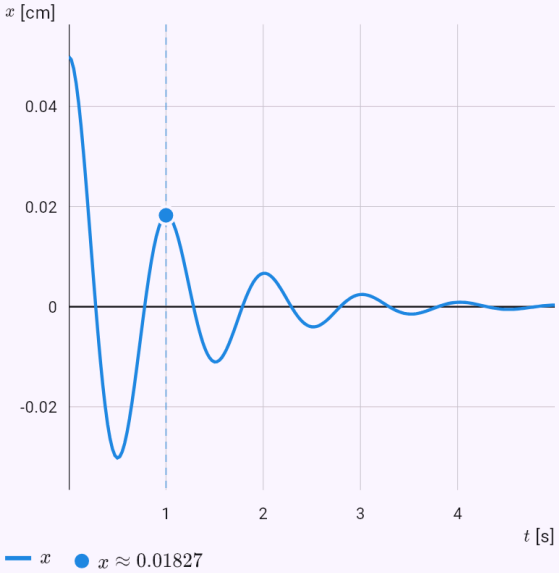
Result

$$x \approx 1.827\,\text{cm}$$

Steps

- Find x; given
  $m = 0.5\,\text{kg}, \quad k = 20\,\text{N/m}, \quad b = 1\,\text{kg/s}, \quad x_0 = 5\,\text{cm},$
- Converting the units: $x_0 = 5\,\text{cm} = 0.05\,\text{m}$
- A body on a spring feels the spring's force −kx (Hooke's law), damping −b·v (a resistance proportional to the speed) and possibly a driving force $F_0 \cdot \cos \omega t$. By Newton's 2nd law $m \cdot x'' = -kx - b \cdot x' + F_0 \cdot \cos \omega t$, that is $m \cdot x'' + b \cdot x' + kx = F_0 \cdot \cos \omega t$ – a linear equation of the 2nd order with constant coefficients. The roots of the characteristic equation tell whether the body oscillates (underdamping) or just returns to the equilibrium. Driving near the natural frequency $\omega_0 = \sqrt{k/m}$ causes resonance. The model's equation:
  $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$
- Natural angular frequency (without damping):
  $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{20}{0.5}} \approx 6.325\,\text{s}^{-1}$
- Damping rate: $\delta = \frac{b}{2m} = \frac{1}{2 \cdot 0.5} \approx 1\,\text{s}^{-1}$
- We look for solutions x = e^(λt); substituting and dividing by e^(λt) gives the characteristic equation:
  $0.5\lambda^2 + \lambda + 20 = 0$
- The roots of the characteristic equation:
  $\lambda_{1,2} = -1 \pm 6.245i$
- Underdamping ($\delta < \omega_0$; $\delta = 1, \quad \omega_0 = 6.325$): complex roots – damped oscillations.
- The homogeneous equation's solution:
  $x_h = e^{-t}(C_1 \cos 6.245t + C_2 \sin 6.245t)$
- We substitute the conditions into the general solution:
  $x(0) = 0.05, \quad x'(0) = 0$, hence
  $C_1 \approx 0.05, \quad C_2 \approx 0.008006$
- The solution of the initial condition (the particular solution):
  $x = e^{-t}(0.05 \cos 6.245t + 0.008006 \sin 6.245t)$
- Substituting: $x(1) \approx 0.01827\,\text{m}$
- Converting the units: $x \approx 0.01827\,\text{m} = 1.827\,\text{cm}$

Graph



The model's quantities in time, computed from the solution; the answer is marked.

An epidemic with R_0 = 2.5 – the peak and the final number of infected.

Input: `model(sir; find=overview[d]; N=10000[pcs]; I0=10[pcs]; R0=2.5; D=7[d]; rates=r0)`

Result

$I_{max} \approx 2340\,\text{ind.}$

$t_{max} \approx 34.1\,\text{d}$

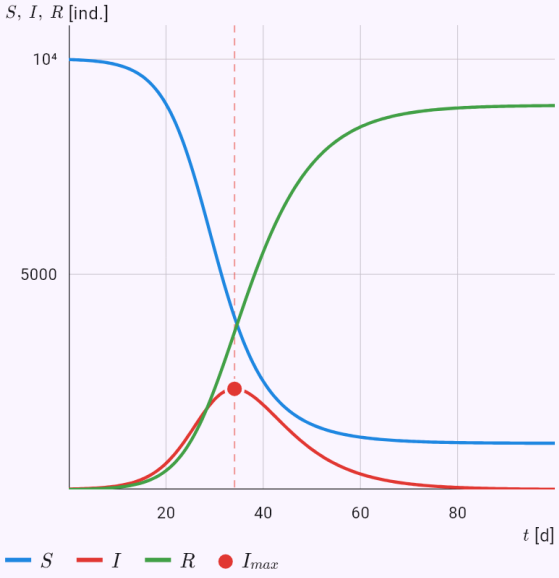
$R_\infty \approx 8930\,\text{ind.}$

Steps

- Find t_{max}; given
  $N = 10\,000\,\text{ind.}, \quad I_0 = 10\,\text{ind.}, \quad R_0 = 2.5, \quad D = 7\,\text{d}$
- The population N is split into the susceptible S, the infected I and the recovered (immune) R. Infections increase in proportion to the encounters of the susceptible with the infected: β·S·I/N; the infected recover at the rate γ = 1/D, D being the illness' duration. The system S' = −βSI/N, I' = βSI/N − γI, R' = γI has no solution in elementary functions – it is solved numerically. Exactly, though, we derive the basic reproduction number R_0 = β/γ, the epidemic's peak, the final number of infected and the herd immunity threshold 1 − 1/R_0. The model's equation
  $$S' = -\frac{\beta S I}{N}$$
  $$I' = \frac{\beta S I}{N} - \gamma I$$
  $$R' = \gamma I$$
- Recovery rate: $\gamma = \frac{1}{D} = \frac{1}{7} \approx 0.1429\,\text{d}^{-1}$
- Transmission rate: $\beta = R_0 \gamma \approx 0.3571\,\text{d}^{-1}$
- The sum S + I + R = N is constant (the equations add up to zero).
- Herd immunity threshold (the immune share at which the illness stops spreading): $1 - \frac{1}{R_0} \approx 60\,\%$
- The epidemic takes off, since initially
  $I' > 0 \Leftrightarrow S > \frac{N}{R_0} \approx 4000$
- The epidemic peaks at S = N/R_0; from the equation for dI/dS
  $\frac{dI}{dS} = -1 + \frac{N}{R_0 S} \Rightarrow I + S - \frac{N}{R_0} \ln S = C \Rightarrow I_{max} = 1$
- Solving the equation numerically (bisection)
  $\ln \frac{S_\infty}{S_0} = \frac{R_0 (S_\infty - N)}{N} \Rightarrow S_\infty \approx 1072\,\text{ind.}$
- In total the illness affects:
  $R_\infty = N - S_\infty \approx 8928\,\text{ind.}\ (89.3\,\%)$
- The system can't be solved in elementary functions – it's solved numerically by the 4th-order Runge–Kutta method with the step 0.05 on $[0,\ 100]$
- › The method: the Runge–Kutta method of the 4th order, 2000 steps
- From the numerical solution: $t_{max} \approx 34.1\,\text{d}$

Graph

In time | Phase plane



The model's quantities in time, computed from the solution; the answer is marked.

Predator and prey – the equilibrium, its stability and the period.

Input: `model(predator; find=overview[mo]; a=0.5[per_mo]; b=0.01[per_pcs_mo]; c=0.4[per_mo];`

d=0.005[per_pcs_mo]; x0=100[pcs]; y0=20[pcs])

Result

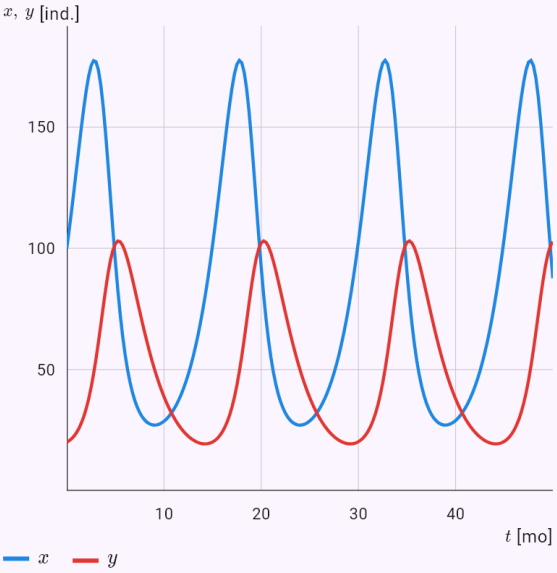
$$[x^*,\, y^*] \approx [80,\, 50]$$
$$T \approx 14.9\,\text{mo}$$

Steps

- Find T; given
  $a = 0.5\,\text{mo}^{-1}, \quad b = 0.01\,(\text{ind.}\cdot\text{mo})^{-1}, \quad c = 0.4\,\text{mo}^{-1}$
- Prey x multiplies without predators (ax), predators y die out without prey (−cy). Encounters, proportional to the product xy, take prey away (−bxy) and give predators food (+dxy). The system x' = ax − bxy, y' = −cy + dxy (Lotka, Volterra) has the equilibrium [c/d; a/b]. Linearized there it has purely imaginary eigenvalues – the populations oscillate around the equilibrium, the predators lagging behind the prey. It's solved numerically; the equilibria and the first integral are derived exactly. The model's equation
  $x' = ax - bxy$
  $y' = -cy + dxy$
- Equilibria – all the derivatives are zero
  $x(a - by) = 0,\ y(dx - c) = 0 \Rightarrow [0,\, 0],\ \left[\frac{c}{d},\, \frac{a}{b}\right] \approx [80,$
- The stability by linearization – the system's Jacobian matrix
  $J = \begin{pmatrix} a - by & -bx \\ dy & dx - c \end{pmatrix}$
- At $[0,\, 0]$ eigenvalues $\lambda_1 = -0.4, \quad \lambda_2 = 0.5$ – a saddle (unstable).
  $\begin{pmatrix} 0.5 & 0 \\ 0 & -0.4 \end{pmatrix}$
- At $[80,\, 50]$ eigenvalues $\lambda_{1,2} = \pm 0.4472i$ – a centre (oscillations around the equilibrium).
  $\begin{pmatrix} 0 & -0.8 \\ 0.25 & 0 \end{pmatrix}$
- Dividing the equations and separating gives a first integral – a quantity constant along a trajectory:
  $V = dx - c\ln x + by - a\ln y = C$
- Since V doesn't change along a trajectory, the trajectories are closed curves – the populations oscillate periodically around the equilibrium.
- Period of small oscillations around the equilibrium (from the eigenvalues): $T \approx \frac{2\pi}{\sqrt{ac}} \approx 14.05\,\text{mo}$
- The system can't be solved in elementary functions – it's solved numerically by the 4th-order Runge–Kutta method with the step 0.0125 on $[0,\, 50]$
- › The method: the Runge–Kutta method of the 4th order, 4000 steps
- From the numerical solution: $T \approx 14.95\,\text{mo}$

Graph

In time | Phase plane



How the quantities change in time.

A conical vessel: the time to empty.

Input: model(vessel; find=T[min]; shape=cone; R=30[cm]; H=60[cm]; a=1[cm2]; mu=0.6;

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h0=50[cm]; g=9.81[m_s2])
```

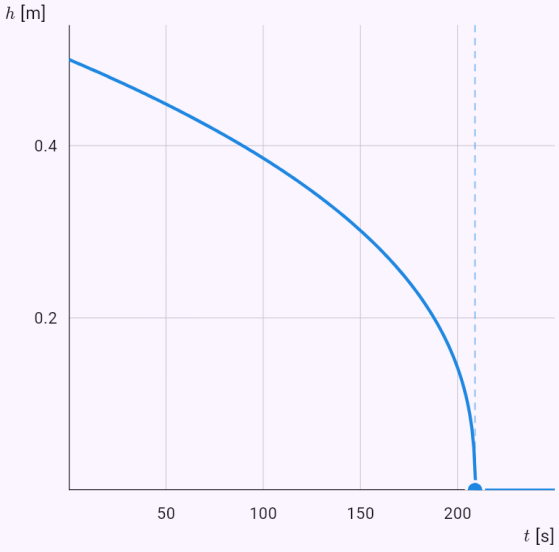
Result

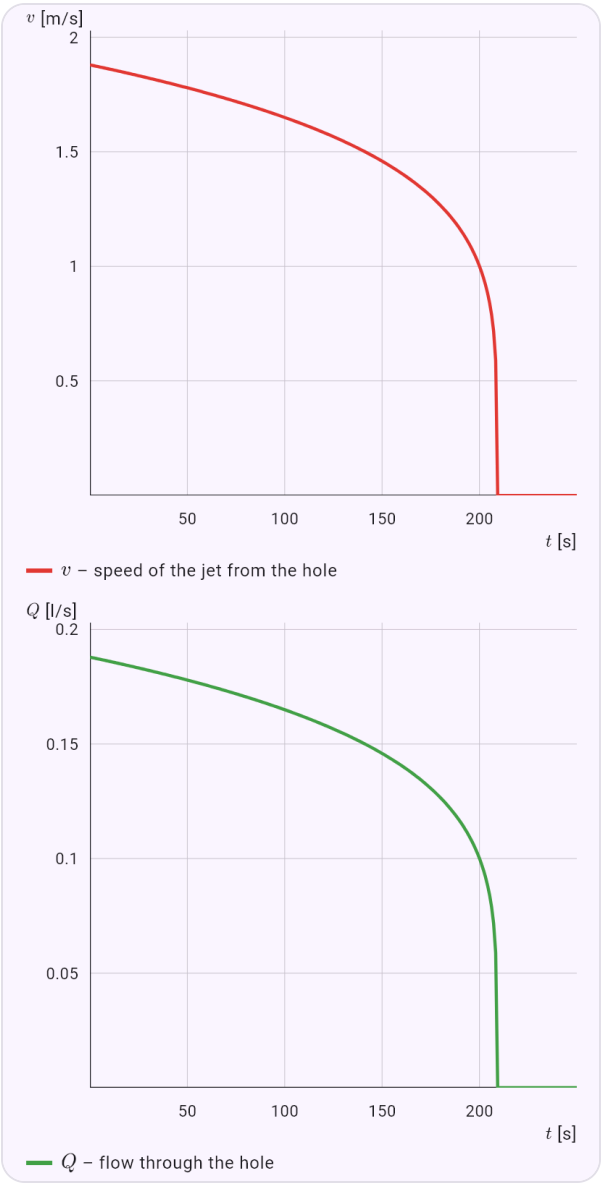
$$T \approx 3.48\,\text{min}$$

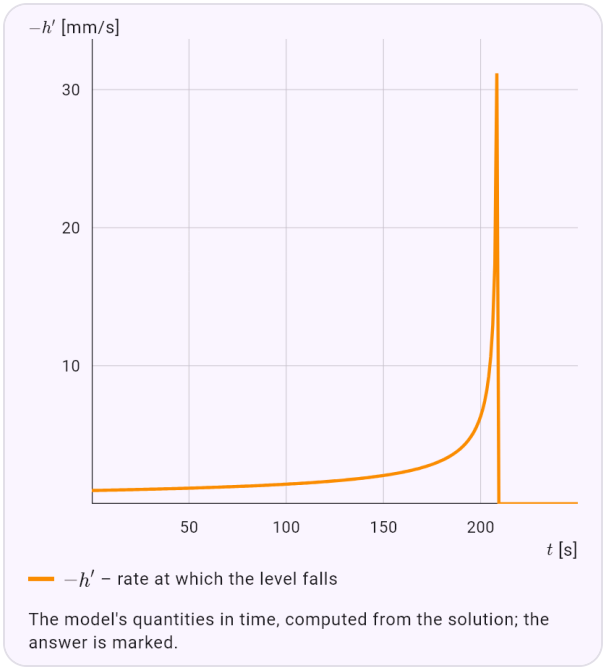
Steps

- Find T; given
  $R = 30\,\text{cm}, \quad H = 60\,\text{cm}, \quad s = 1\,\text{cm}^2, \quad \mu = 0.6, \quad h$
- Converting the units
  $R = 30\,\text{cm} = 0.3\,\text{m}, \quad H = 60\,\text{cm} = 0.6\,\text{m}, \quad s = 1\,\text{cm}$
- A vessel of revolution of the chosen shape has a hole in the bottom. The liquid leaves at a speed from the energy balance: Torricelli's v = μ√(2gh), corrected for a large hole (the s²/S² term), for an outlet pipe of length L (adding the height L, but also friction by the liquid's viscosity – laminar or turbulent) and, in a closed vessel, for the underpressure of the air above the level, which slows the flow down to a stop. In dt the volume s·v·dt leaves and the level falls by dh, so dt = −S(h)/(s·v) dh and the time follows by integration. The graph shows the level, the speed of the jet and the flow. For r = R·(h/H)^p without a pipe or a lid the formula is shown for comparison. With r = k·h^(1/4) the level falls evenly, which is how water clocks worked. The model's equation
  $$S(h)\frac{dh}{dt} = -s\,v, \quad \frac{v^2}{2g\mu^2}\left(1 - \frac{s^2}{S^2(h)}\right) = h$$
- The vessel's cross-section at the height h:
  $$S(h) = \pi R^2 \left(\frac{h}{H}\right)^{2p}, \quad p = 1$$
- Water at 20 °C: $\rho = 998.2\,\text{kg/m}^3, \quad \eta = 1.002\,\text{mPa}\cdot\text{s}$. Liquid at normal pressure from 0 to 100 °C.
- Separating the variables: $dt = -\dfrac{S(h)}{s\,v(h)}\,dh$
- The speed of the jet depends on the level in a complicated way, so the integral is computed numerically.
- For comparison, Torricelli without the large-hole correction: $T_0 = \dfrac{h_0^q}{q\kappa} \approx 209\,\text{s}$
- Expressing the sought quantity: $T = \displaystyle\int_0^{h_0} \frac{S(h)}{s\,v(h)}\,dh$
- Substituting: $T = \displaystyle\int_0^{0.5} \frac{S(h)}{s\,v(h)}\,dh \approx 208.9\,\text{s}$
- Converting the units: $T \approx 208.9\,\text{s} = 3.482\,\text{min}$

Graph







The model's quantities in time, computed from the solution; the answer is marked.

Oil from a spherical vessel through a pipe – the level, the jet's speed, the flow and the rate at which the level falls.

Input: `model(vessel; find=T[min]; shape=sphere; liquid=oil; R=20[cm]; a=1[cm2]; mu=0.6;`

```
h0=30[cm]; L=30[cm]; g=9.81[m_s2])
```

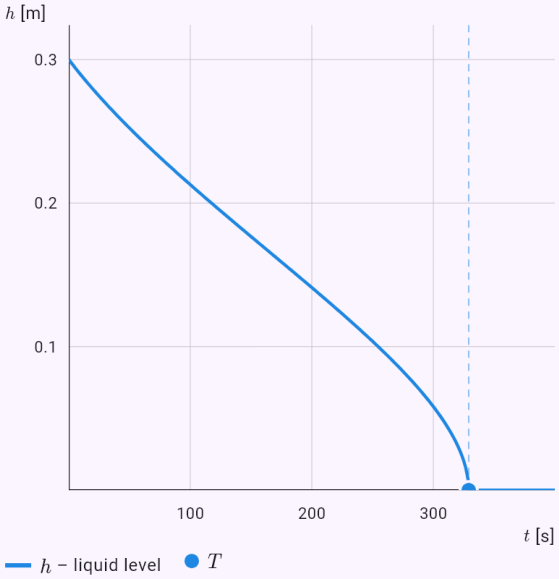
Result

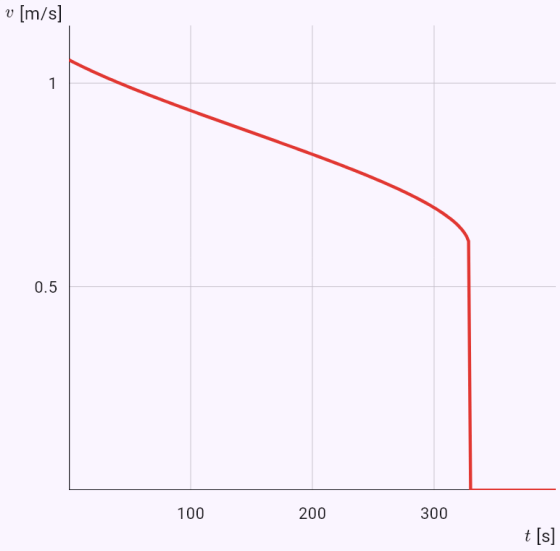
$$T \approx 5.48\,\text{min}$$

Steps

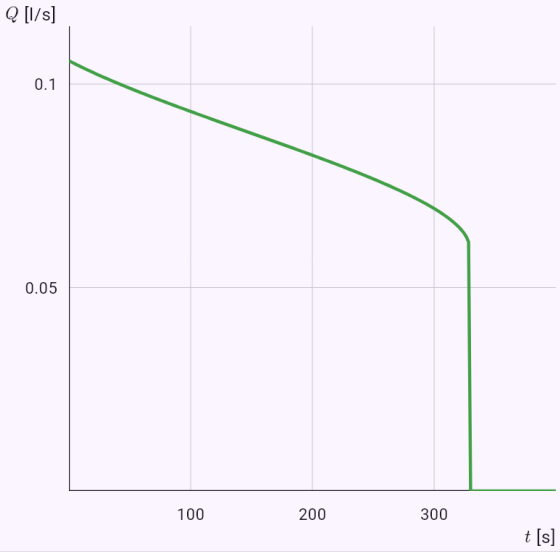
- Find T; given
  $R = 20\,\text{cm}, \quad s = 1\,\text{cm}^2, \quad \mu = 0.6, \quad h_0 = 30\,\text{cm}, \quad L$
- Converting the units
  $R = 20\,\text{cm} = 0.2\,\text{m}, \quad s = 1\,\text{cm}^2 = 10^{-4}\,\text{m}^2, \quad h_0 = 30$
- A vessel of revolution of the chosen shape has a hole in the bottom. The liquid leaves at a speed from the energy balance: Torricelli's v = μ√(2gh), corrected for a large hole (the s²/S² term), for an outlet pipe of length L (adding the height L, but also friction by the liquid's viscosity – laminar or turbulent) and, in a closed vessel, for the underpressure of the air above the level, which slows the flow down to a stop. In dt the volume s·v·dt leaves and the level falls by dh, so dt = −S(h)/(s·v) dh and the time follows by integration. The graph shows the level, the speed of the jet and the flow. For r = R·(h/H)^p without a pipe or a lid the formula is shown for comparison. With r = k·h^(1/4) the level falls evenly, which is how water clocks worked. The model's equation
  $S(h)\dfrac{dh}{dt} = -s\,v, \quad \dfrac{v^2}{2g\mu^2}\left(1 - \dfrac{s^2}{S^2(h)}\right) + h_z(v) = h +$
- The vessel's cross-section at the height h:
  $S(h) = \pi\left(2Rh - h^2\right)$
- Sunflower oil at 20 °C:
  $\rho = 920\,\text{kg/m}^3, \quad \eta = 50\,\text{mPa}\cdot\text{s}$. It sets around −17 °C and doesn't boil – above 200 °C it decomposes; its viscosity falls steeply with temperature.
- Separating the variables: $dt = -\dfrac{S(h)}{s\,v(h)}\,dh$
- The speed of the jet depends on the level in a complicated way, so the integral is computed numerically.
- Expressing the sought quantity: $T = \displaystyle\int_0^{h_0} \frac{S(h)}{s\,v(h)}\,dh$
- Substituting: $T = \displaystyle\int_0^{0.3} \frac{S(h)}{s\,v(h)}\,dh \approx 329.1\,\text{s}$
- Converting the units: $T \approx 329.1\,\text{s} = 5.485\,\text{min}$

Graph

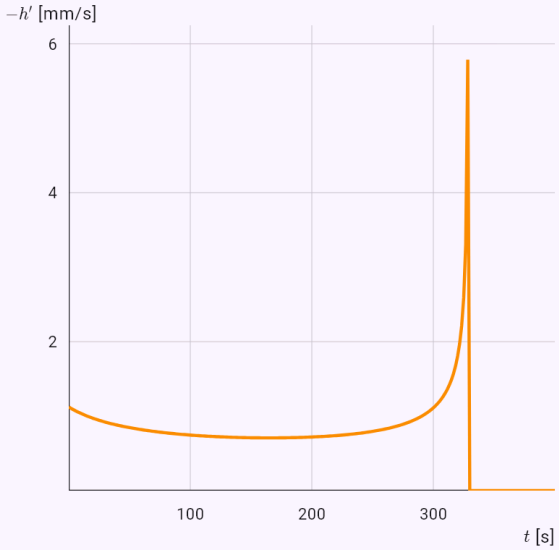




v – speed of the jet from the hole



Q – flow through the hole



$-h'$ – rate at which the level falls

The model's quantities in time, computed from the solution; the answer is marked.

The vessel's shape in which the level falls evenly.

Input: `model(vessel; find=k[cm]; a=1[cm2]; mu=0.6; h0=50[cm]; g=9.81[m_s2]; v=0.001[m_s])`

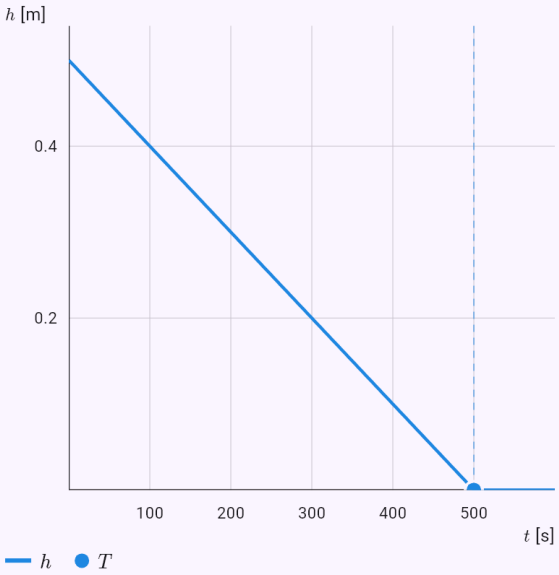
Result

$$r(h_0) \approx 24.46\,\text{cm}$$

Steps

- Find $r(h_0)$; given
  $s = 1\,\text{cm}^2, \quad \mu = 0.6, \quad h_0 = 50\,\text{cm}, \quad g = 9.81\,\text{m/s}^2,$
- Converting the units
  $s = 1\,\text{cm}^2 = 10^{-4}\,\text{m}^2, \quad h_0 = 50\,\text{cm} = 0.5\,\text{m}$
- A vessel of revolution of the chosen shape has a hole in the bottom. The liquid leaves at a speed from the energy balance: Torricelli's v = μ√(2gh), corrected for a large hole (the s²/S² term), for an outlet pipe of length L (adding the height L, but also friction by the liquid's viscosity – laminar or turbulent) and, in a closed vessel, for the underpressure of the air above the level, which slows the flow down to a stop. In dt the volume s·v·dt leaves and the level falls by dh, so dt = −S(h)/(s·v) dh and the time follows by integration. The graph shows the level, the speed of the jet and the flow. For r = R·(h/H)^p without a pipe or a lid the formula is shown for comparison. With r = k·h^(1/4) the level falls evenly, which is how water clocks worked. The model's equation: $\pi r^2(h)\, v = \mu s \sqrt{2gh}$
- Expressing the sought quantity:
  $r(h) = k\, h^{1/4}, \quad k = \sqrt{\dfrac{\mu s \sqrt{2g}}{\pi v}} \approx 0.2909$
- The level falls evenly when the cross-section is proportional to √h, i.e. r = k·h^(1/4). That was the shape of water clocks.
- Substituting: $r(h_0) = 0.2909 \cdot 0.5^{1/4} \approx 0.2446\,\text{m}$
- Converting the units: $r(h_0) \approx 0.2446\,\text{m} = 24.46\,\text{cm}$

Graph



The model's quantities in time, computed from the solution; the answer is marked.

43. Physics

Physics calculations are in the “Applications” section, field Physics. The rocket engine has five modes: thrust, specific impulse, Tsiolkovsky’s equation, a multi-stage rocket and the exhaust velocity from the thermodynamics of an ideal nozzle.

Thrust is computed from the mass flow and the exhaust velocity, optionally with the pressure term; the graph then shows how it grows with altitude in the standard atmosphere. In Tsiolkovsky’s equation you can look for the velocity change, the propellant needed, the initial or final mass, or the exhaust velocity. The results are ideal, without gravity and drag losses.

Kepler's laws have a form of their own with five modes: ellipse, areas, position in time, period and the speed from the vis-viva equation. The graph draws the orbit at true scale and the body moves along it by the slider or the play button. For the second law it shows coloured sectors swept in equal times.

The form on bodies influencing each other computes the barycentre, Lagrange points, tides, orbital resonances and the precession of the perihelion. The barycentre's graph shows both bodies around the [[barycenter|barycentre]] B at true scale. When the larger body's circle is too small to see in the whole picture (the Sun around the barycentre of the Sun and Jupiter), the switch “Detail at the barycentre” shows it enlarged, with the radius R also the body's outline – the barycentre lies just above the Sun's surface. A famous story: Uranus strayed from its predicted path, and in 1846 Le Verrier worked out from these perturbations where an unknown planet had to be. Neptune was then found almost exactly there.

The three-body problem is computed numerically in units where G = 1: the periodic figure-eight solution, the chaotic Pythagorean problem in which two nearly identical runs eventually separate, and your own values. The change of the total energy checks the accuracy.

Nuclear physics in the applications is for peaceful use only: the laboratory, medicine and energy. The Nuclear physics form computes the activity of a sample, radioactive equilibrium, Bateman's equations of a decay series, the binding energy with a graph by mass number, the energy of a reaction and the energy of fuel compared with coal.

The Nuclear reactor form shows neutron generations as a geometric sequence, the reactor period with delayed neutrons and the size of the core in the one-group diffusion model, only for moderated lattices (graphite, light water).

Ballistics computes a projectile's flight for school physics, sport and hunting. In a vacuum there are exact formulas with the steps; with air drag a numerical Runge-Kutta solution including headwind and crosswind, the air density by height and the Coriolis force. The graph shows a side view and a comparison with a vacuum, with a side drift also a top view. For supersonic bullets the drag coefficient comes from an approximate curve by the Mach number. Only the flight is computed, not the effect on a

target, and a spinning bullet's drift isn't included. The opposite task finds both launch angles, flat and steep, for a given distance.

Aeroplane computes performance for civil aviation and teaching: level flight on a parabolic drag polar, the take-off and landing roll numerically with wind and altitude, a turn and a glide. An optional ground effect lessens the induced drag. The model leaves out the compressibility of the air, so for airliners at cruise speed the result is only indicative. Speeds and lengths can also be given in knots and feet.

Atmospheric entry computes a spacecraft's pass through the atmosphere numerically by the Runge-Kutta method: the speed, the flight path angle and the height over a round Earth with the standard atmosphere. The result is a landing, a skip back into space or a limit exceeded, the peak g-load (checked by Allen and Eggers' formula), the heat flux at the [[stagnationPoint|stagnation point]] and the shield's temperature by [[radiativeEquilibrium|radiative equilibrium]]. The Entry corridor mode finds the shallow and the steep limit of the entry angle by bisection – for a return from the Moon the [[entryCorridor|corridor]] comes out about 2° wide, with a note on Apollo 13. Presets: Vostok, Apollo, Soyuz and the Space Shuttle (with a factual note on the loss of Columbia). The only control is a constant [[bankAngle|bank angle]]; the Earth doesn't rotate in the model. An interactive demo where you bank the lift during the flight is at mathtrace.app under “Try it yourself”.

Bodies in water is a form for teaching and civil use: buoyancy and floating including a balloon, pressure with depth, Bernoulli's equation, sinking and rising with an added mass, the range and drop of a speargun for sport fishing, a powered submerged vehicle, ballast and a ship. A sphere's drag coefficient follows the Reynolds number and a bubble has its own correlation. A ship's resistance is the friction by the ITTC-57 line plus a given residual coefficient, so beyond displacement speeds it is only a rough guide. For a ship you can switch on a comparison with the KM ekranoplan: the work against the resistance per tonne and kilometre and a chart of the power by the speed.

Thermonuclear fusion has two forms, both for civil use only (power plants, laboratories, the Sun and stars). The Thermonuclear fusion form computes a reaction's energy (D-T, D-D, D-^{3}He and the Sun's p-p chain) with its share between the products, the energy of fuel compared with uranium fission and with coal, the height of the [[coulombBarrier|Coulomb barrier]] and the [[gamowPeak|Gamow peak]]. A fusion reactor cannot run away like a fission one: fusion has no critical mass, and if something fails, the plasma cools down and the reaction stops by itself.

The Fusion plasma form computes the [[lawsonCriterion|Lawson criterion]] with a graph of the triple product needed at each temperature, the gain Q, the reaction rate $\langle\sigma v\rangle$ as an integral over the Maxwell distribution by Simpson's rule (compared with Bosch and Hale's published fit), the fusion power density, the bremsstrahlung loss and the Larmor radius of a charged particle in a magnetic field. The app does not compute the plasma's dynamics (magnetohydrodynamics).

Particle accelerators have two forms. Relativistic kinematics computes the [[lorentzFactor|Lorentz factor]], speed, energy and momentum from a speed or a kinetic energy, and the time dilation with the range of unstable particles (muon, pion, kaon). The Particle accelerators form compares the collision energy √s of colliding beams and of a fixed target, computes a ring's bending radius and the field needed, and the synchrotron radiation loss. The menu fills in the values of the LHC, LEP or the Tevatron.

Examples

Engine thrust at sea level, compared with vacuum.

Input: `model(rocket; find=F[kN]; mdot=250[kg_s]; ve=3000[m_s]; pe=70[kPa]; pa=101.325[kPa]; Ae=1.5[m2]; mode=thrust; pressure=full)`

Result

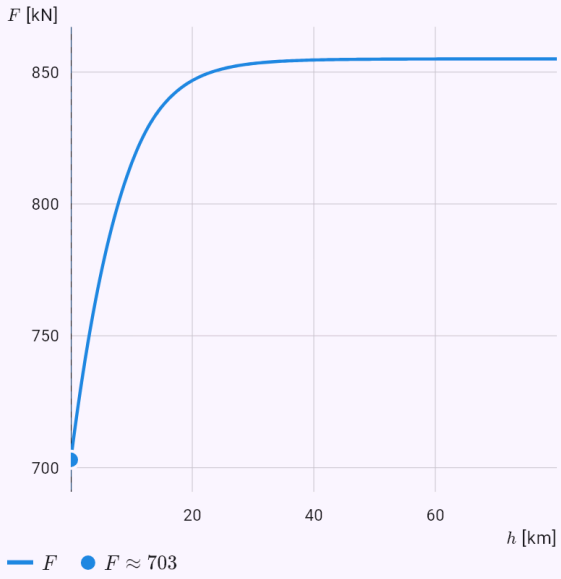
$$F \approx 703\,\text{kN}$$

Steps

- Find F; given
  $\dot{m} = 250\,\text{kg/s}, \quad v_e = 3000\,\text{m/s}, \quad p_e = 70\,\text{kPa}, \quad p_a =$
- Converting the units
  $p_e = 70\,\text{kPa} = 70\,000\,\text{Pa}, \quad p_a = 101.3\,\text{kPa} = 101\,300$
- A rocket is driven by throwing mass away: each second a mass Δm/Δt leaves at speed v_e, and conservation of momentum gives the thrust $F = (\Delta m/\Delta t)\cdot v_e$. If the gas pressure at the nozzle exit p_e differs from the ambient p_a, the term $(p_e - p_a)\cdot A_e$ is added, which is why thrust grows with altitude. The model's equation:
  $F = \dot{m} v_e + (p_e - p_a) A_e$
- Pressure term of the thrust
  $(p_e - p_a) A_e = (70\,000 - 101\,300)\cdot 1.5 \approx -46\,990\,\text{N}$
- The gas is overexpanded in the nozzle (p_e < p_a), so the ambient pressure lowers the thrust. Typical at lift-off for an engine designed for higher altitudes.
- Momentum term of the thrust:
  $\dot{m} v_e = 250 \cdot 3000 \approx 750\,000\,\text{N}$
- Substituting: $F = 750\,000 - 46\,990 \approx 703\,000\,\text{N}$
- Specific impulse
  $$I_{sp} = \frac{F}{\dot{m} g_0} = \frac{703\,000}{250 \cdot 9.80665} \approx 286.7\,\text{s}$$
- Thrust in vacuum: $F_{vac} = \dot{m} v_e + p_e A_e \approx 855\,000\,\text{N}$

- Converting the units: $F \approx 703\,000\,\text{N} = 703\,\text{kN}$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

Exhaust velocity from the specific impulse of RP-1/LOX.

Input: `model(rocket; find=ve[m_s]; Isp=340[s]; mode=isp; propellant=kerosene)`

Result

$$v_e \approx 3334\,\mathrm{m/s}$$

Steps

- Find v_e; given: $I_{sp} = 340\,\mathrm{s}$
- The specific impulse I_sp = v_e/g_0 says for how many seconds 1 kg of propellant can produce a thrust equal to the weight of 1 kg, so it compares propellants. The model's equation: $I_{sp} = \dfrac{v_e}{g_0} = \dfrac{F}{\dot{m} g_0}$
- Standard gravity: $g_0 = 9.80665\,\mathrm{m/s^2}$
- Expressing the sought quantity: $v_e = I_{sp} g_0$
- Substituting: $v_e = 340 \cdot 9.80665 \approx 3334\,\mathrm{m/s}$
- The higher the specific impulse, the less propellant the engine burns for the same thrust. Typical vacuum values: solid propellant ≈ 265 s, RP-1/LOX ≈ 340 s, CH_4/LOX ≈ 365 s, LH_2/LOX ≈ 450 s.

Velocity change by Tsiolkovsky's equation.

Input: `model(rocket; find=dv[m_s]; ve=3000[m_s]; m0=1000[kg]; mf=250[kg]; mode=tsiolkovsky)`

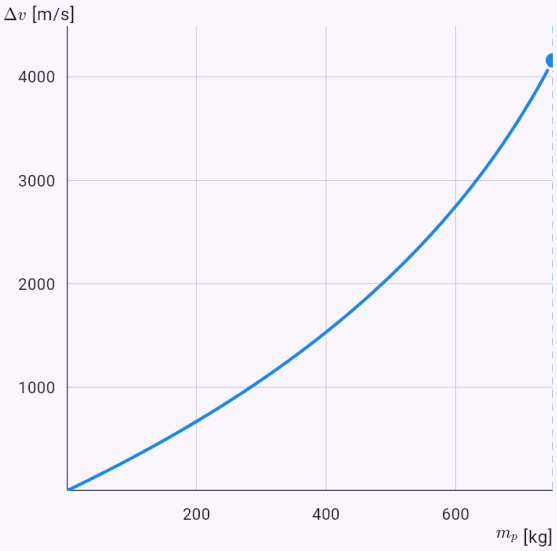
Result

$$\Delta v \approx 4159\,\mathrm{m/s}$$

Steps

- Find Δv; given: $v_e = 3000\,\mathrm{m/s}, \quad m_0 = 1000\,\mathrm{kg}, \quad m_f = 250\,\mathrm{kg}$
- The momentum of the expelled propellant speeds the rocket up: integrating m·dv = −v_e·dm gives Tsiolkovsky's equation Δv = v_e·ln(m_0/m_f). The model's equation: $m\,dv = -v_e\,dm$
- Separating the variables: $dv = -v_e \dfrac{dm}{m}$
- Integrating both sides: $\Delta v = -v_e \displaystyle\int_{m_0}^{m_f} \frac{dm}{m} = v_e \ln \frac{m_0}{m_f}$
- Substituting: $\Delta v = 3000 \cdot \ln \dfrac{1000}{250} \approx 4159\,\mathrm{m/s}$
- Mass ratio: $\dfrac{m_0}{m_f} \approx 4$
- Propellant share of the initial mass: $\dfrac{m_0 - m_f}{m_0} \approx 75\,\%$
- An ideal result without gravity and drag losses. Reaching low Earth orbit actually takes about 9.4 km/s.

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

How much propellant 1 t of rocket needs to reach low orbit.

Input: `model(rocket; find=mp[kg]; dv=9400[m_s]; ve=4400[m_s]; mf=1000[kg]; mode=tsiolkovsky)`

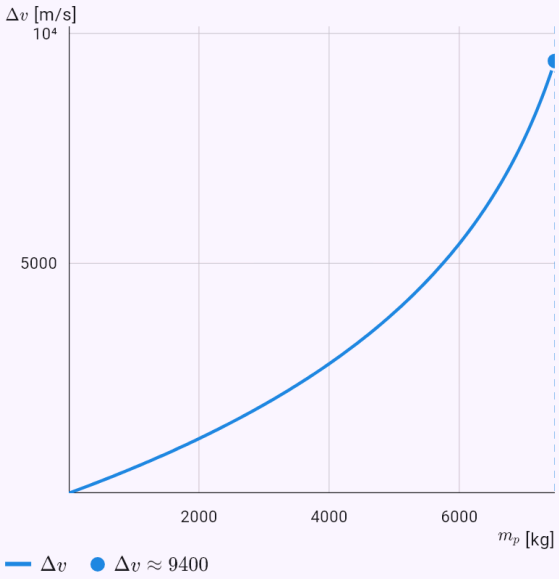
Result

$$m_p \approx 7469\,\text{kg}$$

Steps

- Find m_p; given: $\Delta v = 9400\,\text{m/s}, \quad v_e = 4400\,\text{m/s}, \quad m_f = 1000\,\text{kg}$
- The momentum of the expelled propellant speeds the rocket up: integrating m·dv = −v$_e$·dm gives Tsiolkovsky's equation Δv = v$_e$·ln(m$_0$/m_f). The model's equation: $m\,dv = -v_e\,dm$
- Separating the variables: $dv = -v_e \dfrac{dm}{m}$
- Integrating both sides: $\Delta v = -v_e \displaystyle\int_{m_0}^{m_f} \frac{dm}{m} = v_e \ln \frac{m_0}{m_f}$
- Expressing the sought quantity: $m_p = m_0 - m_f = m_f\left(e^{\frac{\Delta v}{v_e}} - 1\right)$
- Substituting: $m_p = 1000 \cdot \left(e^{\frac{9400}{4400}} - 1\right) \approx 7469\,\text{kg}$
- Mass ratio: $\dfrac{m_0}{m_f} \approx 8.469$
- Propellant share of the initial mass: $\dfrac{m_0 - m_f}{m_0} \approx 88.2\,\%$
- An ideal result without gravity and drag losses. Reaching low Earth orbit actually takes about 9.4 km/s.

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

A three-stage rocket of the Saturn V class and what staging gains.

Input: `model(rocket; find=dv[m_s]; mps=2077 444 107[t]; mss=130 36 11[t]; isps=263 421`

421[s]; mL=45[t]; mode=staging)

Result

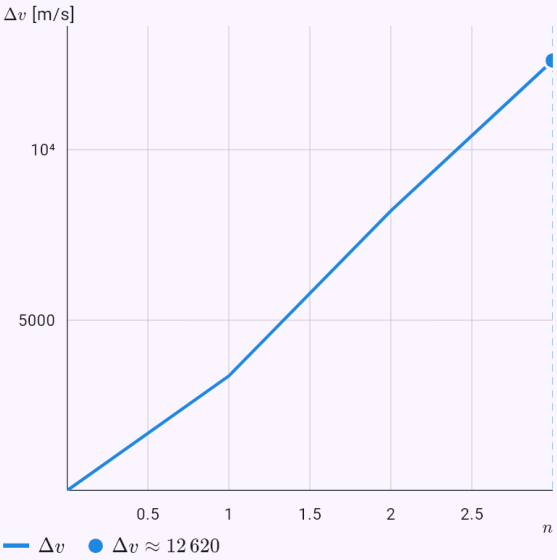
$$\Delta v \approx 12\,620\ \mathrm{m/s}$$

Steps

- Find Δv; given
  $m_p = (2077;\ 444;\ 107)\ \mathrm{t}, \quad m_s = (130;\ 36;\ 11)\ \mathrm{t}, \quad I_{sl}$
- Converting the units
  $m_p = (2\,077\,000;\ 444\,000;\ 107\,000)\ \mathrm{kg}, \quad m_s = (130\,0($
- Each stage adds speed by Tsiolkovsky's equation and its empty tanks are dropped, so the next stage no longer carries useless mass. The velocity gains add up. The model's equation: $\Delta v = \sum_i I_{sp,i}\, g_0 \ln \frac{m_{0,i}}{m_{f,i}}$
- Standard gravity: $g_0 = 9.80665\ \mathrm{m/s^2}$
- Velocity change of the stage
  $\Delta v_1 = 263 \cdot 9.80665 \cdot \ln \frac{2\,850\,000}{773\,000} \approx 3365\ \mathrm{m/s}$
- Velocity change of the stage
  $\Delta v_2 = 421 \cdot 9.80665 \cdot \ln \frac{643\,000}{199\,000} \approx 4842\ \mathrm{m/s}$
- Velocity change of the stage:
  $\Delta v_3 = 421 \cdot 9.80665 \cdot \ln \frac{163\,000}{56\,000} \approx 4411\ \mathrm{m/s}$
- For comparison, one stage with the same mass and propellant
  $\Delta v^* = 296.1 \cdot 9.80665 \cdot \ln \frac{2\,850\,000}{222\,000} \approx 7412\ \mathrm{m/s}$
- By dropping empty stages the rocket gains $\Delta v - \Delta v^* \approx 5206\ \mathrm{m/s}$ more than one stage with the same propellant.

- An ideal result without gravity and drag losses. Reaching low Earth orbit actually takes about 9.4 km/s.

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

Exhaust velocity of a hydrogen engine from nozzle thermodynamics.

Input: model(rocket; find=ve[m_s]; Tc=3250[K]; M=13[g_mol]; kappa=1.2[one]; pr=1000[one];

```
mode=nozzle; propellant=hydrogen)
```

Result

$$v_e \approx 4130\,\mathrm{m/s}$$

Steps

- Find v_e; given

  $$T_c = 3250\,\mathrm{K}, \quad M = 13\,\mathrm{g/mol}, \quad \kappa = 1.2, \quad \frac{p_c}{p_e} = 100$$

- Converting the units: $M = 13\,\mathrm{g/mol} = 0.013\,\mathrm{kg/mol}$
- The exhaust velocity comes from the thermodynamics of an ideal nozzle: it grows with the combustion chamber temperature and falls with the molar mass of the exhaust. The model's equation

  $$v_e = \sqrt{\frac{2\kappa}{\kappa - 1}\,\frac{RT_c}{M}\left[1 - \left(\frac{p_e}{p_c}\right)^{\frac{\kappa-1}{\kappa}}\right]}$$

- Molar gas constant: $R = 8.31446\,\mathrm{J/(mol\,K^{-1})}$
- Expansion term

  $$\left(\frac{p_e}{p_c}\right)^{\frac{\kappa-1}{\kappa}} = \left(\frac{1}{1000}\right)^{0.1667} \approx 0.3162$$

- Substituting

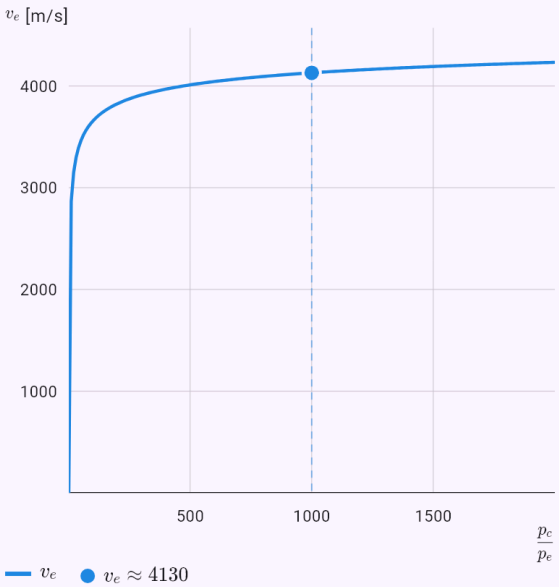
  $$v_e = \sqrt{\frac{2 \cdot 1.2}{1.2 - 1} \cdot \frac{8.31446 \cdot 3250}{0.013} \cdot (1 - 0.3162)} \approx 4130$$

- Specific impulse: $I_{sp} = \dfrac{v_e}{g_0} \approx 421.1\,\mathrm{s}$
- Limiting velocity for expansion into vacuum:

  $$v_{max} = \sqrt{\frac{2\kappa}{\kappa - 1}\,\frac{RT_c}{M}} \approx 4994\,\mathrm{m/s}$$

- The exhaust velocity grows with √(T/M), which is why hydrogen (a light exhaust) has the highest specific impulse.
- Ideal isentropic flow; a real nozzle reaches about 92–98 % of this value.

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

The perihelion of Halley's comet from the semi-major axis and the eccentricity.

Input: `model(orbit; find=rp[au]; a=17.834[au]; e=0.96714[one]; mode=ellipse; body=halley)`

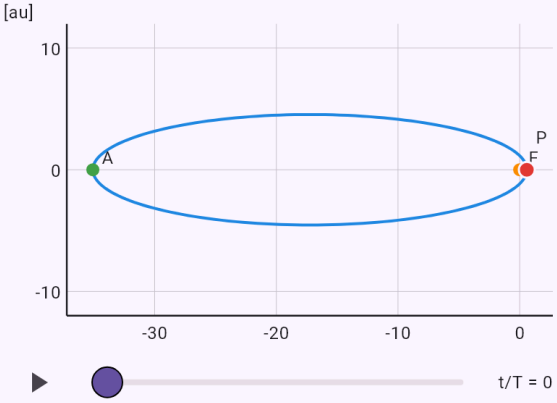
Result

$$r_p \approx 0.586\,\text{au}$$

Steps

- Find r_p; given: $a = 17.83\,\text{au}, \quad e = 0.9671$
- Kepler's first law: a planet moves on an ellipse with the central body at one focus. The distance r depends on the angle θ from the perihelion; it is smallest at the perihelion r_p = a(1 − e) and largest at the aphelion r_a = a(1 + e). The eccentricity e says how elongated the ellipse is. The model's equation: $r(\theta) = \dfrac{a(1-e^2)}{1+e\cos\theta}$
- Expressing the sought quantity: $r_p = a(1-e)$
- Substituting: $r_p = 17.83(1-0.9671) \approx 0.586\,\text{au}$
- Distance at aphelion: $r_a = a(1+e) \approx 35.08\,\text{au}$

Graph



The bodies at true scale. Use the slider or the play button to follow the motion.

The Earth's speed at aphelion from Kepler's second law; sectors of equal time.

Input: `model(orbit; find=va[km_s]; rp=0.9833[au]; ra=1.0167[au]; vp=30.29[km_s]; mode=areas; body=earth)`

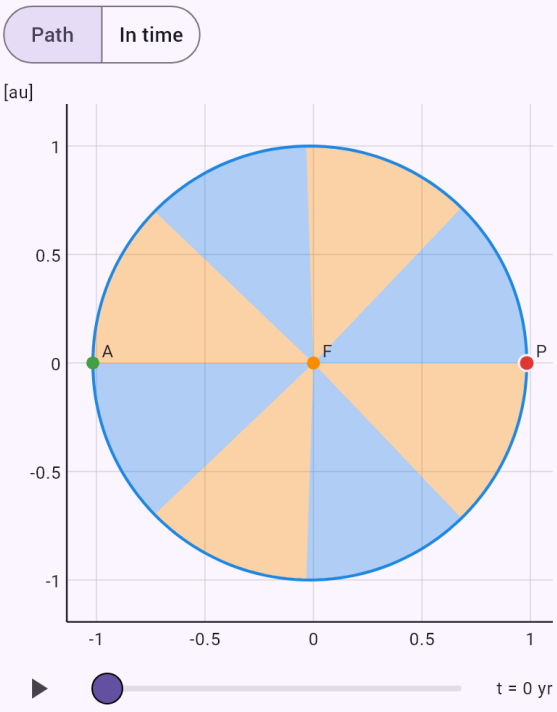
Result

$$v_a \approx 29.29\,\text{km/s}$$

Steps

- Find v_a; given: $r_p = 0.9833\,\text{au}, \quad r_a = 1.017\,\text{au}, \quad v_p = 30.29\,\text{km/s}$
- Kepler's second law: the line from the focus sweeps equal areas in equal times, because angular momentum is conserved. The areal velocity is constant, so a body is fastest at the perihelion: v_p·r_p = v_a·r_a. The coloured sectors in the graph took the same time and have the same area. The model's equation: $\dfrac{dS}{dt} = \dfrac{1}{2}r^2\dot{\theta} = \text{const.}, \quad v_p r_p = v_a r_a$
- Expressing the sought quantity: $v_a = v_p \dfrac{r_p}{r_a}$
- Substituting: $v_a = 30.29 \cdot \dfrac{0.9833}{1.017} \approx 29.29\,\text{km/s}$
- Ratio of the speeds: $\dfrac{v_p}{v_a} = \dfrac{r_a}{r_p} \approx 1.034$
- A body is fastest at perihelion and slowest at aphelion; this is why the Earth moves fastest in January.

Graph



The bodies at true scale. Use the slider or the play button to follow the motion.

Where Halley's comet is 10 years after perihelion (Newton's method).

Input: `model(orbit; find=theta[deg]; a=17.834[au]; e=0.96714[one]; T=75.32[yr]; t=10[yr]; mode=position; body=halley)`

Result

$\theta \approx 168\,^\circ$

Steps

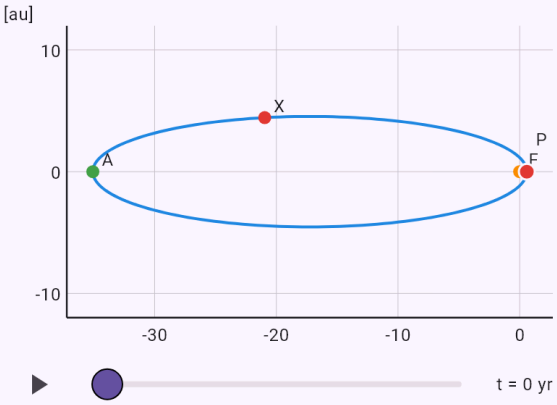
- Find θ; given
  $a = 17.83\,\text{au}, \quad e = 0.9671, \quad T = 75.32\,\text{yr}, \quad t = 10\,\text{y}$
- Where a body will be at time t follows from Kepler's equation M = E − e·sin E. The mean anomaly M grows evenly with time; the eccentric anomaly E cannot be expressed from it by a formula, so Newton's method finds it. The model's equation:
  $M = E - e\sin E, \quad M = \dfrac{2\pi t}{T}$
- Mean anomaly: $M = \dfrac{2\pi \cdot 10}{75.32} \approx 0.8342$
- Kepler's equation has no solution in formulas, so E is found numerically.
- Newton iteration: $E_{k+1} = E_k - \dfrac{E_k - e\sin E_k - M}{1 - e\cos E_k}$
- Step: $E_0 \approx 3.1415927$
- Step: $E_1 \approx 1.9686238$
- Step: $E_2 \approx 1.7919915$
- Step: $E_3 \approx 1.7802638$
- Step: $E_4 \approx 1.7802097$
- Step: $E_5 \approx 1.7802097$
- Step: $E_6 \approx 1.7802097$
- Eccentric anomaly: $E \approx 1.78021$
- True anomaly

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}, \quad \theta \approx 168^\circ$$

- Distance from the focus: $r = a(1 - e\cos E) \approx 21.42\,\text{au}$

Graph

Path | In time



The bodies at true scale. Use the slider or the play button to follow the motion.

The altitude of the geostationary orbit from Kepler's third law.

Input: `model(orbit; find=h[km]; T=86164[s]; M=5.972e24[kg]; R=6378[km]; mode=period;`

`central=earth)`

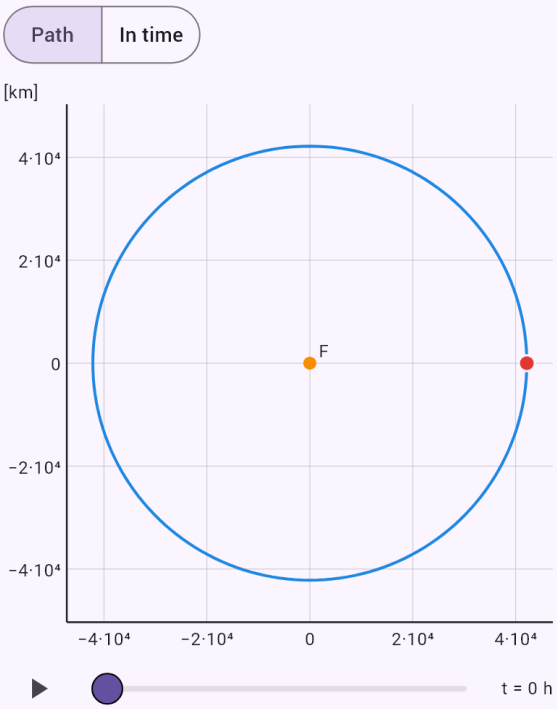
Result

$h \approx 35\,790\,\text{km}$

Steps

- Find h; given
  $T = 86\,160\,\text{s}, \quad M = 5.972 \cdot 10^{24}\,\text{kg}, \quad R = 6378\,\text{km}$
- Converting the units: $R = 6378\,\text{km} = 6\,378\,000\,\text{m}$
- Kepler's third law: the ratio T²/a³ depends only on the mass of the central body. A moon's orbit therefore gives the planet's mass, and a required period gives the orbit's altitude, such as the geostationary one. The model's equation: $\frac{T^2}{a^3} = \frac{4\pi^2}{GM}$
- Expressing the sought quantity: $a = \sqrt[3]{\frac{GMT^2}{4\pi^2}}$
- Substituting
  $a = \sqrt[3]{\frac{6.674 \cdot 10^{-11} \cdot 5.972 \cdot 10^{24} \cdot (86\,160)^2}{4\pi^2}} \approx 4.216 \cdot$
- Expressing the sought quantity: $h = a - R$
- Substituting:
  $h = 4.216 \cdot 10^7 - 6\,378\,000 \approx 3.579 \cdot 10^7\,\text{m}$
- This holds for an orbiting mass negligible next to M; otherwise M + m belongs in it.
- Converting the units: $h \approx 3.579 \cdot 10^7\,\text{m} = 35\,790\,\text{km}$

Graph



The bodies at true scale. Use the slider or the play button to follow the motion.

The speed of Halley's comet at perihelion from the vis-viva equation.

Input: `model(orbit; find=v[km_s]; r=0.586[au]; a=17.834[au]; M=1.989e30[kg]; mode=visviva;`

`central=sun)`

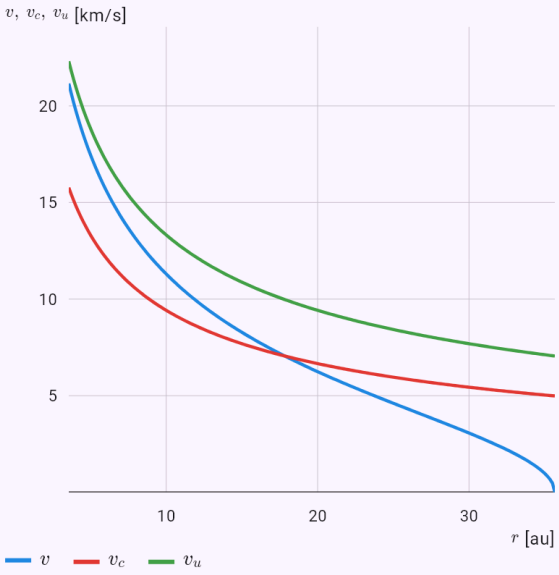
Result

$$v \approx 54.58\,\mathrm{km/s}$$

Steps

- Find v; given
  $r = 0.586\,\mathrm{au}, \quad a = 17.83\,\mathrm{au}, \quad M = 1.989 \cdot 10^{30}\,\mathrm{kg}$
- Converting the units
  $r = 0.586\,\mathrm{au} = 8.766 \cdot 10^{10}\,\mathrm{m}, \quad a = 17.83\,\mathrm{au} = 2.668$
- The vis-viva law v² = GM(2/r − 1/a) follows from conservation of energy and gives the speed anywhere on the orbit. The circular speed keeps a body on a circle; the escape speed is √2 times greater and the body never returns. The model's equation: $v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$
- Substituting
  $v = \sqrt{6.674 \cdot 10^{-11} \cdot 1.989 \cdot 10^{30}\left(\frac{2}{8.766 \cdot 10^{10}} - \frac{}{2.668}\right.}$
- Circular speed at this distance:
  $v_c = \sqrt{\frac{GM}{r}} \approx 38\,910\,\mathrm{m/s}$
- Escape speed at this distance:
  $v_u = \sqrt{\frac{2GM}{r}} \approx 55\,030\,\mathrm{m/s}$
- Converting the units: $v \approx 54\,580\,\mathrm{m/s} = 54.58\,\mathrm{km/s}$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

The barycentre of the Sun and Jupiter lies just above the surface of the Sun; “Detail at the barycentre” shows the Sun circling it.

Input: `model(twobody; find=rb[km]; M=1.989e30[kg]; m=1.898e27[kg]; a=778500000[km];`

R=696000[km]; mode=barycenter; system=sunJupiter)

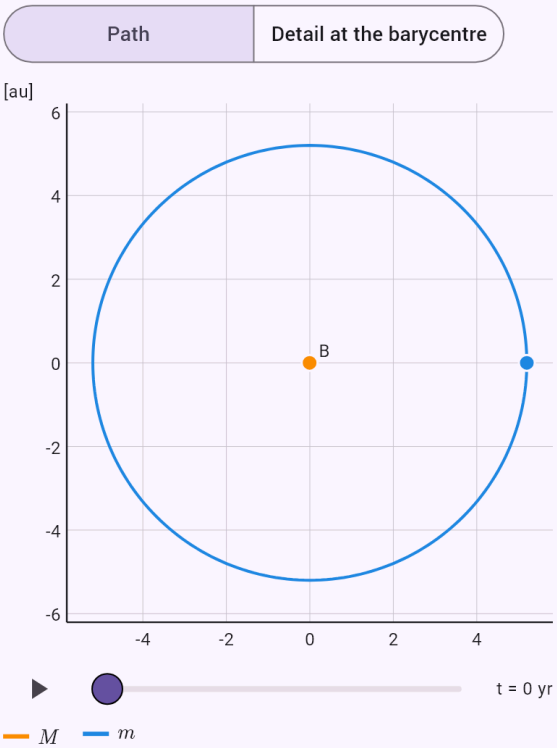
Result

$r_b \approx 742\,200\,\text{km}$

Steps

- Find r_b; given
  $M = 1.989 \cdot 10^{30}\,\text{kg}, \quad m = 1.898 \cdot 10^{27}\,\text{kg}, \quad a = 7.78$
- Converting the units
  $a = 7.785 \cdot 10^{8}\,\text{km} = 7.785 \cdot 10^{11}\,\text{m}, \quad R = 696\,000\,\text{km}$
- In reality both bodies orbit their common centre of mass (barycentre). The two-body problem reduces to the motion of one body with the reduced mass, and Kepler's third law contains the sum M + m. A star's wobble around the barycentre reveals exoplanets. The model's equation
  $r_b = \frac{am}{M+m}, \quad \mu = \frac{Mm}{M+m}, \quad T^2 = \frac{4\pi^2 a^3}{G(M+m)}$
- Substituting
  $r_b = \frac{7.785 \cdot 10^{11} \cdot 1.898 \cdot 10^{27}}{1.989 \cdot 10^{30} + 1.898 \cdot 10^{27}} \approx 7.422 \cdot 10^{8}\,\text{m}$
- Distance of the other body from the barycentre:
  $a - r_b \approx 7.778 \cdot 10^{11}\,\text{m}$
- The barycentre lies outside the body: $\frac{r_b}{R} \approx 1.066$. The larger body itself traces a visible circle.
- Converting the units: $r_b \approx 7.422 \cdot 10^{8}\,\text{m} = 742\,200\,\text{km}$

Graph



The bodies at true scale. Use the slider or the play button to follow the motion.

The distance of L2, where the James Webb telescope works.

Input: `model(twobody; find=rL[Mkm]; M=1.989e30[kg]; m=6.05e24[kg]; a=149.6[Mkm];`

`mode=lagrange; point=L2; system=sunEarth)`

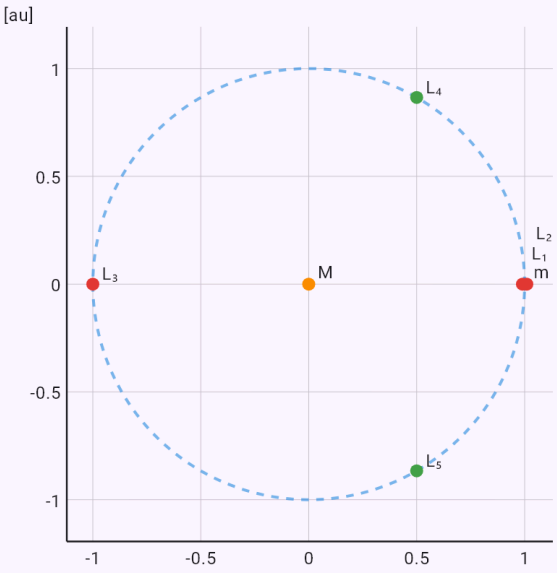
Result

$$r_{L_2} \approx 1.508 \text{ million km}$$

Steps

- Find r_{L_2}; given
  $M = 1.989 \cdot 10^{30}\,\text{kg}, \quad m = 6.05 \cdot 10^{24}\,\text{kg}, \quad a = 149.6$
- In a frame rotating with the body m there are five places where the gravity of both bodies and the centrifugal force balance. L1, L2 and L3 lie on the line through the bodies and are found numerically; L4 and L5 form equilateral triangles with the two bodies. The model's equation
  $$x = \frac{(1-\mu)(x+\mu)}{|x+\mu|^3} + \frac{\mu(x-1+\mu)}{|x-1+\mu|^3}$$
- Mass ratio: $\mu = \dfrac{m}{M+m} \approx 3.042 \cdot 10^{-6}$
- Approximately for small m:
  $r_{L_2} \approx a\sqrt[3]{\dfrac{m}{3M}} \approx 1.503 \text{ million km}$
- The Lagrange point is a root of the equation on the line through the bodies, found by bisection.
- Exactly from the equation: $r_{L_2} \approx 1.508 \text{ million km}$
- L1, L2 and L3 are unstable: a body has to keep its place by corrections, like the James Webb telescope at L2.
- L4 and L5 are stable because M/m > 25; Jupiter keeps its Trojans there.

Graph



The bodies and the notable points at true scale.

The tidal acceleration caused by the Moon.

Input: `model(twobody; find=at[m_s2]; M=7.342e22[kg]; R=6371[km]; d=384400[km]; mode=tide;`

`tideSource=moon)`

Result

$$a_s \approx 1.099 \cdot 10^{-6}\,\mathrm{m/s^2}$$

Steps

- Find a_s; given
  $M = 7.342 \cdot 10^{22}\,\mathrm{kg}, \quad R = 6371\,\mathrm{km}, \quad d = 384\,400\,\mathrm{kr}$
- Converting the units
  $R = 6371\,\mathrm{km} = 6\,371\,000\,\mathrm{m}, \quad d = 384\,400\,\mathrm{km} = 3.84$
- The near side of a body is pulled harder than the far side. The tidal acceleration 2GMR/d³ falls with the cube of the distance, which is why the Moon matters more than the far more massive Sun. The model's equation:
  $a_s \approx \dfrac{2GMR}{d^3}$
- Substituting
  $a_s = \dfrac{2 \cdot 6.674 \cdot 10^{-11} \cdot 7.342 \cdot 10^{22} \cdot 6\,371\,000}{\left(3.844 \cdot 10^{8}\right)^3} \approx 1.099$
- Exactly, the difference of gravity on the near side and at the centre
  $\dfrac{GM}{(d-R)^2} - \dfrac{GM}{d^2} \approx 1.127 \cdot 10^{-6}\,\mathrm{m/s^2}$
- The attracting body raises two tidal bulges, on the near side and on the opposite one, so there are two high tides per rotation.
- Tidal friction slows rotation: the Moon always shows the Earth the same side and recedes by about 3.8 cm a year.

The resonance of the moons Io, Europa and Ganymede.

Input: `model(twobody; find=ratio; Ts=1.769 3.551 7.155[d]; mode=resonance)`

Result

$$T_1 : T_2 : T_3 \approx 1 : 2 : 4$$

Steps

- Given: $T = (1.769;\ 3.551;\ 7.155)\,\mathrm{d}$
- An orbital resonance occurs when periods are in a ratio of small whole numbers; the bodies then meet regularly and influence each other. Examples: Io, Europa and Ganymede 1 : 2 : 4, Neptune and Pluto 2 : 3. The Kirkwood gaps in the asteroid belt lie at resonances with Jupiter. The model's equation:
  $T_1 : T_2 : \ldots \approx p_1 : p_2 : \ldots$
- Largest deviation from a whole-number ratio: $\delta \approx 1.1\,\%$
- The periods are in a ratio of small whole numbers: an orbital resonance.

The precession of the perihelion of Mercury according to general relativity.

Input: `model(twobody; find=dphi[arcsec_cy]; M=1.989e30[kg]; a=57909000[km]; e=0.2056[one];`

`mode=precession)`

Result

$$\Delta\varphi \approx 43\,''/100\ \mathrm{yr}$$

Steps

- Find $\Delta\varphi$; given
  $M = 1.989 \cdot 10^{30}\,\mathrm{kg}, \quad a = 5.791 \cdot 10^{7}\,\mathrm{km}, \quad e = 0.205$
- Converting the units:
  $a = 5.791 \cdot 10^{7}\,\mathrm{km} = 5.791 \cdot 10^{10}\,\mathrm{m}$
- In general relativity the perihelion of an orbit slowly turns by 6πGM/(c²a(1 − e²)) per revolution. For Mercury this is 43" per century, the difference Newtonian mechanics could not explain. The model's equation:
  $\Delta\varphi = \dfrac{6\pi GM}{c^{2}a\left(1-e^{2}\right)}$
- Turn per revolution
  $\Delta\varphi_{1} = \dfrac{6\pi \cdot 6.674 \cdot 10^{-11} \cdot 1.989 \cdot 10^{30}}{\left(2.998 \cdot 10^{8}\right)^{2} \cdot 5.791 \cdot 10^{10}\left(1-(0.2056)^{2}\right)} \approx$
- Revolutions in 100 years: $N = \dfrac{100\,\mathrm{yr}}{T} \approx 415.3$
- Substituting: $\Delta\varphi = N\Delta\varphi_{1} \approx 43\,''/100\ \mathrm{yr}$
- Newtonian mechanics predicts less turning for Mercury; the remaining 43" per century was explained only by general relativity.

The periodic figure-eight solution of three equal bodies.

Input: `model(threebody; find=E; tEnd=6.32591398[one]; scenario=figure8)`

Result

$$E \approx -1.28714$$

Steps

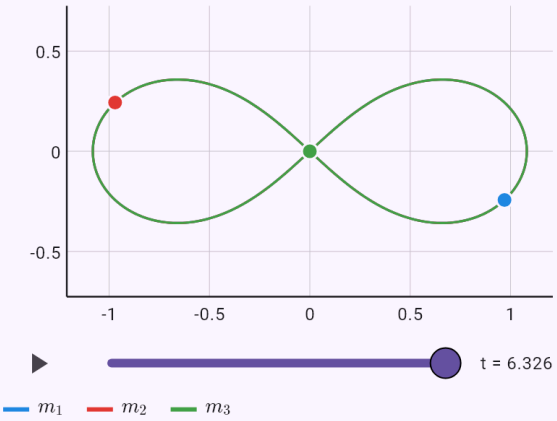
- Find E; given: $t_k = 6.326$
- Three bodies generally have no solution in formulas. There are exceptional periodic orbits, though: in the figure-eight solution (Chenciner and Montgomery, 2000) three equal bodies chase each other along a single curve shaped like an 8. The model's equation

$$\ddot{\vec{r}}_i = \sum_{j \neq i} \frac{m_j(\vec{r}_j - \vec{r}_i)}{|\vec{r}_j - \vec{r}_i|^3}, \quad G = 1$$

- In general three bodies have no solution in formulas, so the motion is computed numerically, in units where G = 1.
- Number of computation steps: $N = 2748$
- Total energy: $E = \sum_i \frac{m_i v_i^2}{2} - \sum_{i<j} \frac{m_i m_j}{r_{ij}} \approx -1.28714$
- Relative change of energy (accuracy check): $\left|\frac{\Delta E}{E}\right| \approx 1.6 \cdot 10^{-12}$
- Three equal bodies travel along one curve shaped like an 8 and return to their places after a period.

- Largest distance from the start after one period: $\max_i |\vec{r}_i(T) - \vec{r}_i(0)| \approx 4.1 \cdot 10^{-8}$

Graph



The bodies at true scale. Use the slider or the play button to follow the motion.

Chaos: when two nearly identical runs separate.

Input: `model(threebody; find=ts; tEnd=60[one]; delta=0.001[one]; scenario=chaos)`

Result

$t_s \approx 42.9$

Steps

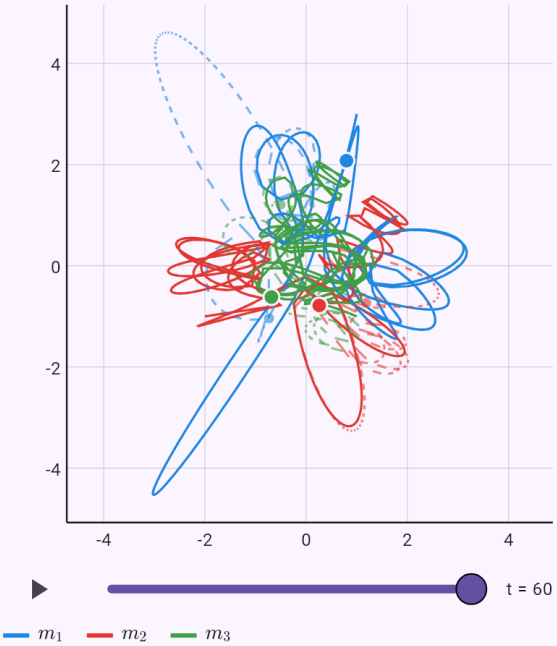
- Find t_s; given: $t_k = 60, \quad \delta = 0.001$
- The three-body problem is chaotic: a tiny change at the start grows until the paths have nothing in common. The Pythagorean problem with masses 3, 4 and 5 is computed twice, the second time with a small offset δ. The model's equation

$$\ddot{\vec{r}}_i = \sum_{j \neq i} \frac{m_j(\vec{r}_j - \vec{r}_i)}{|\vec{r}_j - \vec{r}_i|^3}, \quad G = 1$$

- In general three bodies have no solution in formulas, so the motion is computed numerically, in units where G = 1.
- Number of computation steps: $N = 52\,307$
- Total energy: $E = \sum_i \frac{m_i v_i^2}{2} - \sum_{i<j} \frac{m_i m_j}{r_{ij}} \approx -12.8167$
- Relative change of energy (accuracy check): $\left|\frac{\Delta E}{E}\right| \approx 1.8 \cdot 10^{-8}$
- A second run offset by $\delta = 0.001$ is computed as well. The dashed lines are that run; their separation shows chaos. A straight line of the logarithm in time means exponential growth.
- At time t_s the position of some body differs by more than 1 between the two runs.

Graph

Path | In time



The solid and the dashed lines are two runs with a very slightly different start.

The activity of 1 g of radium, about one curie.

Input: `model(nuclear; find=A[Bq]; m=1[g]; Mm=226[g_mol]; T12=1600[yr]; mode=activity;`

`isotope=ra226)`

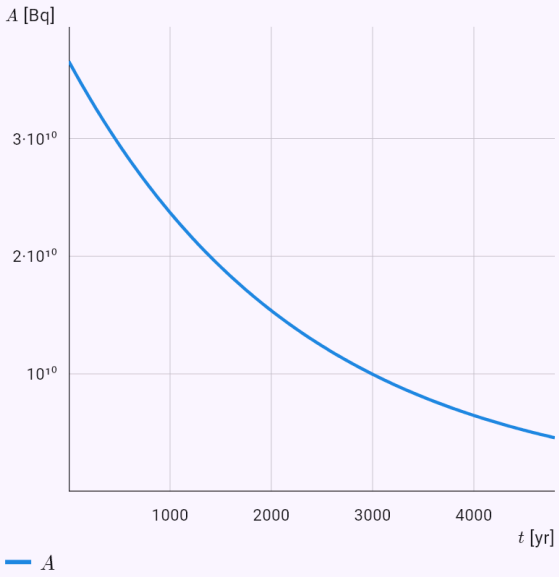
Result

$$A \approx 3.658 \cdot 10^{10}\,\mathrm{Bq}$$

Steps

- Find A; given:
  $m = 1\,\mathrm{g}, \quad M = 226\,\mathrm{g/mol}, \quad T_{1/2} = 1600\,\mathrm{yr}$
- Converting the units: $T_{1/2} = 1600\,\mathrm{yr} = 5.049 \cdot 10^{10}\,\mathrm{s}$
- The activity A = λN says how many nuclei decay per second. The number of nuclei N follows from the mass, the molar mass and Avogadro's constant, the decay constant λ from the half-life. The model's equation:
  $A = \lambda N, \quad \lambda = \frac{\ln 2}{T_{1/2}}, \quad N = \frac{m}{M} N_A$
- In a half-life the amount halves – the decay constant:
  $\lambda = \frac{\ln 2}{T_{1/2}} \approx 1.373 \cdot 10^{-11}\,\mathrm{s^{-1}}$
- Number of nuclei: $N = \frac{1}{226} \cdot 6.022 \cdot 10^{23} \approx 2.665 \cdot 10^{21}$
- Substituting:
  $A = 1.373 \cdot 10^{-11} \cdot 2.665 \cdot 10^{21} \approx 3.658 \cdot 10^{10}\,\mathrm{Bq}$
- A becquerel is one decay per second. The historical unit curie is about the activity of 1 g of radium, 3.7·10¹⁰ Bq.

Graph



How the quantities change in time.

The ratio of activities in a Mo-99 / Tc-99m generator used in hospitals.

Input: `model(nuclear; find=ratio; A0=100[GBq]; TA=66[h]; TB=6.01[h]; mode=equilibrium;`

`generator=moTc)`

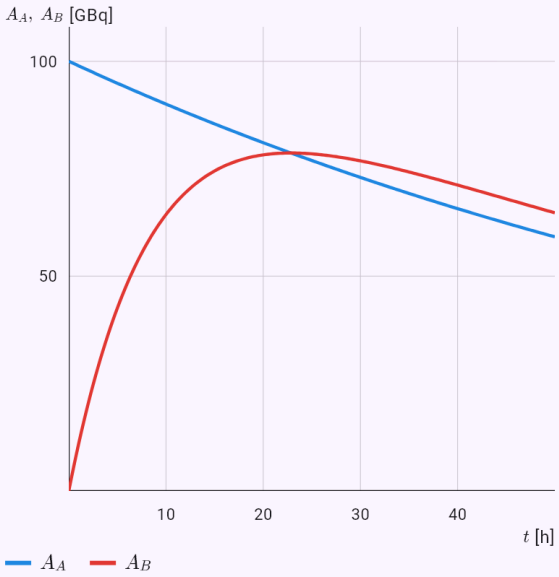
Result

$$\frac{A_B}{A_A} \approx 1.1$$

Steps

- Find $\frac{A_B}{A_A}$; given:
  $A_{A,0} = 100\,\text{GBq}, \quad T_A = 66\,\text{h}, \quad T_B = 6.01\,\text{h}$
- When a short-lived daughter arises from a long-lived parent, its activity catches up with the parent and then follows it in a fixed ratio. Hospital radionuclide generators are based on this. The model's equation
  $A_B(t) = A_{A,0}\frac{\lambda_B}{\lambda_B - \lambda_A}\left(e^{-\lambda_A t} - e^{-\lambda_B t}\right)$
- In a half-life the amount halves – the decay constant:
  $\lambda_A \approx 0.0105, \quad \lambda_B \approx 0.1153$
- Expressing the sought quantity:
  $\frac{A_B}{A_A} = \frac{\lambda_B}{\lambda_B - \lambda_A} = \frac{T_A}{T_A - T_B}$
- Substituting: $\frac{A_B}{A_A} = \frac{66}{66 - 6.01} \approx 1.1$
- Transient equilibrium: after several half-lives of the daughter the ratio of activities is constant.

Graph



How the quantities change in time.

A three-member decay series by Bateman's equations.

Input: `model(nuclear; find=N[pct]; N0=100[pct]; Ts=10 2 5[h]; t=4[h]; mode=bateman)`

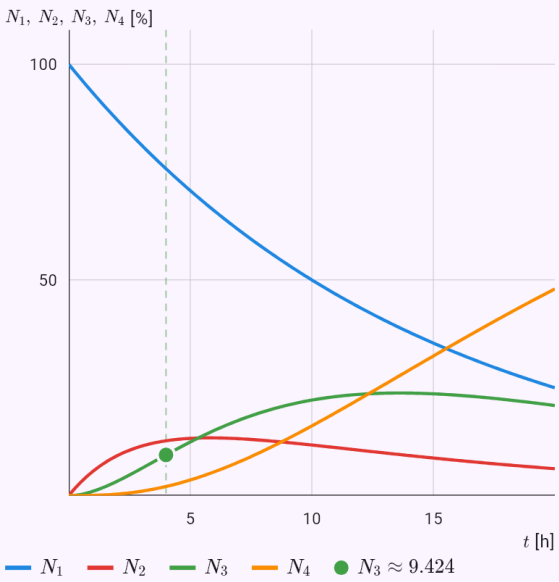
Result

$$N_n \approx 9.424\,\%$$

Steps

- Find N_n; given:
  $N_0 = 100\,\%, \quad T_i = (10;\ 2;\ 5)\,\text{h}, \quad t = 4\,\text{h}$
- Bateman's equations describe a decay series: each member arises from the previous one and decays into the next. It is a system of linear differential equations with a closed-form solution. The model's equation:
  $N_k' = \lambda_{k-1} N_{k-1} - \lambda_k N_k$
- In a half-life the amount halves – the decay constant
  $\lambda_1 \approx 0.06931, \quad \lambda_2 \approx 0.3466, \quad \lambda_3 \approx 0.1386$
- Expressing the sought quantity
  $N_n = N_0 \prod_{i=1}^{n-1} \lambda_i \sum_{j=1}^{n} \frac{e^{-\lambda_j t}}{\prod_{l \neq j} (\lambda_l - \lambda_j)}$
- Substituting: $N_3(4) \approx 9.424\,\%$
- The graph also shows the stable end product; the sum of all members is constant.

Graph



The model's quantities in time, computed from the solution; the answer is marked.

The binding energy per nucleon of iron-56.

Input: `model(nuclear; find=BA[MeV]; Z=26[one]; Anum=56[one]; Ma=55.93493633[u]; mode=binding; nuclide=fe56)`

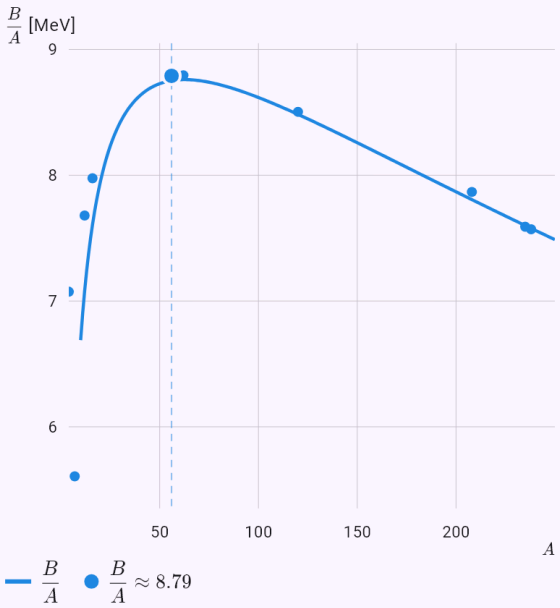
Result

$$\frac{B}{A} \approx 8.7903\,\mathrm{MeV}$$

Steps

- Find $\frac{B}{A}$; given: $Z = 26, \quad A = 56, \quad M_a = 55.93\,\mathrm{u}$
- A nucleus weighs less than its free protons and neutrons. By E = mc², the difference Δm corresponds to the binding energy. Its curve per nucleon peaks near iron, which is why both the fission of heavy nuclei and the fusion of light ones release energy. The model's equation:
  $\Delta m = Z m_H + (A - Z) m_n - M_a, \quad B = \Delta m\, c^2$
- Mass defect
  $\Delta m = 26 \cdot 1.00782503 + 30 \cdot 1.00866492 - 55.934936$
- Energy equivalent: $1\,\mathrm{u} \cdot c^2 = 931.494\,\mathrm{MeV}$
- Substituting: $B \approx 0.528462 \cdot 931.494 \approx 492.26\,\mathrm{MeV}$
- Binding energy per nucleon: $\frac{B}{A} \approx 8.79\,\mathrm{MeV}$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

The energy of the fission of uranium-235.

Input: `model(nuclear; find=Q[MeV]; mode=reaction; reaction=u235ba)`

Result

$$Q \approx 173.3\,\mathrm{MeV}$$

Steps

- The energy of a reaction Q is the difference of the masses before and after, converted by E = mc². A positive Q means energy is released. The model's equation: $Q = \left(\sum m_r - \sum m_p\right) c^2$
- Reaction: $^{235}\mathrm{U} + n \rightarrow {}^{141}\mathrm{Ba} + {}^{92}\mathrm{Kr} + 3n$
- Sum of the masses:
  $\sum m_r \approx 236.0526\,\mathrm{u}, \quad \sum m_p \approx 235.86657\,\mathrm{u}$
- Substituting: $Q \approx 0.18602 \cdot 931.494 \approx 173.3\,\mathrm{MeV}$
- This energy is released at once; with the decays of the fission products it is about 200 MeV per fission.

How much coal 1 g of uranium-235 replaces.

Input: `model(nuclear; find=coal[kg]; m=1[g]; Ef=200[MeV]; Mm=235[g_mol]; q=29[MJ_kg];`

`mode=fuel)`

Result

$$m_c \approx 2832\,\text{kg}$$

Steps

- Find m_c; given
  $m = 1\,\text{g}, \quad E_f = 200\,\text{MeV}, \quad M = 235\,\text{g/mol}, \quad q = 2$
- The fuel's mass gives the number of nuclei, and the energy of one fission gives the total energy, which can be compared with burning coal. The model's equation: $E = \frac{m}{M} N_A E_f$
- Number of nuclei: $N = \frac{1}{235} \cdot 6.022 \cdot 10^{23} \approx 2.563 \cdot 10^{21}$
- Substituting
  $E = 2.563 \cdot 10^{21} \cdot 200 \cdot 1.602 \cdot 10^{-13} \approx 8.212 \cdot 10^{10}\,\text{J}$
- Substituting: $m_c = \frac{E}{q} = \frac{8.212 \cdot 10^{10}}{2.9 \cdot 10^{7}} \approx 2832\,\text{kg}$
- An ideal calculation: a real reactor uses only a few per cent of its fuel and turns about a third of the heat into electricity.

The number of neutrons after 10 generations with k = 1.05.

Input: `model(reactor; find=N; k=1.05[one]; N0=1000[one]; n=10[one]; mode=generations)`

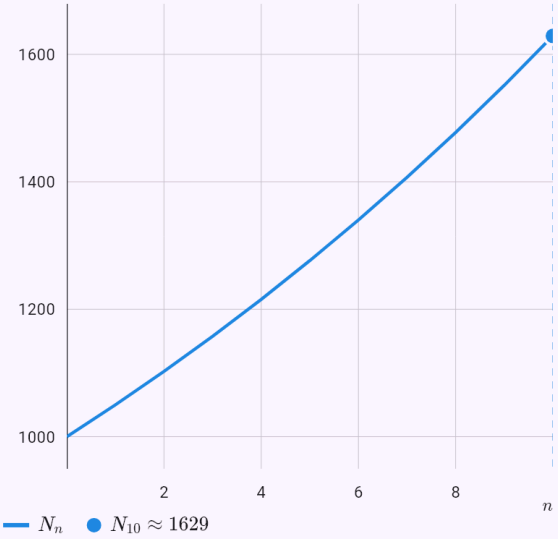
Result

$$N_n \approx 1629$$

Steps

- Find N_n; given: $k = 1.05, \quad N_0 = 1000, \quad n = 10$
- The multiplication factor k says how many neutrons of the next generation one neutron causes. The neutron count is the geometric sequence $N_0 k^n$: with k < 1 the reaction dies out, with k = 1 it runs steadily and with k > 1 it grows. The model's equation: $N_n = N_0 k^n$
- Supercritical state: k > 1, the power grows. A reactor enters this state only briefly and very slightly, when its power is to be raised.
- Substituting: $N_n = 1000 \cdot (1.05)^{10} \approx 1629$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

The sum of all generations in the subcritical state.

Input: `model(reactor; find=Sinf; k=0.9[one]; N0=1000[one]; mode=generations)`

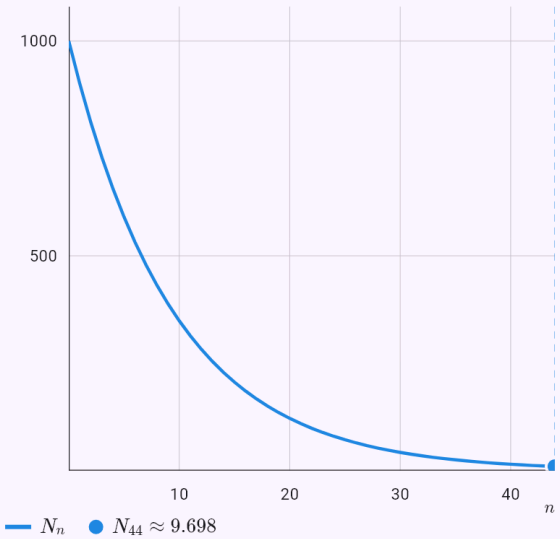
Result

$$S \approx 10\,000$$

Steps

- Find S; given: $k = 0.9, \quad N_0 = 1000$
- The multiplication factor k says how many neutrons of the next generation one neutron causes. The neutron count is the geometric sequence N_0k^n: with k < 1 the reaction dies out, with k = 1 it runs steadily and with k > 1 it grows. The model's equation: $N_n = N_0 k^n$
- Subcritical state: k < 1, the reaction dies out.
- Expressing the sought quantity: $S = \sum_{n=0}^{\infty} N_0 k^n = \frac{N_0}{1-k}$
- Substituting: $S = \frac{1000}{1 - 0.9} \approx 10\,000$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

The reactor period with delayed neutrons.

Input: `model(reactor; find=T[s]; k=1.001[one]; l=0.0001[s]; beta=0.0065[one]; tau=13[s]; mode=kinetics)`

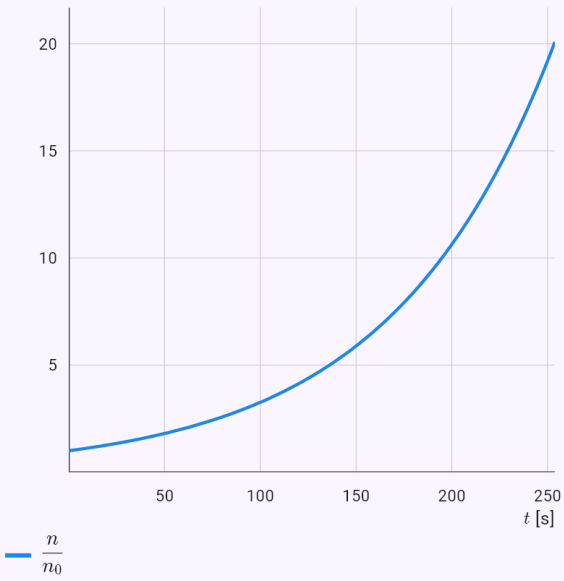
Result

$$T \approx 84.6\,\text{s}$$

Steps

- Find T; given
  $k = 1.001, \quad \ell = 10^{-4}\,\text{s}, \quad \beta = 0.0065, \quad \tau = 13\,\text{s}$
- A reactor's power changes exponentially with the period T = ℓ/(k − 1). Delayed neutrons lengthen the mean generation time about a thousand times, which is what makes a reactor controllable. The model's equation:
  $n(t) = n_0 e^{t/T}, \quad T = \frac{\ell}{k-1}$
- Reactivity: $\rho = \frac{k-1}{k} \approx 9.99 \cdot 10^{-4}$
- Substituting: $T_p = \frac{10^{-4}}{1.001 - 1} \approx 0.1\,\text{s}$
- Mean generation time with delayed neutrons:
  $\ell_{ef} = (1-\beta)\ell + \beta\tau \approx 0.0846\,\text{s}$
- Substituting: $T = \frac{0.0846}{1.001 - 1} \approx 84.6\,\text{s}$
- Without delayed neutrons the period would be a fraction of a second. Thanks to them, control rods have time to respond to changes of power.
- The power grows with this period.

Graph



How the quantities change in time.

The radius of a spherical graphite core.

Input: `model(reactor; find=R[m]; kinf=1.07[one]; M2=650[cm2]; mode=core; lattice=graphite; shape=sphere)`

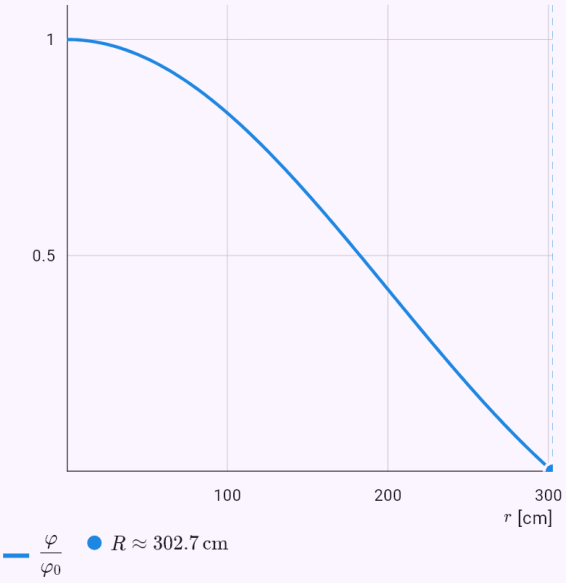
Result

$$R \approx 3.027\,\text{m}$$

Steps

- Find R; given: $k_\infty = 1.07, \quad M^2 = 650\,\text{cm}^2$
- In the one-group diffusion model the shape of the neutron flux must satisfy ∇²φ + B²φ = 0. The size of the reactor core follows from the flux falling to zero at the edge. It is an eigenvalue problem, meant for the moderated lattices of power reactors. The model's equation: $\nabla^2\varphi + B^2\varphi = 0, \quad B^2 = \dfrac{k_\infty - 1}{M^2}$
- Buckling: $B^2 = \dfrac{1.07 - 1}{650} \approx 1.077 \cdot 10^{-4}\,\text{cm}^{-2}$
- Equation with spherical symmetry: $\varphi'' + \dfrac{2}{r}\varphi' + B^2\varphi = 0$
- Substitution: $u = r\varphi \Rightarrow u'' + B^2 u = 0 \Rightarrow \varphi = C\dfrac{\sin Br}{r}$
- Boundary condition: $\varphi(R) = 0 \Rightarrow BR = \pi$
- Substituting: $R = \dfrac{\pi}{B} = \dfrac{\pi}{\sqrt{1.077 \cdot 10^{-4}}} \approx 302.7\,\text{cm}$
- The one-group diffusion model without a reflector: real reactors have a reflector and are therefore smaller. A general shape leads to a partial differential equation.
- Converting the units: $R \approx 302.7\,\text{cm} = 3.027\,\text{m}$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

The edge of a cubic graphite core.

Input: `model(reactor; find=a[cm]; kinf=1.07[one]; M2=650[cm2]; mode=core; lattice=graphite;`

shape=cube)

Result

$$a \approx 524.3\,\text{cm}$$

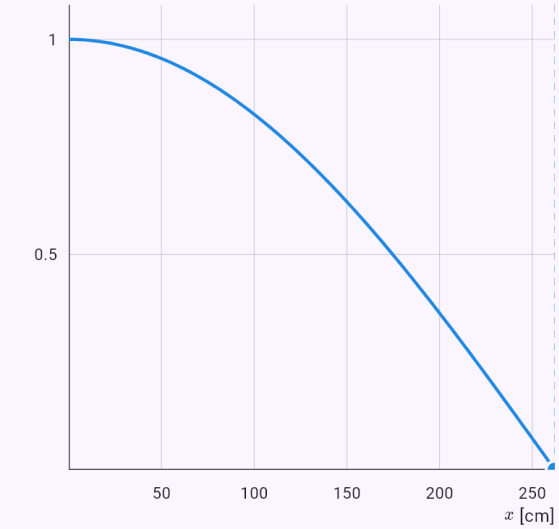
Steps

- Find a; given: $k_\infty = 1.07, \quad M^2 = 650\,\text{cm}^2$
- In the one-group diffusion model the shape of the neutron flux must satisfy ∇²φ + B²φ = 0. The size of the reactor core follows from the flux falling to zero at the edge. It is an eigenvalue problem, meant for the moderated lattices of power reactors. The model's equation: $\nabla^2\varphi + B^2\varphi = 0, \quad B^2 = \dfrac{k_\infty - 1}{M^2}$
- Buckling: $B^2 = \dfrac{1.07 - 1}{650} \approx 1.077 \cdot 10^{-4}\,\text{cm}^{-2}$
- Separation of variables in x, y, z

  $\varphi = \cos\dfrac{\pi x}{a}\cos\dfrac{\pi y}{a}\cos\dfrac{\pi z}{a}, \quad B^2 = 3\left(\dfrac{\pi}{a}\right)^2$
- Substituting

  $a = \dfrac{\sqrt{3}\,\pi}{B} = \dfrac{\sqrt{3}\,\pi}{\sqrt{1.077 \cdot 10^{-4}}} \approx 524.3\,\text{cm}$
- The one-group diffusion model without a reflector: real reactors have a reflector and are therefore smaller. A general shape leads to a partial differential equation.

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

A copper rod at 100 °C with its ends at 20 °C – the middle after 10 minutes.

Input: `model(heatrod; find=T[degC]; alpha=117[mm2_s]; L=50[cm]; T0=100[degC]; Te=20[degC];`

x=25[cm]; t=10[min]; mode=held; rodMaterial=copper)

Result

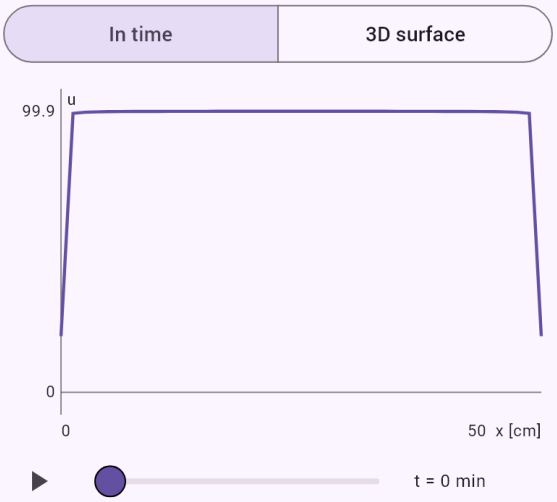
$$T \approx 26.37\,^\circ\mathrm{C}$$

Steps

- Find T; given
  $\alpha = 117\,\mathrm{mm}^2/\mathrm{s}, \quad L = 50\,\mathrm{cm}, \quad T_0 = 100\,^\circ\mathrm{C}, \quad T_e = $
- Converting the units
  $\alpha = 117\,\mathrm{mm}^2/\mathrm{s} = 1.17 \cdot 10^{-4}\,\mathrm{m}^2/\mathrm{s}, \quad L = 50\,\mathrm{cm} = 0.5$
- The temperature in a rod obeys the heat equation T_t = α·T_xx; the thermal diffusivity α = λ/(ρc) tells how fast the temperature evens out. At first the rod has the temperature T_0 everywhere; from t = 0 its ends are held at T_e (or one end is insulated). The solution by separation of variables is a Fourier series; each term decays exponentially, the first one slowest, with the time constant τ. Both ends are held at T_e; the middle of the rod keeps its temperature longest. The model's equation
  $\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}, \quad T(0,t) = T(L,t) = T_e, \quad T(x,0) = T_0$
- Solution by separation of variables (a Fourier series)
  $T = T_e + (T_0 - T_e) \sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi} e^{-(2k-1)^2 t/\tau} \sin \frac{(2k-}{}$
- Time constant of the slowest term
  $\tau = \frac{L^2}{\alpha \pi^2} = \frac{(0.5)^2}{1.17 \cdot 10^{-4} \cdot \pi^2} \approx 216.5\,\mathrm{s}$
- Expressing the sought quantity
  $T(x,t) = T_e + (T_0 - T_e) S, \quad S = \sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi} e^{-(2k-}$

- Sum of the series at the point: $S \approx 0.07967$
- Substituting:
  $T = 20 + (100 - 20) \cdot 0.07967 \approx 26.37\,^\circ\mathrm{C}$

Graph



The solution u at the time t chosen by the slider; the button plays it.

A steel rod with one insulated end – how long until 5 % of the temperature difference remains.

Input: model(heatrod; find=teq[min]; alpha=11.7[mm2_s]; L=30[cm]; T0=200[degC]; Te=20[degC];

p=5[pct]; mode=insulated; rodMaterial=steel)

Result

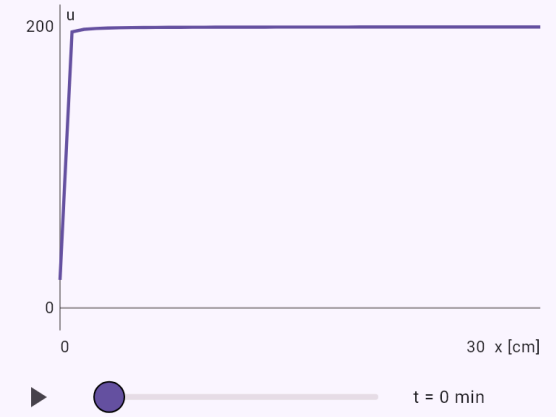
$$t \approx 168.2\,\text{min}$$

Steps

- Find t; given
  $\alpha = 11.7\,\text{mm}^2/\text{s}, \quad L = 30\,\text{cm}, \quad T_0 = 200\,^\circ\text{C}, \quad T_e =$
- Converting the units
  $\alpha = 11.7\,\text{mm}^2/\text{s} = 1.17 \cdot 10^{-5}\,\text{m}^2/\text{s}, \quad L = 30\,\text{cm} = 0.$
- The temperature in a rod obeys the heat equation T_t = α·T_xx; the thermal diffusivity α = λ/(ρc) tells how fast the temperature evens out. At first the rod has the temperature T_0 everywhere; from t = 0 its ends are held at T_e (or one end is insulated). The solution by separation of variables is a Fourier series; each term decays exponentially, the first one slowest, with the time constant τ. The end x = 0 is held at T_e and the end x = L is insulated (no heat passes through it) – the rod behaves like half of a rod twice as long; the insulated end keeps its temperature longest. The model's equation
  $\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}, \quad T(0,t) = T_e, \; \frac{\partial T}{\partial x}(L,t) = 0, \quad T(x,0)$
- Solution by separation of variables (a Fourier series)
  $T = T_e + (T_0 - T_e) \sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi} e^{-(2k-1)^2 t/\tau} \sin \frac{(2k-}{2}$
- Time constant of the slowest term
  $\tau = \frac{(2L)^2}{\alpha \pi^2} = \frac{(2 \cdot 0.3)^2}{1.17 \cdot 10^{-5} \cdot \pi^2} \approx 3118\,\text{s}$
- Expressing the sought quantity
  $\frac{T - T_e}{T_0 - T_e} \approx \frac{4}{\pi} e^{-t/\tau} = p \Rightarrow t \approx \tau \ln \frac{4}{\pi p}$
- Substituting: $t = 3118 \cdot \ln \frac{4}{\pi \cdot 0.05} \approx 10\,090\,\text{s}$
- The insulated end is the hottest. After a while the first term of the series dominates, so its decay is enough; the exact time comes from the whole series.
- Converting the units: $t \approx 10\,090\,\text{s} = 168.2\,\text{min}$

Graph

In time | 3D surface



The solution u at the time t chosen by the slider; the button plays it.

A copper plate 30 × 20 cm from 100 °C with its edges at 20 °C – the temperature in the middle after 2 minutes.

Input: `model(plate; find=T[degC]; alpha=117[mm2_s]; a=30[cm]; b=20[cm]; T0=100[degC];`

```
Te=20[degC]; x=15[cm]; y=10[cm]; t=2[min]; mode=held; rodMaterial=copper)
```

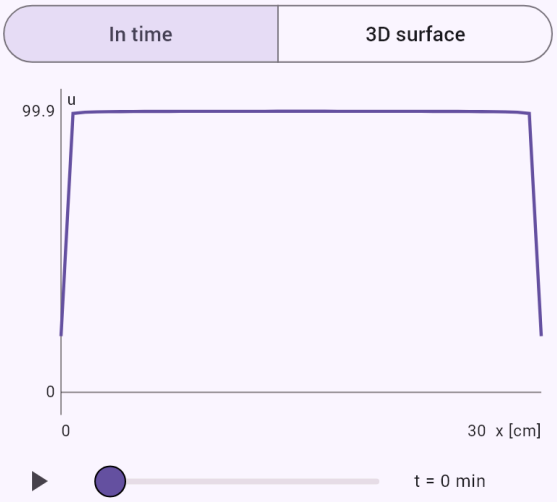
Result

$$T \approx 20.87\,^{\circ}\mathrm{C}$$

Steps

- Find T; given
  $\alpha = 117\,\mathrm{mm}^2/\mathrm{s}, \quad a = 30\,\mathrm{cm}, \quad b = 20\,\mathrm{cm}, \quad T_0 = 10($
- Converting the units
  $\alpha = 117\,\mathrm{mm}^2/\mathrm{s} = 1.17 \cdot 10^{-4}\,\mathrm{m}^2/\mathrm{s}, \quad a = 30\,\mathrm{cm} = 0.3$
- The temperature in a rectangular plate a × b follows the heat equation T_t = α(T_xx + T_yy). The plate starts at T_0 everywhere and from t = 0 its edges are held at T_e. The problem separates into two rods: the remaining share of the temperature difference is the product of the series for a rod of length a and one of length b – so a plate cools faster than either rod. The centre keeps its temperature longest. The model's equation
  $$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right), \quad T(0,y) = T(a,y) = T(x,0$$
- Separation into two rods – the product of two Fourier series
  $$\frac{T - T_e}{T_0 - T_e} = S_a(x,t)\,S_b(y,t), \quad S_L(x,t) = \sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi}$$
- The slowest term's time constant
  $$\tau = \frac{1}{\alpha\pi^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right)} = \frac{1}{1.17 \cdot 10^{-4} \cdot \pi^2\left(\frac{1}{0.3^2} + \frac{1}{0.2^2}\right)} \approx 2$$
- Expressing the sought quantity:
  $T(x,y,t) = T_e + (T_0 - T_e)S_a(x,t)\,S_b(y,t)$
- Both series' sums at the point:
  $S_a \approx 0.2731, \quad S_b \approx 0.03985$
- Substituting
  $T = 20 + (100 - 20) \cdot 0.2731 \cdot 0.03985 \approx 20.87\,^{\circ}\mathrm{C}$
- The graph shows the temperature along the line:
  $y = 10\,\mathrm{cm}$

Graph



The solution u at the time t chosen by the slider; the button plays it.

A concrete slab with one insulated edge – how many hours until 5 % of the temperature difference is left.

Input: `model(plate; find=teq[h]; alpha=0.69[mm2_s]; a=40[cm]; b=20[cm]; T0=60[degC];`

```
Te=20[degC]; p=5[pct]; mode=insulated; rodMaterial=concrete)
```

Result

$$t \approx 11.35\,\mathrm{h}$$

Steps

- Find t; given

  $\alpha = 0.69\,\mathrm{mm}^2/\mathrm{s}, \quad a = 40\,\mathrm{cm}, \quad b = 20\,\mathrm{cm}, \quad T_0 = 60$

- Converting the units

  $\alpha = 0.69\,\mathrm{mm}^2/\mathrm{s} = 6.9 \cdot 10^{-7}\,\mathrm{m}^2/\mathrm{s}, \quad a = 40\,\mathrm{cm} = 0.4$

- The temperature in a rectangular plate a × b follows the heat equation T_t = α(T_xx + T_yy). The plate starts at T_0 everywhere and from t = 0 three edges are held at T_e while the edge y = b is insulated (no heat passes through it). The problem separates into two rods; the insulated edge acts as the axis of symmetry of a plate twice as high. The middle of the insulated edge keeps its temperature longest. The model's equation

  $\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right), \quad T(0,y) = T(a,y) = T(x,0$

- Separation into two rods – the product of two Fourier series

  $\frac{T - T_e}{T_0 - T_e} = S_a(x,t)\,S_{2b}(y,t), \quad S_L(x,t) = \sum_{k=1}^{\infty} \frac{4}{(2k-1)}$

- The slowest term's time constant

  $\tau = \frac{1}{\alpha\pi^2\left(\frac{1}{a^2} + \frac{1}{(2b)^2}\right)} = \frac{1}{6.9 \cdot 10^{-7} \cdot \pi^2\left(\frac{1}{0.4^2} + \frac{1}{(2\cdot 0.2)^2}\right)}$

- Expressing the sought quantity

  $\frac{T - T_e}{T_0 - T_e} \approx \frac{16}{\pi^2} e^{-t/\tau} = p \Rightarrow t \approx \tau \ln \frac{16}{\pi^2 p}$

- Substituting: $t = 11\,750 \cdot \ln \frac{16}{\pi^2 \cdot 0.05} \approx 40\,870\,\mathrm{s}$

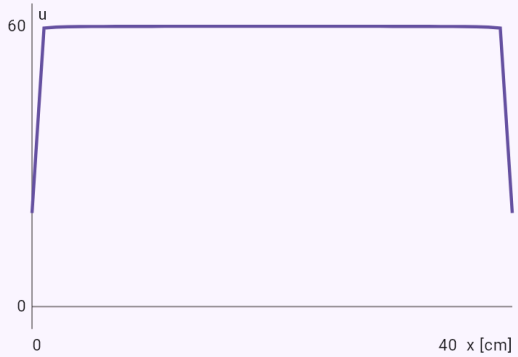
- The middle of the insulated edge is the hottest. After a while the product of both series' first terms (16/π²)·e^(−t/τ) dominates, so its decay is enough; the exact time comes from the whole series.

- The graph shows the temperature along the line: $y = 20\,\mathrm{cm}$

- Converting the units: $t \approx 40\,870\,\mathrm{s} = 11.35\,\mathrm{h}$

Graph

In time | 3D surface



t = 0 h

The solution u at the time t chosen by the slider; the button plays it.

A guitar's high E string – the fundamental frequency.

Input: `model(string; find=f1[Hz]; L=64.8[cm]; F=72[N]; mu=0.4[g_m])`

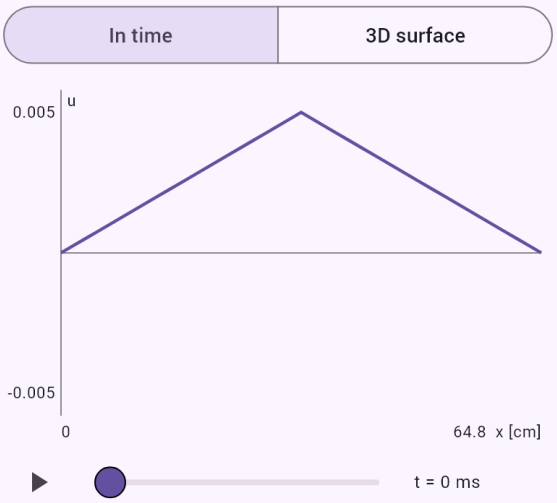
Result

$$f_1 \approx 327.4\,\mathrm{Hz}$$

Steps

- Find f_1; given:
  $L = 64.8\,\mathrm{cm}, \quad F = 72\,\mathrm{N}, \quad \mu = 0.4\,\mathrm{g/m}$
- Converting the units
  $L = 64.8\,\mathrm{cm} = 0.648\,\mathrm{m}, \quad \mu = 0.4\,\mathrm{g/m} = 4 \cdot 10^{-4}\,\mathrm{kg/m}$
- A stretched string with fixed ends obeys the wave equation u_tt = c²·u_xx; the wave speed c = √(F/μ) comes from the tension F and the linear density μ. Separation of variables gives standing waves sin(nπx/L) with the frequencies f_n = n·c/(2L). A plucked string has the shape of a triangle – it is expanded into a Fourier series and each harmonic vibrates at its own frequency. The model's equation
  $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(L,t) = 0$
- Wave speed: $c = \sqrt{\frac{F}{\mu}} = \sqrt{\frac{72}{4 \cdot 10^{-4}}} \approx 424.3\,\mathrm{m/s}$
- The string's modes (standing waves)
  $X_n = \sin\frac{n\pi x}{L}, \quad \omega_n = \frac{n\pi c}{L} \Rightarrow f_n = \frac{nc}{2L}$
- Substituting: $f_1 = \frac{424.3}{2 \cdot 0.648} \approx 327.4\,\mathrm{Hz}$
- The higher frequencies are whole multiples of the fundamental (harmonics) – that is why a string has a clear pitch.

Graph



The solution u at the time t chosen by the slider; the button plays it.

A plucked string – the displacement in the middle after 1 ms.

Input: `model(string; find=u[mm]; L=64.8[cm]; F=72[N]; mu=0.4[g_m]; p=10[cm]; h=3[mm];`

x=32.4[cm]; t=1[ms])

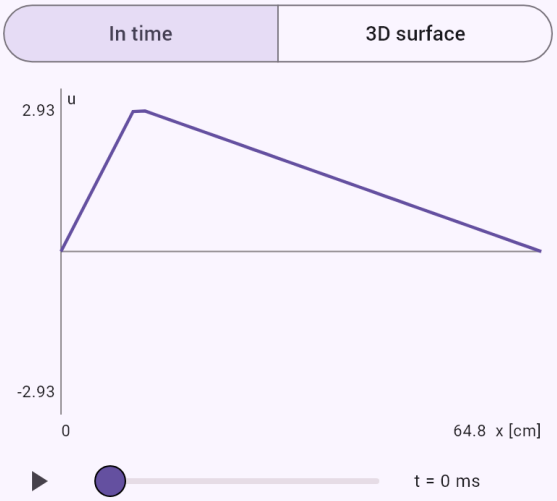
Result

$$u \approx -1.774\,\text{mm}$$

Steps

- Find u; given
  $L = 64.8\,\text{cm}, \quad F = 72\,\text{N}, \quad \mu = 0.4\,\text{g/m}, \quad p = 10\,\text{cm}$
- Converting the units
  $L = 64.8\,\text{cm} = 0.648\,\text{m}, \quad \mu = 0.4\,\text{g/m} = 4 \cdot 10^{-4}\,\text{kg/r}$
- A stretched string with fixed ends obeys the wave equation u_tt = c²·u_xx; the wave speed c = √(F/μ) comes from the tension F and the linear density μ. Separation of variables gives standing waves sin(nπx/L) with the frequencies f_n = n·c/(2L). A plucked string has the shape of a triangle – it is expanded into a Fourier series and each harmonic vibrates at its own frequency. The model's equation
  $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(L,t) = 0$$
- Wave speed: $c = \sqrt{\frac{F}{\mu}} = \sqrt{\frac{72}{4 \cdot 10^{-4}}} \approx 424.3\,\text{m/s}$
- The string's modes (standing waves)
  $$X_n = \sin\frac{n\pi x}{L}, \quad \omega_n = \frac{n\pi c}{L} \Rightarrow f_n = \frac{nc}{2L}$$
- A plucked string – a Fourier series
  $$b_n = \frac{2hL^2}{n^2\pi^2 p(L-p)} \sin\frac{n\pi p}{L}, \quad u = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \cos$$
- Substituting: $u(0.324, 0.001) \approx -0.001774\,\text{m}$
- After one period the string's shape returns:
  $$T = \frac{2L}{c} \approx 0.003055\,\text{s}$$
- Converting the units: $u \approx -0.001774\,\text{m} = -1.774\,\text{mm}$

Graph



The solution u at the time t chosen by the slider; the button plays it.

A drum – the frequency of the mode (1, 1) with one nodal diameter.

Input: model(membrane; find=fmn[Hz]; a=33[cm]; T=3000[N_m]; sigma=0.26[kg_m2]; m=1[one];

```
n=1[one])
```

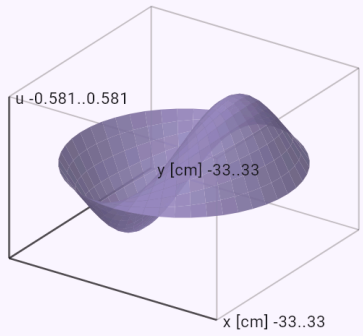
Result

$$f_{mn} \approx 198.5\,\text{Hz}$$

Steps

- Find f_{mn}; given
  $a = 33\,\text{cm}, \quad T = 3000\,\text{N/m}, \quad \sigma = 0.26\,\text{kg/m}^2, \quad m =$
- Converting the units: $a = 33\,\text{cm} = 0.33\,\text{m}$
- A circular membrane fixed at its rim (a drumhead) obeys the wave equation u_tt = c²·Δu with the speed c = √(T/σ). In polar coordinates the variables separate: the angular part is cos(mφ), the radial one a Bessel function J_m(kr). The condition at the rim J_m(ka) = 0 gives k = j_mn/a, where j_mn is the n-th root of J_m, and the frequencies f_mn = j_mn·c/(2πa). The mode (m, n) has m nodal diameters and n − 1 nodal circles. The model's equation
  $$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u, \quad c = \sqrt{\frac{T}{\sigma}}, \quad u(a, \varphi, t) = 0$$
- Wave speed in the membrane:
  $$c = \sqrt{\frac{T}{\sigma}} = \sqrt{\frac{3000}{0.26}} \approx 107.4\,\text{m/s}$$
- Separation in polar coordinates – Bessel's equation
  $u = R(r)\cos(m\varphi)\cos(\omega t): \quad r^2 R'' + rR' + \left(k^2 r^2 - \right.$
- The membrane's rim is fixed:
  $$J_1(ka) = 0 \Rightarrow k = \frac{j_{11}}{a}, \quad j_{11} \approx 3.83171$$
- Substituting: $f_{11} = \dfrac{3.832 \cdot 107.4}{2\pi \cdot 0.33} \approx 198.5\,\text{Hz}$
- The higher modes aren't whole multiples of the fundamental. That is why a drum has no pitch as clear as a string's.
  $$\frac{f_{11}}{f_{01}} \approx 1.59, \quad \frac{f_{21}}{f_{01}} \approx 2.14, \quad \frac{f_{02}}{f_{01}} \approx 2.3$$

Graph



The solution u as a surface over the domain. Turn it by dragging a finger.

Range of a throw in a vacuum.

Input: `model(ballistics; find=R[m]; v0=20[m_s]; alpha=45[deg]; mode=vacuum)`

Result

$$R \approx 40.77\,\mathrm{m}$$

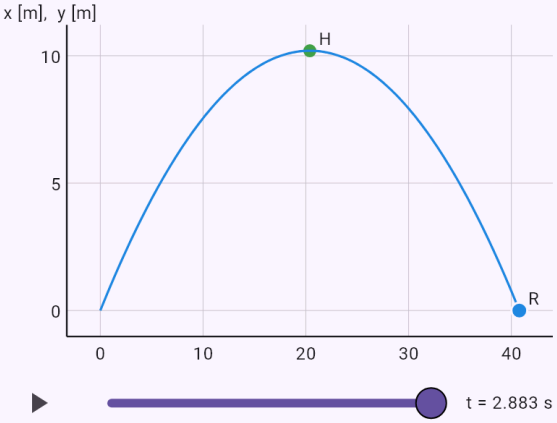
Steps

- Find R; given: $v_0 = 20\,\mathrm{m/s}, \quad \alpha = 45^\circ$
- Without air drag the motion splits into two independent ones: uniform horizontally and uniformly accelerated vertically. The path is a parabola and every value has an exact formula. The model's equation:
  $$x = v_0 t\cos\alpha, \quad y = h_0 + v_0 t\sin\alpha - \frac{1}{2}gt^2$$
- Components of the launch velocity
  $$v_{0x} = 20\cos 45^\circ \approx 14.14\,\mathrm{m/s}$$
  $$v_{0y} = 20\sin 45^\circ \approx 14.14\,\mathrm{m/s}$$
- Time of flight
  $$T = \frac{v_{0y} + \sqrt{v_{0y}^2 + 2g(h_0 - h_k)}}{g}$$
  $$= \frac{14.14 + \sqrt{(14.14)^2 + 2\cdot 9.81(0-0)}}{9.81} \approx 2.883\,\mathrm{s}$$
- Expressing the sought quantity: $R = v_{0x}T$
- Substituting: $R = 14.14 \cdot 2.883 \approx 40.77\,\mathrm{m}$
- With equal launch and landing heights the school formulas hold
  $$R = \frac{v_0^2 \sin 2\alpha}{g}$$
  $$H - h_0 = \frac{v_0^2 \sin^2\alpha}{2g}$$
  $$T = \frac{2v_0\sin\alpha}{g}$$

- The greatest range is at 45°; the angles α and 90° − α give the same range.

Graph

Path | In time



The axes have different scales. Use the slider or the play button to follow the flight in time.

Landing speed of a throw from a height.

Input: `model(ballistics; find=v[m_s]; v0=20[m_s]; alpha=30[deg]; h0=10[m]; mode=vacuum)`

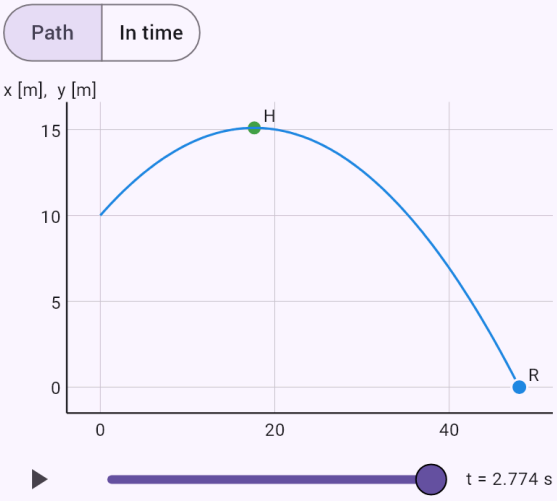
Result

$$v_k \approx 24.42\,\mathrm{m/s}$$

Steps

- Find v_k; given: $v_0 = 20\,\mathrm{m/s}, \quad \alpha = 30\,^\circ, \quad h_0 = 10\,\mathrm{m}$
- Without air drag the motion splits into two independent ones: uniform horizontally and uniformly accelerated vertically. The path is a parabola and every value has an exact formula. The model's equation:
  $$x = v_0 t\cos\alpha, \quad y = h_0 + v_0 t\sin\alpha - \frac{1}{2}gt^2$$
- Components of the launch velocity
  $$v_{0x} = 20\cos 30^\circ \approx 17.32\,\mathrm{m/s}$$
  $$v_{0y} = 20\sin 30^\circ \approx 10\,\mathrm{m/s}$$
- Expressing the sought quantity:
  $$v_k = \sqrt{v_0^2 + 2g(h_0 - h_k)}$$
- Substituting
  $$v_k = \sqrt{(20)^2 + 2\cdot 9.81(10 - 0)} \approx 24.42\,\mathrm{m/s}$$
- This follows from the conservation of mechanical energy and doesn't depend on the angle.

Graph



The axes have different scales. Use the slider or the play button to follow the flight in time.

Two launch angles for a distance of 200 m.

Input: `model(ballistics; find=alpha; v0=50[m_s]; d=200[m]; mode=aim)`

Result

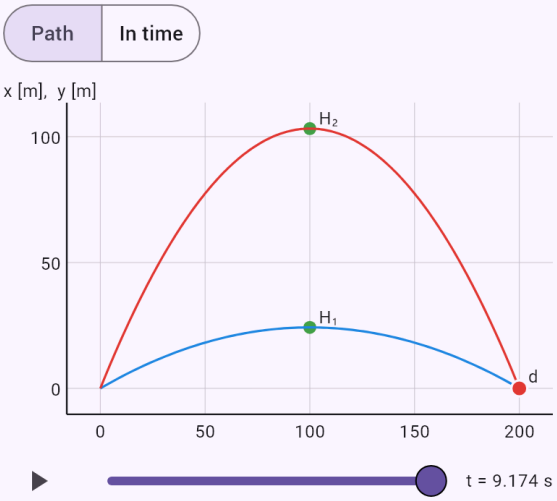
$\alpha_1 \approx 25.85^\circ$

$\alpha_2 \approx 64.15^\circ$

Steps

- Find α; given: $v_0 = 50\,\mathrm{m/s}, \quad d = 200\,\mathrm{m}$
- The target's distance and height are put into the equation of the path. With u = tan α it becomes a quadratic equation, usually with two roots: a flat and a steep path. The model's equation:
  $h_k = h_0 + d\tan\alpha - \frac{gd^2}{2v_0^2}\left(1+\tan^2\alpha\right)$
- Quadratic equation for tan α
  $u = \tan\alpha: \quad au^2 - du + c = 0$
  $a = \frac{gd^2}{2v_0^2} = \frac{9.81\cdot(200)^2}{2(50)^2} \approx 78.48$
  $c = a + h_k - h_0 \approx 78.48$
- Discriminant:
  $D = d^2 - 4ac = (200)^2 - 4\cdot 78.48\cdot 78.48 \approx 15\,360$
- Roots: $u_{1,2} = \frac{d \mp \sqrt{D}}{2a} \approx 0.4845;\ 2.064$
- Solution
  $\alpha_1 \approx 25.85^\circ, \quad T_1 \approx 4.445\,\mathrm{s}, \quad v_1 \approx 50\,\mathrm{m/s}$
- Solution
  $\alpha_2 \approx 64.15^\circ, \quad T_2 \approx 9.174\,\mathrm{s}, \quad v_2 \approx 50\,\mathrm{m/s}$
- On level ground the two angles add up to 90°.

Graph



The axes have different scales. Use the slider or the play button to follow the flight in time.

Range of a football with air drag, compared with a vacuum.

Input: `model(ballistics; find=R[m]; v0=25[m_s]; alpha=30[deg]; h0=0[m]; m=430[g]; S=38000[mm2]; cd=0.22[one]; rho=1.225[kg_m3]; mode=drag; draglaw=quadratic;`

`projectile=ball; cdlaw=constant; air=constant; coriolis=off)`

Result

$R \approx 38.65\,\mathrm{m}$

Steps

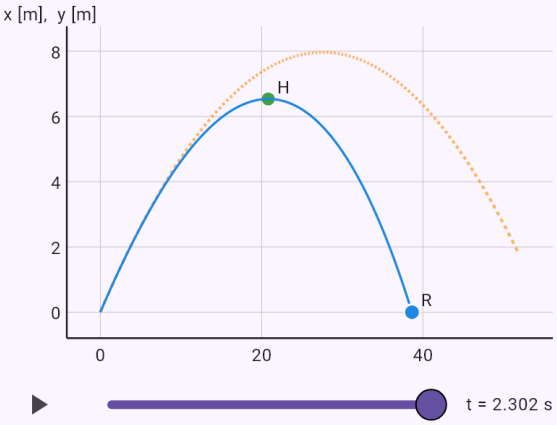
- Find R; given
  $v_0 = 25\,\mathrm{m/s}, \quad \alpha = 30\,^\circ, \quad h_0 = 0\,\mathrm{m}, \quad m = 430\,\mathrm{g}, \quad S$
- Converting the units
  $m = 430\,\mathrm{g} = 0.43\,\mathrm{kg}, \quad S = 38\,000\,\mathrm{mm}^2 = 0.038\,\mathrm{m}^2$
- Gravity and air drag act on the body, the drag opposing its velocity relative to the air, so the wind enters the equations too. The velocity's components are coupled and no exact solution exists, so the system is solved numerically with the 4th-order Runge-Kutta method. The model's equation: $m\dot{\vec{v}} = m\vec{g} - k|\vec{v} - \vec{w}|(\vec{v} - \vec{w})$
- Components of the launch velocity
  $v_{0x} = 25\cos 30^\circ \approx 21.65\,\mathrm{m/s}$
  $v_{0y} = 25\sin 30^\circ \approx 12.5\,\mathrm{m/s}$
- Air-drag constant (drag coefficient):
  $k = \frac{1}{2}\rho c_d S \approx 0.005121\,\mathrm{kg/m}$
- Terminal velocity of a free fall: $v_t = \sqrt{\frac{mg}{k}} \approx 28.7\,\mathrm{m/s}$
- Air drag against gravity at launch: $\frac{F_o}{F_g} \approx 0.7587$
- The system of six equations for x, y, z and the velocity's components was solved by the 4th-order Runge-Kutta method; the number of steps: $N = 1807$
- Course of the flight

t [s]	x [m]	y [m]	v [m/s]
0	0	0	25
0.46	9.358	4.405	20.31
0.921	17.72	6.367	17.37
1.38	25.32	6.163	15.99
1.84	32.28	3.985	15.96
2.3	38.65	0	16.88

- Key values
  $H \approx 6.534\,\mathrm{m}$
  $T \approx 2.302\,\mathrm{s}$
  $R \approx 38.65\,\mathrm{m}$
  $v_k \approx 16.88\,\mathrm{m/s}$
  $\varphi \approx 38.44^\circ$
- The same throw in a vacuum
  $R_v \approx 55.17\,\mathrm{m}, \quad H_v \approx 7.964\,\mathrm{m}, \quad T_v \approx 2.548\,\mathrm{s}$
- Range in air as a share of the vacuum's: $\frac{R}{R_v} \approx 70\,\%$

Graph

Path | In time



The solid line is the path in air, the dashed one the path in a vacuum. The axes have different scales.

Side drift of a ball in the wind, with a side and a top view.

Input: `model(ballistics; find=z[m]; v0=25[m_s]; alpha=30[deg]; h0=0[m]; m=430[g]; S=38000[mm2]; cd=0.22[one]; rho=1.225[kg_m3]; wc=5[m_s]; mode=drag; draglaw=quadratic;`

`projectile=ball; cdlaw=constant; air=constant; coriolis=off)`

Result

$z \approx 2.621\,\mathrm{m}$

Steps

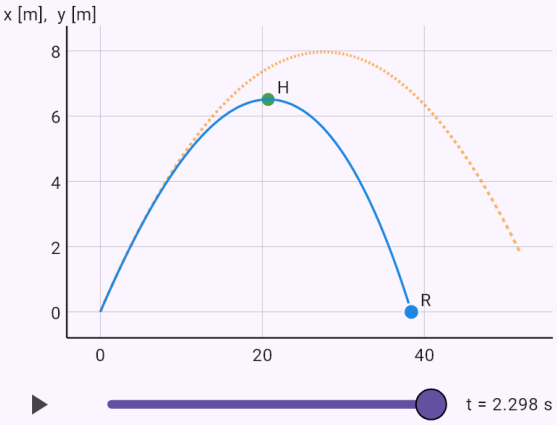
- Find z; given
  $v_0 = 25\,\mathrm{m/s}, \quad \alpha = 30^\circ, \quad h_0 = 0\,\mathrm{m}, \quad m = 430\,\mathrm{g}, \quad S$
- Converting the units
  $m = 430\,\mathrm{g} = 0.43\,\mathrm{kg}, \quad S = 38\,000\,\mathrm{mm}^2 = 0.038\,\mathrm{m}^2$
- Gravity and air drag act on the body, the drag opposing its velocity relative to the air, so the wind enters the equations too. The velocity's components are coupled and no exact solution exists, so the system is solved numerically with the 4th-order Runge-Kutta method. The model's equation: $m\dot{\vec{v}} = m\vec{g} - k|\vec{v} - \vec{w}|(\vec{v} - \vec{w})$
- Components of the launch velocity
  $v_{0x} = 25\cos 30^\circ \approx 21.65\,\mathrm{m/s}$
  $v_{0y} = 25\sin 30^\circ \approx 12.5\,\mathrm{m/s}$
- Air-drag constant (drag coefficient):
  $k = \frac{1}{2}\rho c_d S \approx 0.005121\,\mathrm{kg/m}$
- Terminal velocity of a free fall: $v_t = \sqrt{\frac{mg}{k}} \approx 28.7\,\mathrm{m/s}$
- Air drag against gravity at launch: $\frac{F_o}{F_g} \approx 0.789$
- The system of six equations for x, y, z and the velocity's components was solved by the 4th-order Runge-Kutta method; the number of steps: $N = 1804$
- Course of the flight

t [s]	x [m]	y [m]	v [m/s]	z [m]
0	0	0	25	0
0.46	9.33	4.393	20.27	0.143
0.919	17.65	6.346	17.33	0.519
1.38	25.19	6.139	15.96	1.07
1.84	32.09	3.968	15.95	1.78
2.3	38.4	0	16.89	2.62

- Key values
  $H \approx 6.51\,\mathrm{m}$
  $T \approx 2.298\,\mathrm{s}$
  $R \approx 38.4\,\mathrm{m}$
  $v_k \approx 16.89\,\mathrm{m/s}$
  $\varphi \approx 38.29^\circ$
  $z \approx 2.621\,\mathrm{m}$
- The same throw in a vacuum
  $R_v \approx 55.17\,\mathrm{m}, \quad H_v \approx 7.964\,\mathrm{m}, \quad T_v \approx 2.548\,\mathrm{s}$
- Range in air as a share of the vacuum's: $\frac{R}{R_v} \approx 69.6\,\%$
- Effect of the wind: $\Delta R \approx -0.2474\,\mathrm{m}, \quad z \approx 2.621\,\mathrm{m}$

Graph

Side view | Top view | In time



The solid line is the path in air, the dashed one the path in a vacuum. The axes have different scales.

A smallbore sporting rifle: the speed at the end of the flight, the G1 curve.

Input: `model(ballistics; find=v[m_s]; v0=326[m_s]; alpha=0.5[deg]; h0=1.4[m]; m=2.59[g]; S=25.2[mm2]; i=0.96[one]; rho=1.225[kg_m3]; mode=drag; draglaw=quadratic;`

`projectile=smallbore; cdlaw=bullet; air=constant; coriolis=off)`

Result

$$v_k \approx 227.7\,\text{m/s}$$

Steps

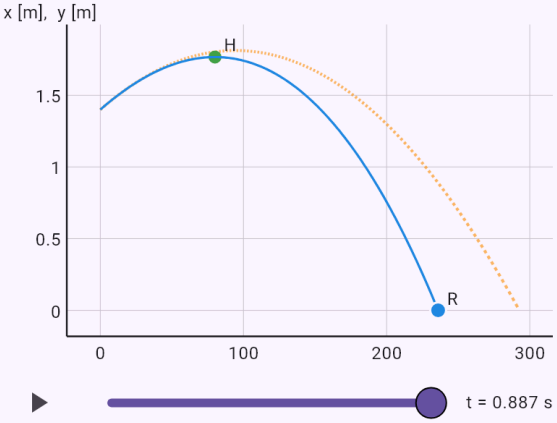
- Find v_k; given
  $v_0 = 326\,\text{m/s}, \quad \alpha = 0.5\,^\circ, \quad h_0 = 1.4\,\text{m}, \quad m = 2.59\,\text{g}$
- Converting the units
  $m = 2.59\,\text{g} = 0.00259\,\text{kg}, \quad S = 25.2\,\text{mm}^2 = 2.52 \cdot 10^{-}$
- Gravity and air drag act on the body, the drag opposing its velocity relative to the air, so the wind enters the equations too. The velocity's components are coupled and no exact solution exists, so the system is solved numerically with the 4th-order Runge-Kutta method. The model's equation: $m\dot{\vec{v}} = m\vec{g} - k|\vec{v} - \vec{w}|(\vec{v} - \vec{w})$
- Components of the launch velocity
  $v_{0x} = 326 \cos 0.5^\circ \approx 326\,\text{m/s}$
  $v_{0y} = 326 \sin 0.5^\circ \approx 2.845\,\text{m/s}$
- Mach number and the drag coefficient at launch:
  $M_0 = \dfrac{v_0}{a} \approx 0.958, \quad c_d(M_0) \approx 0.4031$
- Air-drag constant (drag coefficient):
  $k = \dfrac{1}{2}\rho c_d S \approx 6.223 \cdot 10^{-6}\,\text{kg/m}$
- Terminal velocity of a free fall: $v_t = \sqrt{\dfrac{mg}{k}} \approx 80.76\,\text{m/s}$
- Air drag against gravity at launch: $\dfrac{F_o}{F_g} \approx 26.03$
- The system of six equations for x, y, z and the velocity's components was solved by the 4th-order Runge-Kutta method; the number of steps: $N = 73$
- Course of the flight

t [s]	x [m]	y [m]	v [m/s]
0	0	1.4	326
0.177	54.58	1.727	292.2
0.355	104.4	1.729	270.3
0.532	150.8	1.431	253.5
0.71	194.5	0.8512	239.6
0.887	236	0	227.7

- Key values
  $H \approx 1.767\,\text{m}$
  $T \approx 0.8874\,\text{s}$
  $R \approx 236\,\text{m}$
  $v_k \approx 227.7\,\text{m/s}$
  $\varphi \approx 1.394^\circ$
- The same throw in a vacuum
  $R_v \approx 292.7\,\text{m}, \quad H_v \approx 1.812\,\text{m}, \quad T_v \approx 0.8979\,\text{s}$
- Range in air as a share of the vacuum's: $\dfrac{R}{R_v} \approx 80.6\,\%$
- The drag coefficient comes from an approximate curve by the Mach number; for a bullet the G1 curve times the form factor. The maker's tables take precedence. A spinning bullet's drift isn't included.

Graph

Path | In time



The solid line is the path in air, the dashed one the path in a vacuum. The axes have different scales.

A hunting rifle: the range with the air density by height.

Input: `model(ballistics; find=R[m]; v0=860[m_s]; alpha=5[deg]; h0=1.5[m]; m=9.72[g]; S=48.1[mm2]; i=0.554[one]; alt=300[m]; mode=drag; draglaw=quadratic; projectile=rifle;`

```
cdlaw=bullet; air=altitude; coriolis=off)
```

Result

$$R \approx 2365\,\text{m}$$

Steps

- Find R; given

  $v_0 = 860\,\text{m/s}, \quad \alpha = 5^\circ, \quad h_0 = 1.5\,\text{m}, \quad m = 9.72\,\text{g},$

- Converting the units

  $m = 9.72\,\text{g} = 0.00972\,\text{kg}, \quad S = 48.1\,\text{mm}^2 = 4.81 \cdot 10^{-}$

- Gravity and air drag act on the body, the drag opposing its velocity relative to the air, so the wind enters the equations too. The velocity's components are coupled and no exact solution exists, so the system is solved numerically with the 4th-order Runge-Kutta method. The model's equation: $m\dot{\vec{v}} = m\vec{g} - k|\vec{v} - \vec{w}|(\vec{v} - \vec{w})$

- Components of the launch velocity

  $v_{0x} = 860 \cos 5^\circ \approx 856.7\,\text{m/s}$
  $v_{0y} = 860 \sin 5^\circ \approx 74.95\,\text{m/s}$

- Air density by the standard atmosphere:
  $\rho(h_n + h_0) \approx 1.19\,\text{kg/m}^3$

- Mach number and the drag coefficient at launch:
  $M_0 = \dfrac{v_0}{a} \approx 2.536, \quad c_d(M_0) \approx 0.2979$

- Air-drag constant (drag coefficient):
  $k = \dfrac{1}{2}\rho c_d S \approx 8.527 \cdot 10^{-6}\,\text{kg/m}$

- Terminal velocity of a free fall: $v_t = \sqrt{\dfrac{mg}{k}} \approx 151.3\,\text{m/s}$

- Air drag against gravity at launch: $\dfrac{F_o}{F_g} \approx 66.14$

- The system of six equations for x, y, z and the velocity's components was solved by the 4th-order Runge-Kutta method; the number of steps: $N = 335$

- Course of the flight

t [s]	x [m]	y [m]	v [m/s]
0	0	1.5	860
1.54	847.3	66.67	367.7
3.08	1325	83.84	274
4.62	1715	76.3	236.4
6.17	2058	47.72	211.3
7.71	2365	0	192.8

- Key values

  $H \approx 84.15\,\text{m}$
  $T \approx 7.707\,\text{s}$
  $R \approx 2365\,\text{m}$
  $v_k \approx 192.8\,\text{m/s}$
  $\varphi \approx 11.02^\circ$

- The same throw in a vacuum

  $R_v \approx 13\,110\,\text{m}, \quad H_v \approx 287.8\,\text{m}, \quad T_v \approx 15.3\,\text{s}$

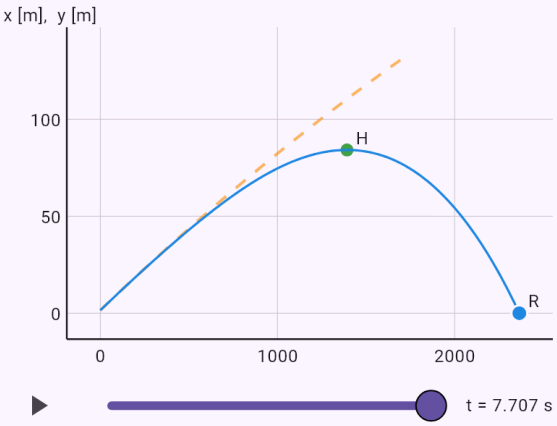
- Range in air as a share of the vacuum's: $\dfrac{R}{R_v} \approx 18\,\%$

- The drag coefficient comes from an approximate curve by the Mach number; for a bullet the G1 curve times the form factor. The maker's tables take precedence. A spinning bullet's drift isn't included.

- During the flight the speed fell below the speed of sound; the drag coefficient changes most in this region.

Graph

Path | In time



The solid line is the path in air, the dashed one the path in a vacuum. The axes have different scales.

A cannonball from school physics: the deviation due to the Coriolis force.

Input: `model(ballistics; find=z[m]; v0=439[m_s]; alpha=30[deg]; h0=1[m]; m=5580[g]; S=10200[mm2]; i=1[one]; rho=1.225[kg_m3]; lat=50[deg]; az=0[deg]; mode=drag;`

`draglaw=quadratic; projectile=cannonball; cdlaw=sphere; air=constant; coriolis=on)`

Result

$z \approx 2.119\,\text{m}$

Steps

- Find z; given

  $v_0 = 439\,\text{m/s}, \quad \alpha = 30^\circ, \quad h_0 = 1\,\text{m}, \quad m = 5580\,\text{g},$

- Converting the units

  $m = 5580\,\text{g} = 5.58\,\text{kg}, \quad S = 10\,200\,\text{mm}^2 = 0.0102\,\text{m}^2$

- Gravity and air drag act on the body, the drag opposing its velocity relative to the air, so the wind enters the equations too. The velocity's components are coupled and no exact solution exists, so the system is solved numerically with the 4th-order Runge-Kutta method. The model's equation

  $m\dot{\vec{v}} = m\vec{g} - k|\vec{v} - \vec{w}|(\vec{v} - \vec{w}) - 2m\,\vec{\Omega} \times \vec{v}$

- Components of the launch velocity

  $v_{0x} = 439 \cos 30^\circ \approx 380.2\,\text{m/s}$
  $v_{0y} = 439 \sin 30^\circ \approx 219.5\,\text{m/s}$

- Mach number and the drag coefficient at launch:

  $M_0 = \dfrac{v_0}{a} \approx 1.29, \quad c_d(M_0) \approx 0.9795$

- Air-drag constant (drag coefficient):

  $k = \dfrac{1}{2}\rho c_d S \approx 0.006119\,\text{kg/m}$

- Terminal velocity of a free fall: $v_t = \sqrt{\dfrac{mg}{k}} \approx 137.1\,\text{m/s}$

- Air drag against gravity at launch: $\dfrac{F_o}{F_g} \approx 21.54$

- Earth's angular velocity (Coriolis force):

  $\Omega \approx 7.292 \cdot 10^{-5}\,\text{rad/s}$

- The system of six equations for x, y, z and the velocity's components was solved by the 4th-order Runge-Kutta method; the number of steps: $N = 1023$

- Course of the flight

t [s]	x [m]	y [m]	v [m/s]	z [m]
0	0	1	439	0
4.46	1014	509.8	172.8	0.122
8.91	1617	648.6	115	0.408
13.4	2069	580.7	95.94	0.839
17.8	2425	353	97.3	1.41
22.3	2706	0	106.4	2.12

- Key values

  $H \approx 650.1\,\text{m}$
  $T \approx 22.28\,\text{s}$
  $R \approx 2706\,\text{m}$
  $v_k \approx 106.4\,\text{m/s}$
  $\varphi \approx 58.47^\circ$
  $z \approx 2.119\,\text{m}$

- The same throw in a vacuum

  $R_v \approx 17\,020\,\text{m}, \quad H_v \approx 2457\,\text{m}, \quad T_v \approx 44.75\,\text{s}$

- Range in air as a share of the vacuum's: $\dfrac{R}{R_v} \approx 15.9\,\%$

- Effect of the Coriolis force:

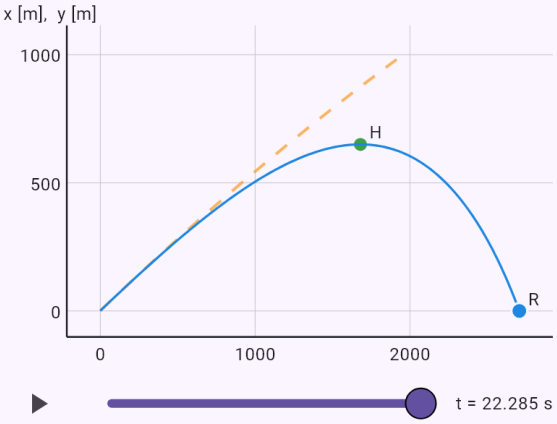
  $\Delta R \approx -6.153 \cdot 10^{-4}\,\text{m}, \quad \Delta z \approx 2.119\,\text{m}$

- The drag coefficient comes from an approximate curve by the Mach number; for a bullet the G1 curve times the form factor. The maker's tables take precedence. A spinning bullet's drift isn't included.

- During the flight the speed fell below the speed of sound; the drag coefficient changes most in this region.

Graph

Side view | Top view | In time



The solid line is the path in air, the dashed one the path in a vacuum. The axes have different scales.

A flat and a steep angle for a ball with air drag.

Input: `model(ballistics; find=alpha; v0=25[m_s]; d=35[m]; h0=0[m]; m=430[g]; S=38000[mm2]; cd=0.22[one]; rho=1.225[kg_m3]; mode=aimdrag; draglaw=quadratic; projectile=ball;`

cdlaw=constant; air=constant; coriolis=off)

Result

$\alpha_1 \approx 24.3^\circ$

$\alpha_2 \approx 59.84^\circ$

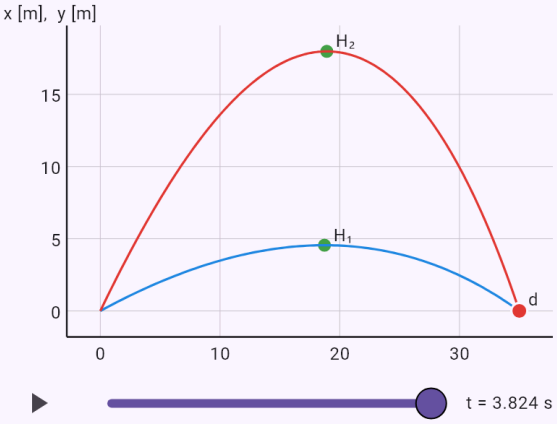
Steps

- Find α; given
  $v_0 = 25\,\mathrm{m/s}, \quad d = 35\,\mathrm{m}, \quad h_0 = 0\,\mathrm{m}, \quad m = 430\,\mathrm{g},$
- Converting the units
  $m = 430\,\mathrm{g} = 0.43\,\mathrm{kg}, \quad S = 38\,000\,\mathrm{mm}^2 = 0.038\,\mathrm{m}^2$
- With air drag there is no formula. The computation tries different angles and watches how high the body flies at the target's distance. The angles at which it passes exactly through the target are found by bisection. The model's equation: $m\dot{\vec{v}} = m\vec{g} - k|\vec{v} - \vec{w}|(\vec{v} - \vec{w})$
- Air-drag constant (drag coefficient):
  $k = \frac{1}{2}\rho c_d S \approx 0.005121\,\mathrm{kg/m}$
- Terminal velocity of a free fall: $v_t = \sqrt{\frac{mg}{k}} \approx 28.7\,\mathrm{m/s}$
- Air drag against gravity at launch: $\frac{F_o}{F_g} \approx 0.7587$
- The angles were found by the shooting method: bisection on the height at the distance d, that is, finding a root of $y(d;\,\alpha) - h_k = 0$
- Solution
  $\alpha_1 \approx 24.3^\circ, \quad T_1 \approx 1.922\,\mathrm{s}, \quad v_1 \approx 17.2\,\mathrm{m/s}$
- Solution

$\alpha_2 \approx 59.84^\circ, \quad T_2 \approx 3.824\,\mathrm{s}, \quad v_2 \approx 17.98\,\mathrm{m/s}$

Graph

Path | In time



The axes have different scales. Use the slider or the play button to follow the flight in time.

The stall speed of a light aeroplane.

Input: model(aircraft; find=vs[km_h]; m=1157[kg]; S=16.2[m2]; CLmax=1.54[one];

`rho=1.225[kg_m3]; mode=level; aircraft=light; air=constant)`

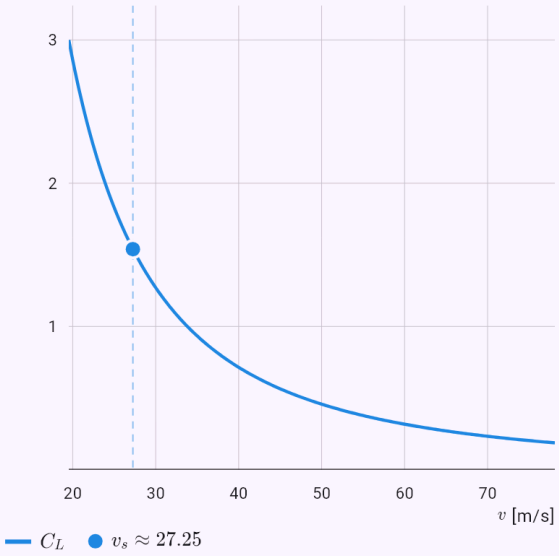
Result

$$v_s \approx 98.11\,\mathrm{km/h}$$

Steps

- Find v_s; given
  $m = 1157\,\mathrm{kg}, \quad S = 16.2\,\mathrm{m^2}, \quad C_{L,max} = 1.54, \quad \rho = 1$
- In level flight the lift equals the weight and the thrust equals the drag. The drag has two parts: the parasite drag grows with the square of the speed and the induced drag falls with it, so there is a speed of least drag. The model's equation:
  $L = \frac{1}{2}\rho v^2 S C_L = W, \quad D = \frac{1}{2}\rho v^2 S\left(C_{D0} + kC_L^2\right)$
- The aeroplane's weight:
  $W = mg = 1157 \cdot 9.81 \approx 11.35\,\mathrm{kN}$
- Expressing the sought quantity:
  v_s = \sqrt{\frac{2W}{\rhoSC_{L,max}}}
- Substituting: $v_s = \sqrt{\frac{2 \cdot 11\,350}{1.225 \cdot 16.2 \cdot 1.54}} \approx 27.25\,\mathrm{m/s}$
- Converting the units: $v_s \approx 27.25\,\mathrm{m/s} = 98.11\,\mathrm{km/h}$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

The drag in level flight and a chart of the parasite and induced drag.

Input: `model(aircraft; find=D[kN]; v=200[km_h]; m=1157[kg]; S=16.2[m2]; b=11[m]; CD0=0.03[one]; e=0.75[one]; rho=1.225[kg_m3]; CLmax=1.54[one]; mode=level; aircraft=light;`

`air=constant; ground=off)`

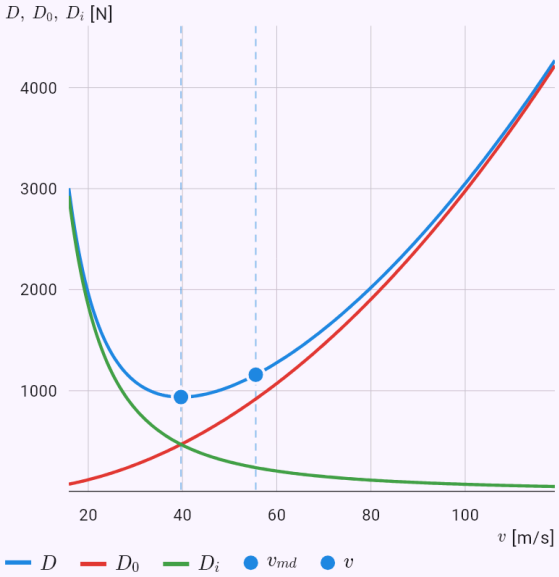
Result

$$D \approx 1.158\,\text{kN}$$

Steps

- Find D; given
  $v = 200\,\text{km/h}, \quad m = 1157\,\text{kg}, \quad S = 16.2\,\text{m}^2, \quad b = 1$
- Converting the units: $v = 200\,\text{km/h} = 55.56\,\text{m/s}$
- In level flight the lift equals the weight and the thrust equals the drag. The drag has two parts: the parasite drag grows with the square of the speed and the induced drag falls with it, so there is a speed of least drag. The model's equation:
  $L = \frac{1}{2}\rho v^2 S C_L = W, \quad D = \frac{1}{2}\rho v^2 S\left(C_{D0} + k C_L^2\right)$
- The aeroplane's weight:
  $W = mg = 1157 \cdot 9.81 \approx 11.35\,\text{kN}$
- Aspect ratio: $\Lambda = \frac{b^2}{S} = \frac{(11)^2}{16.2} \approx 7.469$
- Induced drag factor (Oswald factor e, polar):
  $k = \frac{1}{\pi e \Lambda} = \frac{1}{\pi \cdot 0.75 \cdot 7.469} \approx 0.05682$
- Dynamic pressure
  $q = \frac{1}{2}\rho v^2 = \frac{1}{2} \cdot 1.225 \cdot (55.56)^2 \approx 1890\,\text{Pa}$
- The lift coefficient needed:
  $C_L = \frac{W}{qS} = \frac{11\,350}{1890 \cdot 16.2} \approx 0.3706$
- Parasite and induced drag:
  $D_0 = qSC_{D0} \approx 918.8\,\text{N}, \quad D_i = qSkC_L^2 \approx 239\,\text{N}$
- The glide ratio at this speed: $E = \frac{L}{D} = \frac{W}{D} \approx 9.803$
- Expressing the sought quantity: $D = D_0 + D_i$
- Substituting: $D \approx 1158\,\text{N}$
- Converting the units: $D \approx 1158\,\text{N} = 1.158\,\text{kN}$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

The KM ekranoplan: how much drag the ground effect saves.

Input: `model(aircraft; find=D[kN]; v=430[km_h]; m=544000[kg]; S=662.5[m2]; b=37.6[m]; CD0=0.03[one]; e=0.8[one]; CLmax=1.5[one]; hw=4[m]; rho=1.225[kg_m3]; mode=level;`

aircraft=km; air=constant; ground=on)

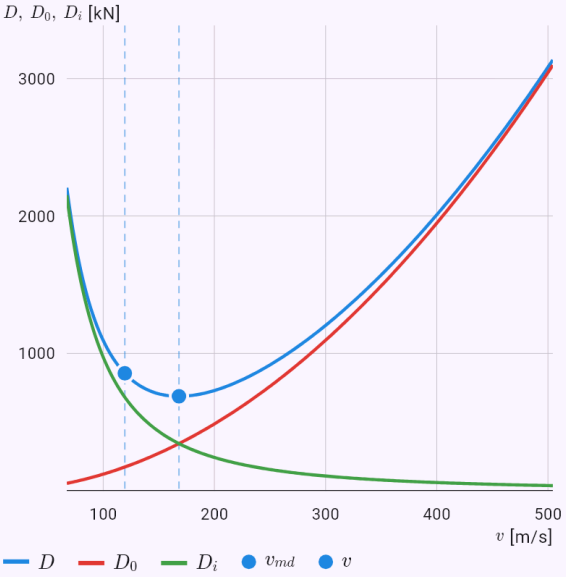
Result

$$D \approx 855.6\,\text{kN}$$

Steps

- Find D; given
  $v = 430\,\text{km/h}, \quad m = 544\,000\,\text{kg}, \quad S = 662.5\,\text{m}^2, \quad b$
- Converting the units: $v = 430\,\text{km/h} = 119.4\,\text{m/s}$
- In level flight the lift equals the weight and the thrust equals the drag. The drag has two parts: the parasite drag grows with the square of the speed and the induced drag falls with it, so there is a speed of least drag. The model's equation:
  $L = \frac{1}{2}\rho v^2 S C_L = W, \quad D = \frac{1}{2}\rho v^2 S\left(C_{D0} + kC_L^2\right)$
- The aeroplane's weight:
  $W = mg = 544\,000 \cdot 9.81 \approx 5337\,\text{kN}$
- Aspect ratio: $\Lambda = \frac{b^2}{S} = \frac{(37.6)^2}{662.5} \approx 2.134$
- Induced drag factor (Oswald factor e, polar):
  $k = \frac{1}{\pi e \Lambda} = \frac{1}{\pi \cdot 0.8 \cdot 2.134} \approx 0.1865$
- The ground effect lessens the induced drag
  $\phi_g = \frac{(16h_w/b)^2}{1 + (16h_w/b)^2} \approx 0.7434, \quad k_g = \phi_g k \approx 0.1386$
- The gain of lift at the same angle of attack (a lifting-line estimate)
  $\frac{C_{L\alpha,g}}{C_{L\alpha}} \approx \frac{1 + 2/\Lambda}{1 + 2\phi_g/\Lambda} \approx 1.142$
- McCormick's empirical formula for ordinary wings. For a low aspect ratio wing with end plates, as on an ekranoplan, the real effect is stronger.
- Dynamic pressure
  $q = \frac{1}{2}\rho v^2 = \frac{1}{2} \cdot 1.225 \cdot (119.4)^2 \approx 8739\,\text{Pa}$
- The lift coefficient needed:
  $C_L = \frac{W}{qS} = \frac{5\,337\,000}{8739 \cdot 662.5} \approx 0.9218$
- Parasite and induced drag:
  $D_0 = qSC_{D0} \approx 173.7\,\text{kN}, \quad D_i = qSkC_L^2 \approx 681.9\,\text{kN}$
- The glide ratio at this speed: $E = \frac{L}{D} = \frac{W}{D} \approx 6.238$
- Drag saved by the ground effect:
  $\Delta D \approx 235.4\,\text{kN}, \quad \frac{\Delta D}{D} \approx 21.6\,\%$
- Expressing the sought quantity: $D = D_0 + D_i$
- Substituting: $D \approx 855.6\,\text{kN}$
- Converting the units: $D \approx 855\,600\,\text{N} = 855.6\,\text{kN}$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

The take-off roll of a light aeroplane with a chart of the speed and distance.

Input: model(aircraft; find=s[m]; m=1157[kg]; S=16.2[m2]; b=11[m]; CD0=0.03[one]; e=0.75[one]; rho=1.225[kg_m3]; dCD=0.022[one]; CLto=1.73[one]; CLg=0.4[one]; kv=1.1[one];

```
T=2.1[kN]; mu=0.03[one]; mode=takeoff; aircraft=light; air=constant; ground=off)
```

Result

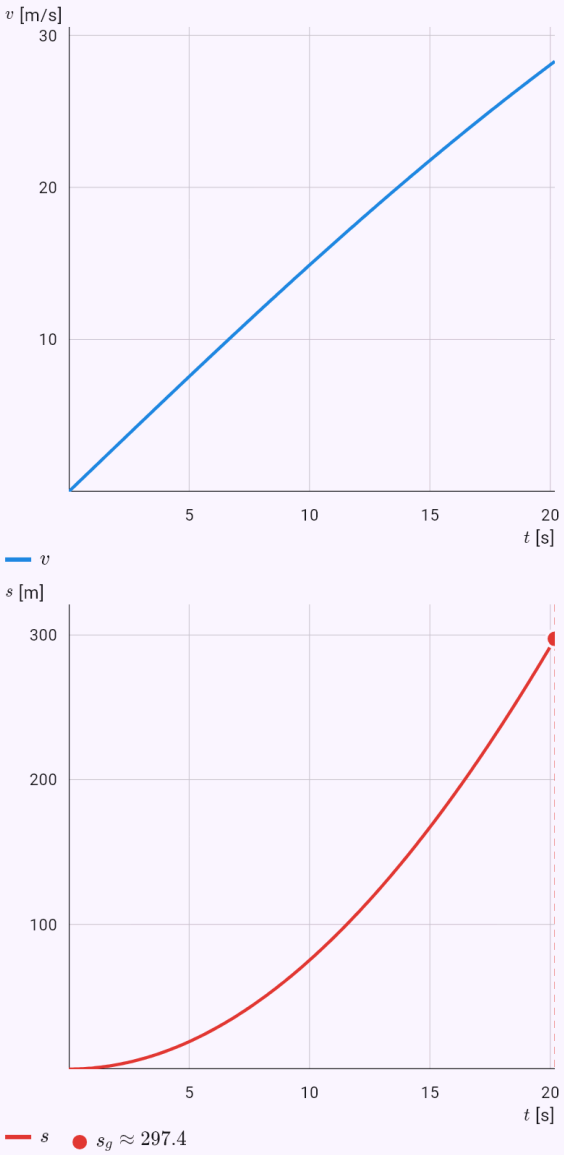
$$s_g \approx 297.4\,\mathrm{m}$$

Steps

- Find s_g; given
  $m = 1157\,\mathrm{kg}, \quad S = 16.2\,\mathrm{m^2}, \quad b = 11\,\mathrm{m}, \quad C_{D0} = 0.0$
- Converting the units: $T = 2.1\,\mathrm{kN} = 2100\,\mathrm{N}$
- The aeroplane accelerates from rest to the lift-off speed under the thrust, the air drag and the rolling friction, which lessens as the lift grows. The equation is solved numerically with the 4th-order Runge-Kutta method; the thrust is a constant average. The model's equation
  $m\dot{v} = T - \frac{1}{2}\rho v_a^2 S C_D - \mu\left(W - \frac{1}{2}\rho v_a^2 S C_{Lg}\right), \quad v_a = v$
- The aeroplane's weight:
  $W = mg = 1157 \cdot 9.81 \approx 11.35\,\mathrm{kN}$
- Stall speed
  $$v_s = \sqrt{\frac{2W}{\rho S C_{L,max}}} = \sqrt{\frac{2 \cdot 11\,350}{1.225 \cdot 16.2 \cdot 1.73}} \approx 25.71\,\mathrm{m/s}$$
- Lift-off speed relative to the air:
  $v_{LOF} = k_v v_s = 1.1 \cdot 25.71 \approx 28.29\,\mathrm{m/s}$
- Aspect ratio: $\Lambda = \frac{b^2}{S} = \frac{(11)^2}{16.2} \approx 7.469$
- Induced drag factor (Oswald factor e, polar):
  $k = \frac{1}{\pi e \Lambda} = \frac{1}{\pi \cdot 0.75 \cdot 7.469} \approx 0.05682$
- Drag coefficient on the runway:
  $C_D = C_{D0} + \Delta C_D + k C_{Lg}^2 \approx 0.06109$
- The acceleration at the start and at the end
  $a_0 \approx 1.521\,\mathrm{m/s^2}, \quad a_k \approx 1.184\,\mathrm{m/s^2}$
- The equation was solved with the 4th-order Runge-Kutta method, steps: $N = 2172$
- The roll in detail

t [s]	s [m]	v [m/s]
0	0	0
4.04	12.38	6.12
8.08	49.25	12.11
12.1	109.9	17.87
16.2	193.1	23.28
20.2	297.4	28.29

Graph



The model's quantities in time, computed from the solution; the answer is marked.

The turn radius at a bank of 30°.

Input: `model(aircraft; find=R[m]; v=200[km_h]; phi=30[deg]; mode=turn)`

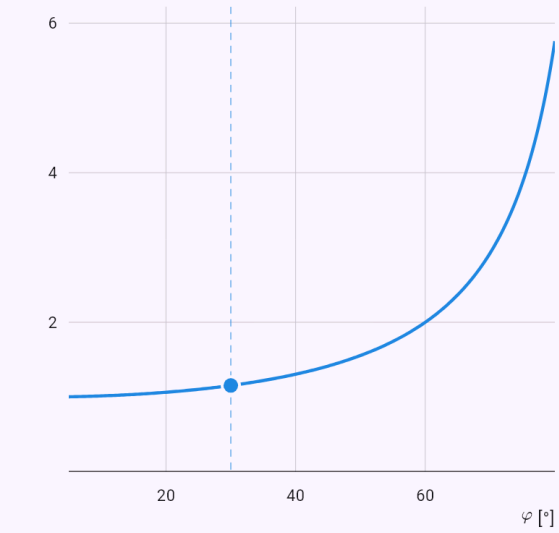
Result

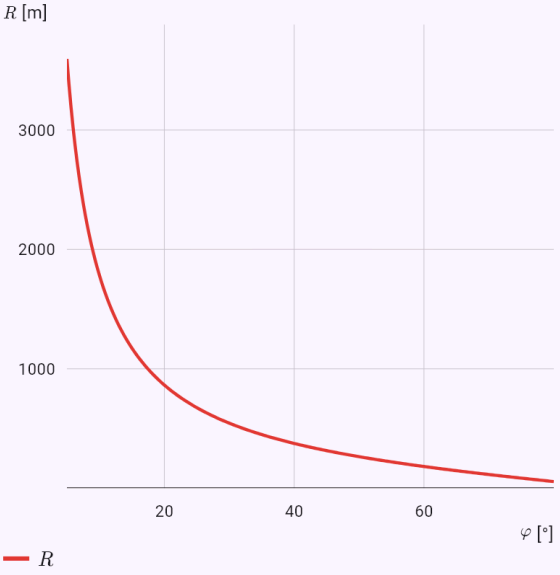
$$R \approx 544.9\,\mathrm{m}$$

Steps

- Find R; given: $v = 200\,\mathrm{km/h}, \quad \varphi = 30^\circ$
- Converting the units: $v = 200\,\mathrm{km/h} = 55.56\,\mathrm{m/s}$
- In a coordinated turn the vertical part of the lift carries the weight and the horizontal part curves the path. The load factor depends only on the bank, the radius on the speed as well. The model's equation:
  $L\cos\varphi = W, \quad L\sin\varphi = \dfrac{mv^2}{R}$
- Expressing the sought quantity: $R = \dfrac{v^2}{g\tan\varphi}$
- Substituting: $R = \dfrac{(55.56)^2}{9.81\cdot\tan 30^\circ} \approx 544.9\,\mathrm{m}$

Graph





The graph computed from the model's solution; the dots are the values given or the answer.

A glider's range from a height of 1000 m.

Input: `model(aircraft; find=R[km]; h=1000[m]; m=600[kg]; S=17.95[m2]; b=17[m]; CD0=0.0088[one]; e=0.8[one]; rho=1.225[kg_m3]; mode=glide; aircraft=glider; air=constant;`

`glidespeed=best)`

Result

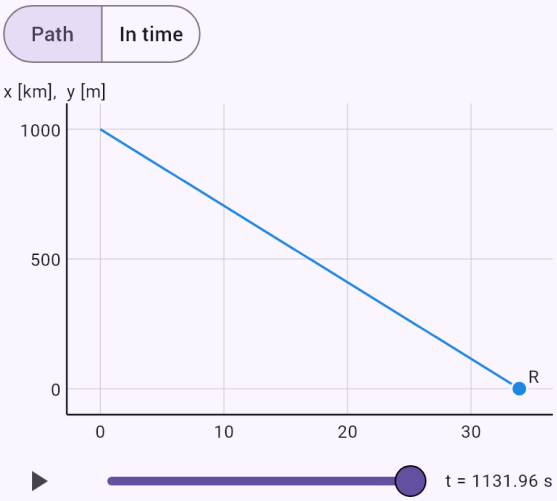
$$R \approx 33.91\,\text{km}$$

Steps

- Find R; given
  $h = 1000\,\text{m}, \quad m = 600\,\text{kg}, \quad S = 17.95\,\text{m}^2, \quad b = 17\,\text{m}$
- Without thrust the aeroplane descends along a straight line whose slope is set by the ratio of lift to drag. The range from a height is the height times the glide ratio, greatest at the best-glide speed. The model's equation:
  $\tan\gamma = \frac{D}{L} = \frac{1}{E}, \quad R = hE, \quad v_y = \frac{v}{E}$
- The aeroplane's weight:
  $W = mg = 600 \cdot 9.81 \approx 5886\,\text{N}$
- Aspect ratio: $\Lambda = \frac{b^2}{S} = \frac{(17)^2}{17.95} \approx 16.1$
- Induced drag factor (Oswald factor e, polar):
  $k = \frac{1}{\pi e \Lambda} = \frac{1}{\pi \cdot 0.8 \cdot 16.1} \approx 0.02471$
- Lift coefficient at the best glide:
  $C_L^* = \sqrt{\frac{C_{D0}}{k}} = \sqrt{\frac{0.0088}{0.02471}} \approx 0.5967$
- Best-glide speed
  $v = \sqrt{\frac{2W}{\rho S C_L^*}} = \sqrt{\frac{2 \cdot 5886}{1.225 \cdot 17.95 \cdot 0.5967}} \approx 29.95\,\text{m/s}$
- The greatest glide ratio and the least drag:
  $E_{max} = \frac{1}{2\sqrt{C_{D0} k}} = \frac{1}{2\sqrt{0.0088 \cdot 0.02471}} \approx 33.91$
- Glide angle: $\gamma = \arctan\frac{1}{E} \approx 1.689°$
- A small glide angle is assumed, where the lift practically equals the weight, and a constant air density.

- Expressing the sought quantity: $R = hE$
- Substituting: $R = 1000 \cdot 33.91 \approx 33\,910\,\text{m}$
- Converting the units: $R \approx 33\,910\,\text{m} = 33.91\,\text{km}$

Graph



The axes have different scales. Use the slider or the play button to follow the flight in time.

Apollo 13: the g-load of an entry at −6.20°.

Input: `model(reentry; mode=flight; find=amax[gn]; vehicle=apollo; body=capsule; m=5500[kg]; d=3.91[m]; CD=1.29[one]; LD=0.3[one]; Rn=4.69[m]; v0=11.03[km_s]; hk=7[km]; nlim=10[gn];`

gamma=-6.2[deg]; sigma=55[deg])

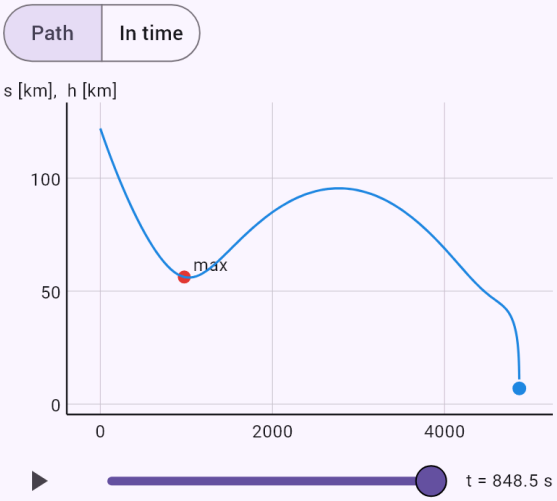
Result

$$a_{max} \approx 6.53\,\mathrm{g}$$

Steps

- Find a_{max}; given
  $m = 5500\,\mathrm{kg}, \quad d = 3.91\,\mathrm{m}, \quad C_D = 1.29, \quad L/D = 0.$
- One pass through the atmosphere from the entry height h_0 (122 km by default) with the given flight path angle γ_0 and a constant bank angle σ down to the height h_k (the parachutes opening, the gliding flight for the Shuttle). The peak g-load is compared with Allen and Eggers' formula for a ballistic entry. The heat flux at the stagnation point follows from the Sutton–Graves formula q = k·√(ρ/R_n)·v³ and the shield's surface temperature from radiative equilibrium εσT⁴ = q. The model's equation
  $$\dot v = -\frac{D}{m} - g\sin\gamma, \quad \dot\gamma = \frac{L\cos\sigma}{mv} - \left(\frac{g}{v} - \frac{v}{r}\right)\cos\gamma,$$
- Frontal area and ballistic coefficient
  $$A = \frac{\pi d^2}{4} = 12.01\,\mathrm{m}^2, \quad \beta = \frac{m}{C_D A} = 355.1\,\mathrm{kg/m}^2$$
- Air density by the standard atmosphere (isothermal above 86 km)
  $\rho(122\,\mathrm{km}) \approx 1.19\cdot 10^{-8}\,\mathrm{kg/m}^3, \quad \rho(60\,\mathrm{km}) \approx 3.1\cdot 10^{-}$
- Vertical component of the lift at the bank angle σ:
  $L\cos\sigma = 0.172\,D \quad (\sigma = 55^\circ)$
- Capture: the spacecraft descended to the height h_k:
  $h = h_k, \quad t \approx 848.5\,\mathrm{s}, \quad s \approx 4869\,\mathrm{km}$
- Peak g-load (numerically)
  $a_{max} \approx 6.53\,g \quad (t \approx 90.5\,\mathrm{s},\ h \approx 56.4\,\mathrm{km})$
- A ballistic entry without lift would give much more by Allen and Eggers – lift lowers the g-load
  $$a_{max} \approx \frac{v_0^2 \sin|\gamma_0|}{2eH} = \frac{11\,030^2 \sin 6.2^\circ}{2e\cdot 7200} \approx 34.2\,g$$
- Peak heat flux at the stagnation point (Sutton–Graves) and the total heat per m²
  $q_{max} = k\sqrt{\frac{\rho}{R_n}}\,v^3 \approx 1.63\,\mathrm{MW/m}^2 \quad (h \approx 58.2\,\mathrm{km},\ v \approx$
- Shield surface temperature by radiative equilibrium (ε = 0.85)
  $$\varepsilon\sigma T^4 = q_{max}: \quad T = \sqrt[4]{\frac{q_{max}}{\varepsilon\sigma}} \approx 2410\,\mathrm{K} \approx 2137\,^\circ\mathrm{C}$$
- Simplifications: the Earth is round and doesn't rotate, C_D and L/D are constant and the bank σ is the same for the whole flight (real guidance varies it). The heat flux is convective only; returning from the Moon, the glowing shock layer's radiation adds more. An ablative shield carries part of the heat away by vaporising, so the radiative equilibrium temperature is an upper estimate.

Graph



The axes have different scales. Use the slider or the play button to follow the flight in time.

The Apollo capsule's entry corridor returning from the Moon.

Input: model(reentry; mode=corridor; find=width[deg]; vehicle=apollo; body=capsule; m=5500[kg]; d=3.91[m]; CD=1.29[one]; LD=0.3[one]; Rn=4.69[m]; v0=11.03[km_s]; hk=7[km];

`nlim=10[gn])`

Result

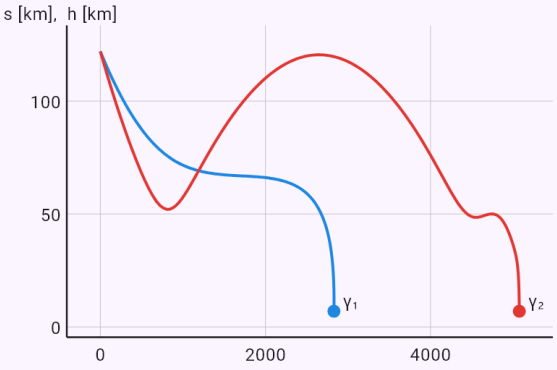
$$\gamma_1 \approx -4.98^\circ$$
$$\gamma_2 \approx -7.2^\circ$$
$$\Delta\gamma \approx 2.22^\circ$$

Steps

- Find $\Delta\gamma$; given
  $m = 5500\,\text{kg}, \quad d = 3.91\,\text{m}, \quad C_D = 1.29, \quad L/D = 0.$
- The limits of the entry angle by bisection. The shallow limit is the shallowest entry that, with the lift turned down (σ = 180°), doesn't climb back above the entry height; the steep limit the steepest entry that, with the lift up (σ = 0°), stays within the given g-load (and heat flux, if given). For a return from the Moon the corridor comes out only about 2° wide. Apollo 13 (April 1970) came back after its oxygen tank explosion with the command module powered down to save energy. The crew corrected the path by a manual burn of the lunar module's engine, aimed by the Earth in the window sight, and the last correction targeted the centre of the corridor. The planned entry angle was −6.20°; the command module's computer was powered up again only before entry. The model's equation
  $$\dot v = -\frac{D}{m} - g\sin\gamma, \quad \dot\gamma = \frac{L\cos\sigma}{mv} - \left(\frac{g}{v} - \frac{v}{r}\right)\cos\gamma,$$
- Frontal area and ballistic coefficient
  $$A = \frac{\pi d^2}{4} = 12.01\,\text{m}^2, \quad \beta = \frac{m}{C_D A} = 355.1\,\text{kg/m}^2$$
- Air density by the standard atmosphere (isothermal above 86 km)
  $\rho(122\,\text{km}) \approx 1.19\cdot10^{-8}\,\text{kg/m}^3, \quad \rho(60\,\text{km}) \approx 3.1\cdot10^{-}$
- Shallow limit of the corridor (lift down):
  $\gamma_1 \approx -4.98^\circ \quad (\sigma = 180^\circ)$
- Steep limit of the corridor (lift up):
  $\gamma_2 \approx -7.2^\circ \quad (\sigma = 0^\circ,\ a_{max} = 10\,g)$
- Width of the entry corridor: $\Delta\gamma = |\gamma_2| - |\gamma_1| \approx 2.22^\circ$
- Simplifications: the Earth is round and doesn't rotate, C_D and L/D are constant and the bank σ is the same for the whole flight (real guidance varies it). The heat flux is convective only; returning from the Moon, the glowing shock layer's radiation adds more. An ablative shield carries part of the heat away by vaporising, so the radiative equilibrium temperature is an upper estimate.

Graph

Path | In time



The axes have different scales.

Vostok: a ballistic descent without lift and 8–9 g.

Input: `model(reentry; mode=flight; find=amax[gn]; vehicle=vostok; body=capsule; m=2460[kg]; d=2.3[m]; CD=1[one]; LD=0[one]; Rn=1.15[m]; v0=7.8[km_s]; gamma=-1.5[deg]; sigma=0[deg];`

hk=7[km]; nlim=10[gn])

Result

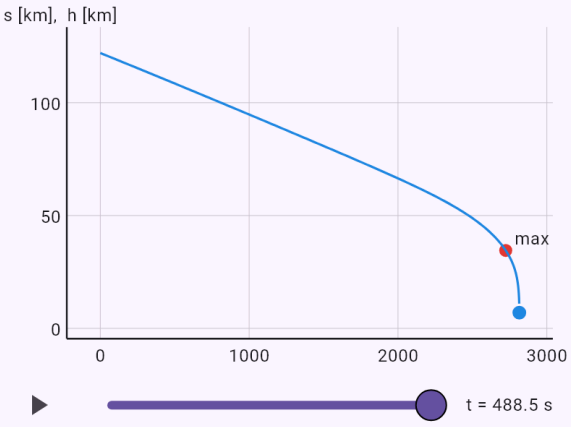
$a_{max} \approx 8.98\,\mathrm{g}$

Steps

- Find a_{max}; given
  $m = 2460\,\mathrm{kg}, \quad d = 2.3\,\mathrm{m}, \quad C_D = 1, \quad L/D = 0, \quad R$
- One pass through the atmosphere from the entry height h_0 (122 km by default) with the given flight path angle γ_0 and a constant bank angle σ down to the height h_k (the parachutes opening, the gliding flight for the Shuttle). The peak g-load is compared with Allen and Eggers' formula for a ballistic entry. The heat flux at the stagnation point follows from the Sutton–Graves formula q = k·√(ρ/R_n)·v³ and the shield's surface temperature from radiative equilibrium εσT⁴ = q. The model's equation
  $\dot{v} = -\frac{D}{m} - g\sin\gamma, \quad \dot{\gamma} = \frac{L\cos\sigma}{mv} - \left(\frac{g}{v} - \frac{v}{r}\right)\cos\gamma,$
- Frontal area and ballistic coefficient
  $A = \frac{\pi d^2}{4} = 4.155\,\mathrm{m}^2, \quad \beta = \frac{m}{C_D A} = 592.1\,\mathrm{kg/m}^2$
- Air density by the standard atmosphere (isothermal above 86 km)
  $\rho(122\,\mathrm{km}) \approx 1.19\cdot 10^{-8}\,\mathrm{kg/m}^3, \quad \rho(60\,\mathrm{km}) \approx 3.1\cdot 10^{-}$
- Capture: the spacecraft descended to the height h_k:
  $h = h_k, \quad t \approx 488.5\,\mathrm{s}, \quad s \approx 2815\,\mathrm{km}$
- Peak g-load (numerically)
  $a_{max} \approx 8.98\,g \quad (t \approx 375.5\,\mathrm{s},\ h \approx 34.4\,\mathrm{km})$
- Check by Allen and Eggers' formula for a ballistic entry (no gravity, an exponential atmosphere of scale height H)
  $a_{max} \approx \frac{v_0^2 \sin|\gamma_0|}{2eH} = \frac{7800^2 \sin 1.5^\circ}{2e\cdot 7200} \approx 4.15\,g$
- Peak heat flux at the stagnation point (Sutton–Graves) and the total heat per m²
  $q_{max} = k\sqrt{\frac{\rho}{R_n}}\,v^3 \approx 1.42\,\mathrm{MW/m}^2 \quad (h \approx 49.4\,\mathrm{km},\ v \approx$
- Shield surface temperature by radiative equilibrium (ε = 0.85)
  $\varepsilon\sigma T^4 = q_{max}: \quad T = \sqrt[4]{\frac{q_{max}}{\varepsilon\sigma}} \approx 2329\,\mathrm{K} \approx 2055\,^\circ\mathrm{C}$
- Simplifications: the Earth is round and doesn't rotate, C_D and L/D are constant and the bank σ is the same for the whole flight (real guidance varies it). The heat flux is convective only; returning from the Moon, the glowing shock layer's radiation adds more. An ablative shield carries part of the heat away by vaporising, so the radiative equilibrium temperature is an upper estimate.

Graph

Path | In time



The axes have different scales. Use the slider or the play button to follow the flight in time.

The Space Shuttle: the shield's temperature by radiative equilibrium.

Input: `model(reentry; mode=flight; find=Tmax[degC]; vehicle=shuttle; body=winged; m=90000[kg]; A=250[m2]; CD=0.84[one]; LD=1[one]; Rn=1[m]; v0=7.8[km_s]; gamma=-1.2[deg];`

```
sigma=60[deg]; hk=25[km]; nlim=10[gn])
```

Result

$$T_{max} \approx 1660\,^{\circ}\mathrm{C}$$

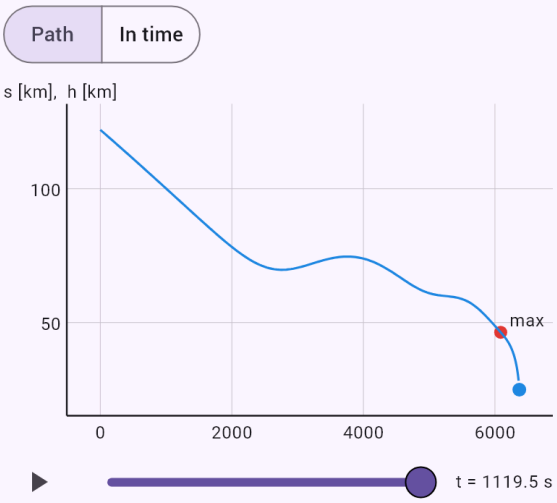
Steps

- Find T_{max}; given
  $m = 90\,000\,\mathrm{kg}, \quad A = 250\,\mathrm{m}^2, \quad C_D = 0.84, \quad L/D =$
- One pass through the atmosphere from the entry height h_0 (122 km by default) with the given flight path angle γ_0 and a constant bank angle σ down to the height h_k (the parachutes opening, the gliding flight for the Shuttle). The peak g-load is compared with Allen and Eggers' formula for a ballistic entry. The heat flux at the stagnation point follows from the Sutton–Graves formula q = k·√(ρ/R_n)·v³ and the shield's surface temperature from radiative equilibrium εσT⁴ = q. The model's equation
  $$\dot{v} = -\frac{D}{m} - g\sin\gamma, \quad \dot{\gamma} = \frac{L\cos\sigma}{mv} - \left(\frac{g}{v} - \frac{v}{r}\right)\cos\gamma,$$
- The Space Shuttle entered at an angle of attack of about 40° and dumped surplus lift by banking about its long axis (up to about 70°); it landed as an unpowered glider. The preset's values are approximate. How extremely the thermal protection is loaded was shown by the loss of Columbia (mission STS-107). At launch on 16 January 2003 a piece of insulating foam from the external tank broke through a reinforced carbon-carbon (RCC) panel on the left wing's leading edge. The leading edges heat up to about 1,650 °C (around 3,000 °F) during entry. On 1 February 2003 hot plasma entered the wing through the breach, the structure melted and the orbiter broke up over Texas; all seven crew members were lost.
- Frontal area and ballistic coefficient:
  $A = 250\,\mathrm{m}^2, \quad \beta = \frac{m}{C_D A} = 428.6\,\mathrm{kg/m}^2$
- Air density by the standard atmosphere (isothermal above 86 km)
  $\rho(122\,\mathrm{km}) \approx 1.19 \cdot 10^{-8}\,\mathrm{kg/m}^3, \quad \rho(60\,\mathrm{km}) \approx 3.1 \cdot 10^{-}$
- Vertical component of the lift at the bank angle σ:
  $L\cos\sigma = 0.5\,D \quad (\sigma = 60^{\circ})$
- Capture: the spacecraft descended to the height h_k:
  $h = h_k, \quad t \approx 1120\,\mathrm{s}, \quad s \approx 6371\,\mathrm{km}$
- Peak g-load (numerically)
  $a_{max} \approx 2.34\,g \quad (t \approx 941.5\,\mathrm{s},\ h \approx 46.2\,\mathrm{km})$
- A ballistic entry without lift would give much more by Allen and Eggers – lift lowers the g-load

$$a_{max} \approx \frac{v_0^2 \sin|\gamma_0|}{2eH} = \frac{7800^2 \sin 1.2^{\circ}}{2e \cdot 7200} \approx 3.32\,g$$

- Peak heat flux at the stagnation point (Sutton–Graves) and the total heat per m²
  $$q_{max} = k\sqrt{\frac{\rho}{R_n}}\,v^3 \approx 0.667\,\mathrm{MW/m}^2 \quad (h \approx 70\,\mathrm{km},\ v \approx$$
- Shield surface temperature by radiative equilibrium (ε = 0.85)
  $$\varepsilon\sigma T^4 = q_{max}: \quad T = \sqrt[4]{\frac{q_{max}}{\varepsilon\sigma}} \approx 1928\,\mathrm{K} \approx 1655\,^{\circ}\mathrm{C}$$
- Simplifications: the Earth is round and doesn't rotate, C_D and L/D are constant and the bank σ is the same for the whole flight (real guidance varies it). The heat flux is convective only; returning from the Moon, the glowing shock layer's radiation adds more. An ablative shield carries part of the heat away by vaporising, so the radiative equilibrium temperature is an upper estimate.

Graph



The axes have different scales. Use the slider or the play button to follow the flight in time.

How much of an iceberg is under the surface.

Input: `model(water; find=phi[pct]; rhob=917[kg_m3]; rhof=1025[kg_m3]; mode=float; fluid=sea;`

`body=ice)`

Result

$$\varphi \approx 89.46\,\%$$

Steps

- Find φ; given: $\rho_t = 917\,\mathrm{kg/m^3}, \quad \rho_f = 1025\,\mathrm{kg/m^3}$
- A body in a fluid feels a buoyant force equal to the weight of the displaced fluid. A floating body is submerged by the ratio of the densities. The same law carries a balloon in air. The model's equation: $F_b = \rho_f V g$
- Expressing the sought quantity: $\varphi = \dfrac{\rho_t}{\rho_f}$
- Substituting: $\varphi = \dfrac{917}{1025} \approx 0.8946$
- Converting the units: $\varphi \approx 0.8946 \;= 89.46\,\%$

What a hot-air balloon can carry.

Input: `model(water; find=mp[kg]; V=2800[m3]; rhog=0.946[kg_m3]; m0=350[kg]; rhof=1.225[kg_m3]; mode=float; fluid=air; gas=hotair)`

Result

$$m_p \approx 431.2\,\mathrm{kg}$$

Steps

- Find m_p; given $V = 2800\,\mathrm{m^3}, \quad \rho_g = 0.946\,\mathrm{kg/m^3}, \quad m_0 = 350\,\mathrm{kg}, \quad \rho$
- A body in a fluid feels a buoyant force equal to the weight of the displaced fluid. A floating body is submerged by the ratio of the densities. The same law carries a balloon in air. The model's equation: $F_b = \rho_f V g$
- Mass the gas can carry $m_v = V(\rho_f - \rho_g) = 2800(1.225 - 0.946) \approx 781.2\,\mathrm{kg}$
- Expressing the sought quantity: $m_p = m_v - m_0$
- Substituting: $m_p = 781.2 - 350 \approx 431.2\,\mathrm{kg}$

The pressure at the bottom of the Mariana Trench.

Input: `model(water; find=p[bar]; h=10935[m]; rhof=1025[kg_m3]; mode=pressure; water=sea)`

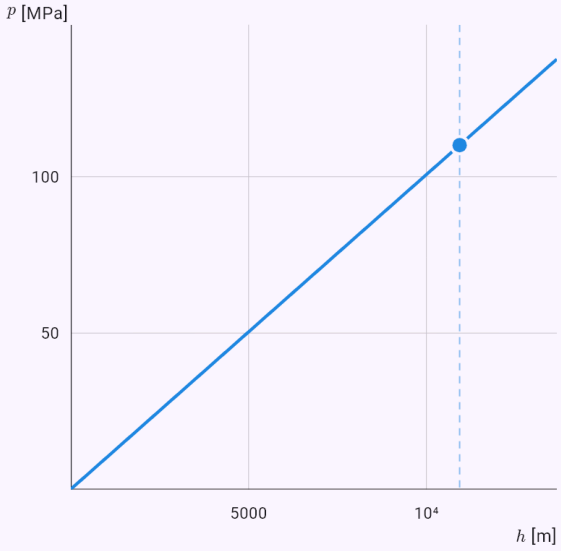
Result

$$p \approx 1101 \text{ bar}$$

Steps

- Find p; given: $h = 10\,940\,\text{m}, \quad \rho_f = 1025\,\text{kg/m}^3$
- Pressure grows linearly with depth, by the density times g times the depth. The air pressure adds to it at the surface. Water is taken as incompressible. The model's equation: $p = p_0 + \rho_f g h$
- At great depth water compresses slightly and its density also changes with temperature and salinity; the result can differ by about 2 %.
- Expressing the sought quantity: $p = p_0 + \rho_f g h$
- Substituting: $p = 101\,300 + 1025 \cdot 9.81 \cdot 10\,940 \approx 110.1\,\text{MPa}$
- How many times the atmospheric pressure: $\frac{p}{p_{atm}} \approx 1086$
- Converting the units: $p \approx 1.101 \cdot 10^8\,\text{Pa} = 1101\,\text{bar}$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

The pressure in a narrowing of a pipe.

Input: `model(water; find=p2[kPa]; A1=20[cm2]; A2=5[cm2]; v1=2[m_s]; p1=200[kPa];`

`rhof=998[kg_m3]; mode=bernoulli; water=fresh)`

Result

$$p_2 \approx 170.1\,\text{kPa}$$

Steps

- Find p_2; given
  $A_1 = 20\,\text{cm}^2, \quad A_2 = 5\,\text{cm}^2, \quad v_1 = 2\,\text{m/s}, \quad p_1 = 200$
- Converting the units
  $A_1 = 20\,\text{cm}^2 = 0.002\,\text{m}^2, \quad A_2 = 5\,\text{cm}^2 = 5 \cdot 10^{-4}\,\text{m}^2,$
- Along a streamline of an ideal liquid the sum of the pressure and the kinetic and potential energy of a volume is conserved. In a narrowing the speed rises and the pressure falls; if it reaches the vapour pressure, cavitation sets in. The model's equation:
  $p + \frac{1}{2}\rho_f v^2 + \rho_f g h = \text{const}$
- Continuity equation: $v_2 = v_1 \frac{A_1}{A_2} = 2 \cdot \frac{0.002}{5 \cdot 10^{-4}} \approx 8\,\text{m/s}$
- Expressing the sought quantity:
  $p_2 = p_1 + \frac{1}{2}\rho_f \left(v_1^2 - v_2^2\right) + \rho_f g (h_1 - h_2)$
- Substituting
  $p_2 = 200\,000 + \frac{1}{2} \cdot 998 \left((2)^2 - (8)^2\right) + 998 \cdot 9.81(0 - (0$
- Converting the units: $p_2 \approx 170\,100\,\text{Pa} = 170.1\,\text{kPa}$

A stone's terminal velocity in water and a chart of the start.

Input: `model(water; find=vt[m_s]; d=50[mm]; rhob=2700[kg_m3]; rhof=998[kg_m3];`

`nu=0.000001[m2_s]; Ca=0.5[one]; mode=motion; water=fresh; object=stone; shape=sphere)`

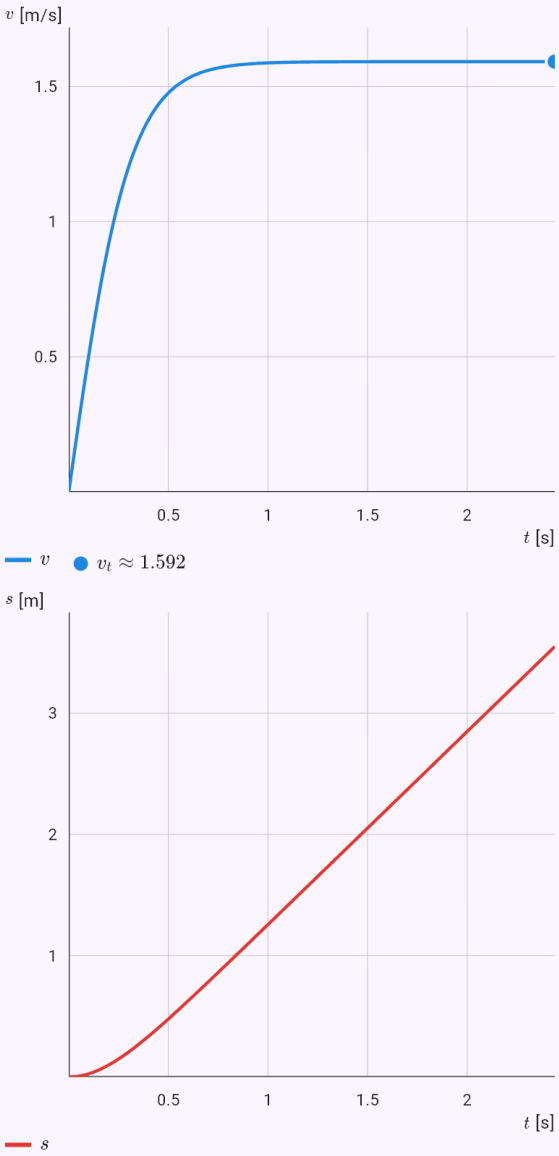
Result

$$v_t \approx 1.592\,\mathrm{m/s}$$

Steps

- Find v_t; given
  $d = 50\,\mathrm{mm}, \quad \rho_t = 2700\,\mathrm{kg/m^3}, \quad \rho_f = 998\,\mathrm{kg/m^3}, \quad$
- Converting the units: $d = 50\,\mathrm{mm} = 0.05\,\mathrm{m}$
- A body sinks or rises until the drag balances the difference of its weight and buoyancy, reaching its terminal velocity. Starting off, it must also set the surrounding water moving, as the added mass describes. The equation is solved numerically by the fourth-order Runge-Kutta method. The model's equation:
  $(m + C_a\rho_f V)\dot{v} = |mg - \rho_f V g| - \frac{1}{2}\rho_f v^2 A C_d$
- Volume, section and mass of the sphere
  $V = \frac{\pi d^3}{6} \approx 6.545 \cdot 10^{-5}\,\mathrm{m^3}, \quad A = \frac{\pi d^2}{4} \approx 0.001963\,\mathrm{m}$
- The net force points down, the body sinks:
  $F = |mg - \rho_f V g| \approx 1.093\,\mathrm{N}$
- Mass with the added mass:
  $m^* = m + C_a\rho_f V \approx 0.2094\,\mathrm{kg}, \quad C_a = 0.5$
- Reynolds number and the drag coefficient at the terminal velocity: $Re = \frac{v_t d}{\nu} \approx 79\,610, \quad C_d \approx 0.44$
- Terminal velocity: $v_t \approx 1.592\,\mathrm{m/s}$
- Time constant of the start with and without the added mass
  $\tau = \frac{m^* v_t}{F} \approx 0.305\,\mathrm{s}, \quad \tau_0 = \frac{m v_t}{F} \approx 0.2575\,\mathrm{s}$
- The added mass only changes the start. The terminal velocity doesn't depend on it, as nothing accelerates in steady motion.
- Substituting: $v_t \approx 1.592\,\mathrm{m/s}$

Graph



The model's quantities in time, computed from the solution; the answer is marked.

How long a bubble rises from a depth of 10 m.

Input: `model(water; find=t[s]; H=10[m]; d=5[mm]; rhob=1.2[kg_m3]; rhof=998[kg_m3];`

nu=0.000001[m2_s]; Ca=0.5[one]; mode=motion; water=fresh; object=bubble; shape=bubble)

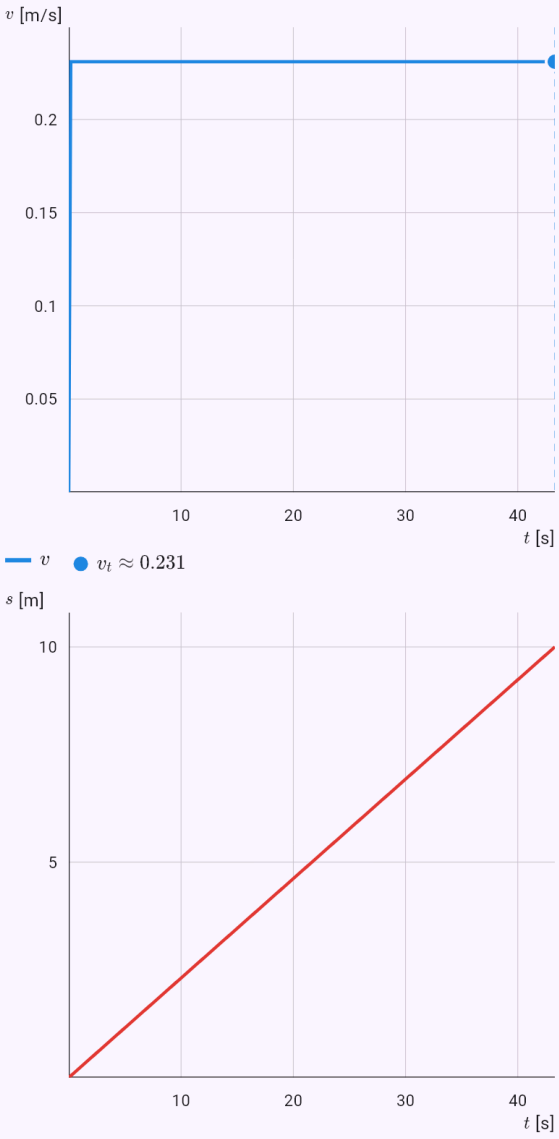
Result

$t \approx 43.3\,\mathrm{s}$

Steps

- Find t; given
  $H = 10\,\mathrm{m}, \quad d = 5\,\mathrm{mm}, \quad \rho_t = 1.2\,\mathrm{kg/m^3}, \quad \rho_f = 998$
- Converting the units: $d = 5\,\mathrm{mm} = 0.005\,\mathrm{m}$
- A body sinks or rises until the drag balances the difference of its weight and buoyancy, reaching its terminal velocity. Starting off, it must also set the surrounding water moving, as the added mass describes. The equation is solved numerically by the fourth-order Runge-Kutta method. The model's equation:
  $(m + C_a\rho_f V)\dot{v} = |mg - \rho_f V g| - \frac{1}{2}\rho_f v^2 A C_d$
- Volume, section and mass of the sphere
  $V = \frac{\pi d^3}{6} \approx 6.545 \cdot 10^{-8}\,\mathrm{m^3}, \quad A = \frac{\pi d^2}{4} \approx 1.963 \cdot 10^{-5}$
- The net force points up, the body rises:
  $F = |mg - \rho_f V g| \approx 6.4 \cdot 10^{-4}\,\mathrm{N}$
- Mass with the added mass:
  $m^* = m + C_a\rho_f V \approx 3.274 \cdot 10^{-5}\,\mathrm{kg}, \quad C_a = 0.5$
- The Eötvös number says how much the bubble deforms:
  $Eo = \frac{g\Delta\rho d^2}{\sigma} \approx 3.395$
- Reynolds number and the drag coefficient at the terminal velocity: $Re = \frac{v_t d}{\nu} \approx 1155, \quad C_d \approx 1.224$
- Terminal velocity: $v_t \approx 0.231\,\mathrm{m/s}$
- Time constant of the start with and without the added mass
  $\tau = \frac{m^* v_t}{F} \approx 0.01182\,\mathrm{s}, \quad \tau_0 = \frac{m v_t}{F} \approx 2.835 \cdot 10^{-5}\,\mathrm{s}$
- The added mass only changes the start. The terminal velocity doesn't depend on it, as nothing accelerates in steady motion.
- Tomiyama's correlation holds for ordinary, slightly contaminated water. In clean water small bubbles rise faster, and large ones break up or wobble.
- The equation was solved by the fourth-order Runge-Kutta method, number of steps: $N = 3363$

Graph



The model's quantities in time, computed from the solution; the answer is marked.

A speargun shaft's drop at 5 m.

Input: model(water; find=drop[cm]; x=5[m]; v0=20[m_s]; m=0.39[kg]; d=7[mm]; L=1.3[m];

rhof=1025[kg_m3]; nu=0.00000119[m2_s]; mode=harpoon; water=sea)

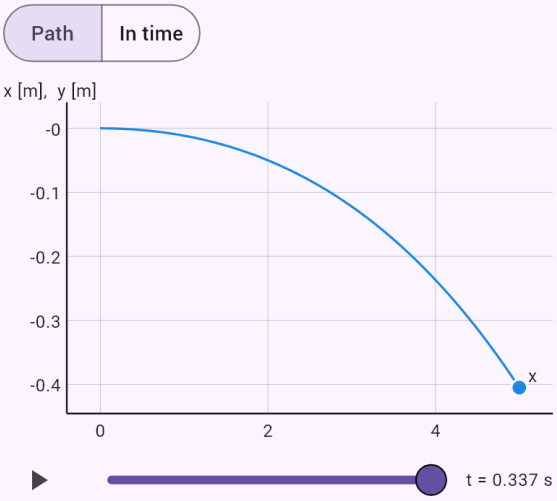
Result

$$\Delta y \approx 40.5\,\text{cm}$$

Steps

- Find Δy; given
  $x = 5\,\text{m}, \quad v_0 = 20\,\text{m/s}, \quad m = 0.39\,\text{kg}, \quad d = 7\,\text{mm},$
- Converting the units: $d = 7\,\text{mm} = 0.007\,\text{m}$
- A speargun for sport fishing: a slender shaft is slowed mainly by the friction along its surface, so it loses its speed within a few metres. The drop comes from the weight less the buoyancy. The equations are solved numerically. The model's equation
  $$m\dot{\vec{v}} = -\frac{1}{2}\rho_f A C_d |\vec{v}|\vec{v} - mg\left(1 - \frac{\rho_f}{\rho_s}\right)\vec{j}$$
- Friction coefficient by the ITTC-57 line
  $Re = \frac{v_0 L}{\nu} \approx 2.185 \cdot 10^7, \quad C_f = \frac{0.075}{(\log_{10} Re - 2)^2} \approx 0.($
- Drag coefficient of the shaft: the nose and the friction along it: $C_d = C_n + 4C_f\frac{L}{d} \approx 2.254$
- The way over which the speed falls to about 37 %, and the downward acceleration in water
  $\ell = \frac{2m}{\rho_f A C_d} \approx 8.772\,\text{m}, \quad g^* = g\left(1 - \frac{\rho_f}{\rho_s}\right) \approx 8.529\,\text{m/}$
- The equation was solved by the fourth-order Runge-Kutta method, number of steps: $N = 1538$
- The model leaves out the drag of the line, which shortens the real range further. The shaft's added mass along its axis is negligible.
- Converting the units: $\Delta y \approx 0.405\,\text{m} = 40.5\,\text{cm}$

Graph



The axes have different scales. Use the slider or the play button to follow the flight in time.

An underwater drone's range on its battery.

Input: `model(water; find=Rg[km]; v=1.5[m_s]; A=0.02835[m2]; Cd=0.23[one]; eta=0.6[one];`

`Ph=26[W]; En=1[kWh]; rhof=1025[kg_m3]; mode=powered; water=sea; vehicle=auv)`

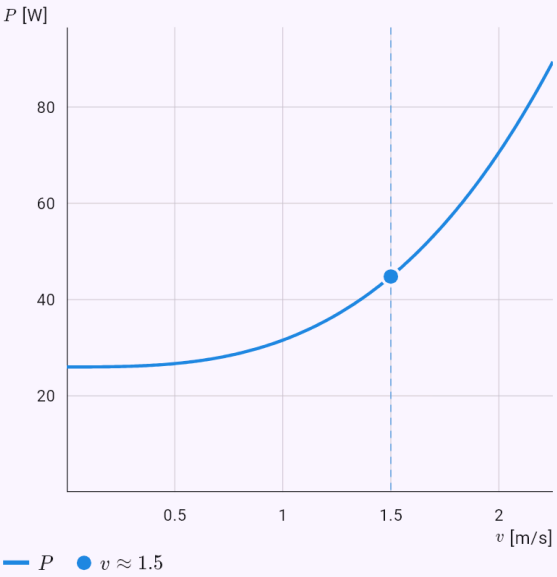
Result

$$R \approx 120.5\,\mathrm{km}$$

Steps

- Find R; given
  $v = 1.5\,\mathrm{m/s}, \quad A = 0.02835\,\mathrm{m^2}, \quad C_d = 0.23, \quad \eta = 0.$
- Converting the units: $E = 1\,\mathrm{kWh} = 3\,600\,000\,\mathrm{J}$
- In steady travel the thrust equals the drag. The drag grows with the square of the speed and the power with its cube. With a battery the constant load of the instruments also counts, so there is a speed of the longest range. The model's equation:
  $T = D = \frac{1}{2}\rho_f v^2 A C_d, \quad P = \frac{Dv}{\eta} + P_h$
- The factor K of the drag D = Kv²
  $K = \frac{1}{2}\rho_f A C_d = \frac{1}{2} \cdot 1025 \cdot 0.02835 \cdot 0.23 \approx 3.342\,\mathrm{kg/n}$
- Power needed: $P = \frac{Kv^3}{\eta} + P_h \approx 44.8\,\mathrm{W}$
- Speed of the longest range: $v_{opt} = \sqrt[3]{\frac{P_h \eta}{2K}} \approx 1.326\,\mathrm{m/s}$
- Battery endurance: $t_E = \frac{E}{P} \approx 22.32\,\mathrm{h}$
- Expressing the sought quantity: $R = v t_E$
- Substituting: $R = 1.5 \cdot 80\,360 \approx 120.5\,\mathrm{km}$
- Converting the units: $R \approx 120\,500\,\mathrm{m} = 120.5\,\mathrm{km}$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

A historical submarine: the power grows with the cube of the speed.

Input: `model(water; find=P[kW]; v=20.7[m_s]; A=70.9[m2]; Cd=0.065[one]; eta=0.7[one];`

`rhof=1025[kg_m3]; mode=powered; water=sea; vehicle=lira)`

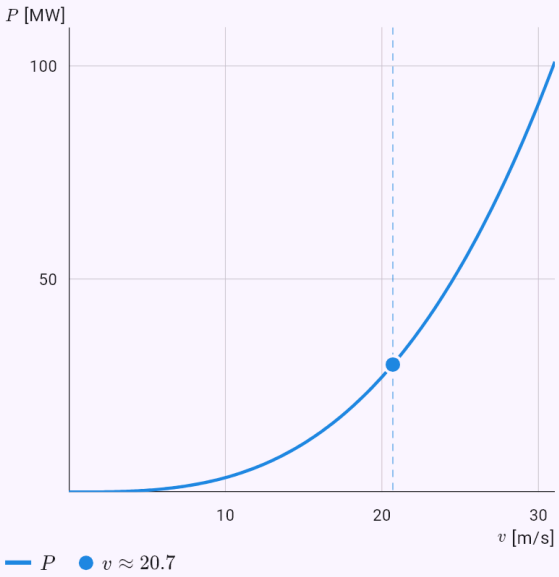
Result

$P \approx 29\,930\,\mathrm{kW}$

Steps

- Find P; given
  $v = 20.7\,\mathrm{m/s}, \quad A = 70.9\,\mathrm{m^2}, \quad C_d = 0.065, \quad \eta = 0.7$
- In steady travel the thrust equals the drag. The drag grows with the square of the speed and the power with its cube. With a battery the constant load of the instruments also counts, so there is a speed of the longest range. The model's equation:
  $T = D = \frac{1}{2}\rho_f v^2 A C_d, \quad P = \frac{Dv}{\eta} + P_h$
- The factor K of the drag D = Kv²
  $K = \frac{1}{2}\rho_f A C_d = \frac{1}{2} \cdot 1025 \cdot 70.9 \cdot 0.065 \approx 2362\,\mathrm{kg/m}$
- Expressing the sought quantity: $P = \frac{Kv^3}{\eta} + P_h$
- Substituting: $P = \frac{2362 \cdot (20.7)^3}{0.7} + 0 \approx 29.93\,\mathrm{MW}$
- The power grows with the cube of the speed: twice the speed takes eight times the power:
  $\frac{P(2v) - P_h}{P(v) - P_h} = 2^3 = 8$
- Converting the units: $P \approx 2.993 \cdot 10^7\,\mathrm{W} = 29\,930\,\mathrm{kW}$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

How much water a submarine must take in to hover.

Input: `model(water; find=mb[t]; V=3073[m3]; m0=2300000[kg]; rhof=1025[kg_m3]; mode=ballast; water=sea; boat=lira)`

Result

$m_b \approx 849.8\,\mathrm{t}$

Steps

- Find m_b; given
  $V = 3073\,\mathrm{m^3}, \quad m_0 = 2\,300\,000\,\mathrm{kg}, \quad \rho_f = 1025\,\mathrm{kg/m}$
- A submerged craft hovers when its mass with the ballast equals the mass of the displaced water. Letting water out or in makes it rise or sink. Blowing a tank at depth takes more air, as it must overcome the surrounding pressure. The model's equation: $\rho_f V = m_0 + m_b$
- Expressing the sought quantity: $m_b = \rho_f V - m_0$
- Substituting:
  $m_b = 1025 \cdot 3073 - 2\,300\,000 \approx 849\,800\,\mathrm{kg}$
- Volume of the ballast water: $V_b = \frac{m_b}{\rho_f} \approx 829.1\,\mathrm{m^3}$
- Converting the units: $m_b \approx 849\,800\,\mathrm{kg} = 849.8\,\mathrm{t}$

A container ship's power with its resistance split into parts.

Input: `model(water; find=P[kW]; v=12.35[m_s]; L=230[m]; B=32.2[m]; Tdr=10.8[m]; Cb=0.6505[one]; S=9424[m2]; k=0.1[one]; Cr=0.000444[one]; eta=0.7[one]; rhof=1025[kg_m3]; nu=0.00000119[m2_s]; mode=ship; water=sea; ship=kcs; compare=off)`

Result

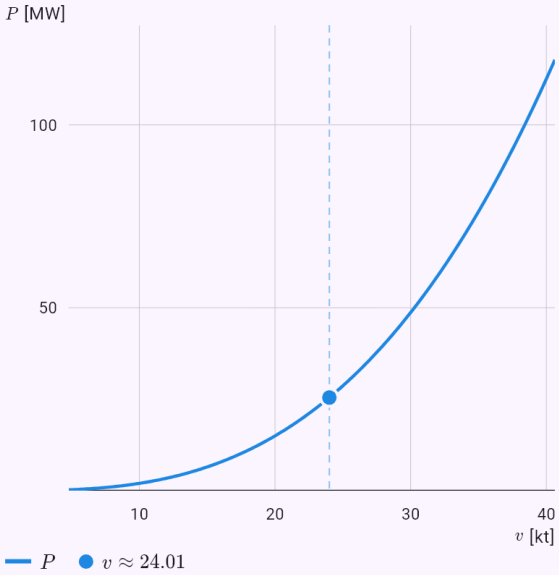
$$P \approx 25\,470\,\mathrm{kW}$$

Steps

- Find P; given
  $v = 12.35\,\mathrm{m/s}, \quad L = 230\,\mathrm{m}, \quad B = 32.2\,\mathrm{m}, \quad T = 10.$
- The draught follows from the displacement and the transverse stability from the metacentre. The resistance has a friction part by the ITTC-57 line and a residual part from a given coefficient. The Froude number and the hull speed show when the waves along the hull start to raise the resistance sharply. The model's equation
  $$Fn = \frac{v}{\sqrt{gL}}, \quad R = \frac{1}{2}\rho_f v^2 S((1+k)C_F + C_R)$$
- Froude and Reynolds numbers:
  $$Fn = \frac{v}{\sqrt{gL}} \approx 0.26, \quad Re = \frac{vL}{\nu} \approx 2.387 \cdot 10^9$$
- Friction coefficient by the ITTC-57 line:
  $$C_F = \frac{0.075}{(\log_{10} Re - 2)^2} \approx 0.001378$$
- Friction and residual resistance
  $R_F = qS(1+k)C_F \approx 1.116\,\mathrm{MN}, \quad R_R = qSC_R \approx 327$
- The residual coefficient is taken as constant. In reality it changes with the speed and comes from model tests.
- Total resistance: $R = R_F + R_R \approx 1.444\,\mathrm{MN}$
- Expressing the sought quantity: $P = \frac{Rv}{\eta}$
- Substituting: $P = \frac{1\,444\,000 \cdot 12.35}{0.7} \approx 25.47\,\mathrm{MW}$

- Converting the units: $P \approx 2.547 \cdot 10^7\,\mathrm{W} = 25\,470\,\mathrm{kW}$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

Metacentric height: a ship's stability.

Input: `model(water; find=GM[m]; L=230[m]; B=32.2[m]; Tdr=10.8[m]; Cb=0.6505[one];`

`Cw=0.82[one]; KG=13.81[m]; rhof=1025[kg_m3]; mode=ship; water=sea; ship=kcs)`

Result

$$GM \approx 0.6038\,\mathrm{m}$$

Steps

- Find GM; given
  $L = 230\,\mathrm{m}, \quad B = 32.2\,\mathrm{m}, \quad T = 10.8\,\mathrm{m}, \quad C_B = 0.65$
- The draught follows from the displacement and the transverse stability from the metacentre. The resistance has a friction part by the ITTC-57 line and a residual part from a given coefficient. The Froude number and the hull speed show when the waves along the hull start to raise the resistance sharply. The model's equation
  $Fn = \frac{v}{\sqrt{gL}}, \quad R = \frac{1}{2}\rho_f v^2 S((1+k)C_F + C_R)$
- Displaced volume and waterplane area (block coefficient)
  $\nabla = C_B LBT \approx 52\,030\,\mathrm{m}^3, \quad A_W = C_W LB \approx 6073\,\mathrm{m}^2$
- Height of the centre of buoyancy above the keel (Morrish's formula)
  $KB = T - \frac{1}{3}\left(\frac{T}{2} + \frac{\nabla}{A_W}\right) \approx 6.144\,\mathrm{m}$
- Height of the metacentre above the centre of buoyancy:
  $BM = \frac{I}{\nabla} = \frac{C_W^2 LB^3}{12\nabla} \approx 8.27\,\mathrm{m}$
- Expressing the sought quantity:
  $GM = KB + BM - KG$
- Substituting: $GM = 6.144 + 8.27 - 13.81 \approx 0.6038\,\mathrm{m}$
- The metacentre is above the centre of gravity: the ship is transversely stable.
- This holds for small heel angles. The waterplane's moment is an estimate from its coefficient, exact for a rectangle; for a real ship the result is only a first estimate.

A 544 t ship sailing beside the KM ekranoplan flying just above the water.

Input: `model(water; find=P[kW]; v=6.17[m_s]; m=544000[kg]; L=40[m]; B=9[m]; Tdr=2.6[m]; Cb=0.567[one]; S=381[m2]; k=0.2[one]; Cr=0.0015[one]; eta=0.55[one]; rhof=1025[kg_m3];`

```
nu=0.00000119[m2_s]; mode=ship; water=sea; ship=coaster; compare=on)
```

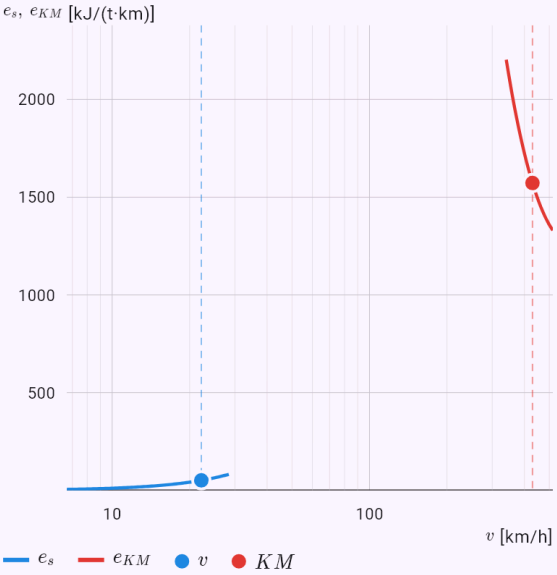
Result

$$P \approx 313.2\,\mathrm{kW}$$

Steps

- Find P; given
  $v = 6.17\,\mathrm{m/s}, \quad m = 544\,000\,\mathrm{kg}, \quad L = 40\,\mathrm{m}, \quad B = 9$
- The draught follows from the displacement and the transverse stability from the metacentre. The resistance has a friction part by the ITTC-57 line and a residual part from a given coefficient. The Froude number and the hull speed show when the waves along the hull start to raise the resistance sharply. The model's equation
  $$Fn = \frac{v}{\sqrt{gL}}, \quad R = \frac{1}{2}\rho_f v^2 S((1+k)C_F + C_R)$$
- Froude and Reynolds numbers
  $$Fn = \frac{v}{\sqrt{gL}} \approx 0.3115, \quad Re = \frac{vL}{\nu} \approx 2.074 \cdot 10^8$$
- Friction coefficient by the ITTC-57 line:
  $$C_F = \frac{0.075}{(\log_{10} Re - 2)^2} \approx 0.00188$$
- Friction and residual resistance
  $R_F = qS(1+k)C_F \approx 16.77\,\mathrm{kN}, \quad R_R = qSC_R \approx 11.1$
- The residual coefficient is taken as constant. In reality it changes with the speed and comes from model tests.
- The KM ekranoplan at cruise (the same computation as in the Aeroplane form)
  $m = 544\,\mathrm{t}, \quad v = 430\,\mathrm{km/h}, \quad D \approx 855.6\,\mathrm{kN}, \quad Dv \approx$
- Work against the resistance per tonne and kilometre: the ship and KM
  $$e_s = \frac{R}{m} \approx 51.32\,\mathrm{kJ/(t \cdot km)}, \quad e_{KM} = \frac{D}{m} \approx 1573\,\mathrm{kJ/(}$$
- How many times more energy KM needs and how many times faster it is: $\frac{e_{KM}}{e_s} \approx 30.6, \quad \frac{v_{KM}}{v} \approx 19.4$
- KM is computed in the ground effect by McCormick's factor. The comparison is of the work against the resistance; it leaves out the engines' efficiency, waves and weather.
- Total resistance: $R = R_F + R_R \approx 27.92\,\mathrm{kN}$
- Expressing the sought quantity: $P = \frac{Rv}{\eta}$
- Substituting: $P = \frac{27\,920 \cdot 6.17}{0.55} \approx 313.2\,\mathrm{kW}$
- Converting the units: $P \approx 313\,200\,\mathrm{W} = 313.2\,\mathrm{kW}$

Graph



The horizontal axis is logarithmic because the speeds differ many times over. The dots are the given speed of the ship and the cruise of the ekranoplan.

The neutron's energy from D-T – shared by the conservation of momentum.

Input: `model(fusion; find=En[MeV]; mode=fusionq; fusion=dt)`

Result

$$E_n \approx 14.05\,\mathrm{MeV}$$

Steps

- The energy of one reaction from the atomic masses. With two products the energy is shared by the conservation of momentum: the lighter particle takes the larger part. The model's equation:
  $Q = \left(\sum m_r - \sum m_p\right)c^2$
- Reaction: ${}^{2}\mathrm{H} + {}^{3}\mathrm{H} \to {}^{4}\mathrm{He} + n$
- Masses before and after:
  $\sum m_r \approx 5.03015106\,\mathrm{u}, \quad \sum m_p \approx 5.01126817\,\mathrm{u}$
- Substituting:
  $Q = \Delta m\,c^2 \approx 0.018883 \cdot 931.494 \approx 17.59\,\mathrm{MeV}$
- The products fly apart with opposite momenta of the same size, so their kinetic energies are in the inverse ratio of their masses.
- Expressing the sought quantity: $E_n = Q\dfrac{m_{\mathrm{He}}}{m_{\mathrm{He}} + m_n}$
- Substituting:
  $E_n = 17.59 \cdot \dfrac{4.0026}{4.0026 + 1.00866} \approx 14.05\,\mathrm{MeV}$
- Energy of the helium nucleus:
  $E_{\mathrm{He}} = Q - E_n \approx 3.54\,\mathrm{MeV}$

How much coal 1 g of D-T fuel replaces.

Input: `model(fusion; find=coal[t]; m=1[g]; q=29[MJ_kg]; mode=fusionfuel; fusion=dt)`

Result

$$m_c \approx 11.63\,\mathrm{t}$$

Steps

- Find m_c; given: $m = 1\,\mathrm{g}, \quad q = 29\,\mathrm{MJ/kg}$
- Converting the units: $m = 1\,\mathrm{g} = 0.001\,\mathrm{kg}$
- How much energy a given amount of fuel releases, compared with the fission of uranium-235 (about 200 MeV per nucleus) and with burning coal. The model's equation: $e_f = \dfrac{Q}{\sum m_r}, \quad E = e_f\, m$
- Reaction: ${}^{2}\mathrm{H} + {}^{3}\mathrm{H} \to {}^{4}\mathrm{He} + n$
- Energy of one reaction: $Q \approx 17.59\,\mathrm{MeV} = 2.818 \cdot 10^{-12}\,\mathrm{J}$
- Substituting $e_f = \dfrac{2.818 \cdot 10^{-12}}{8.353 \cdot 10^{-27}} \approx 3.374 \cdot 10^{14}\,\mathrm{J/kg}$
- Substituting: $E = 3.374 \cdot 10^{14} \cdot 0.001 \approx 3.374 \cdot 10^{11}\,\mathrm{J}$
- Substituting: $m_c = \dfrac{3.374 \cdot 10^{11}}{2.9 \cdot 10^{7}} \approx 11\,630\,\mathrm{kg}$
- Fusion has no critical mass. If something fails, the plasma cools down and the reaction stops by itself; the reactor cannot run away like a fission one.
- Converting the units: $m_c \approx 11\,630\,\mathrm{kg} = 11.63\,\mathrm{t}$

The Gamow peak of the p-p chain in the Sun's core.

Input: `model(fusion; find=E0[keV]; Z1=1[one]; Z2=1[one]; A1=1[one]; A2=1[one]; T=1.34[keV];`

```
mode=gamow; fusion=pp)
```

Result

$$E_0 \approx 6.035\,\mathrm{keV}$$

Steps

- Find E_0; given:
  $Z_1 = 1, \quad Z_2 = 1, \quad A_1 = 1, \quad A_2 = 1, \quad k_B T = 1.34\,\mathrm{keV}$
- In reality the nuclei tunnel through the barrier. The tunnelling probability e^(−√(E_G/E)) rises steeply with energy, while the fast nuclei of the Maxwell distribution become steeply rarer. Their product has a sharp Gamow peak E_0, where the reactions actually happen. The Gamow factor is taken as a ready formula, since deriving it belongs to quantum mechanics. The model's equation

$$P(E) \approx e^{-\sqrt{E_G/E}}, \quad E_G = 2\mu c^2 (\pi \alpha Z_1 Z_2)^2, \quad E_0 = \left($$

- Reduced mass

$$\mu c^2 = \frac{A_1 A_2}{A_1 + A_2} \cdot 931\,494\,\mathrm{keV} \approx 465\,700\,\mathrm{keV}$$

- Substituting

$$E_G = 2 \cdot 465\,700 \cdot (\pi \cdot 0.007297 \cdot 1 \cdot 1)^2 \approx 489.6\,\mathrm{keV}$$

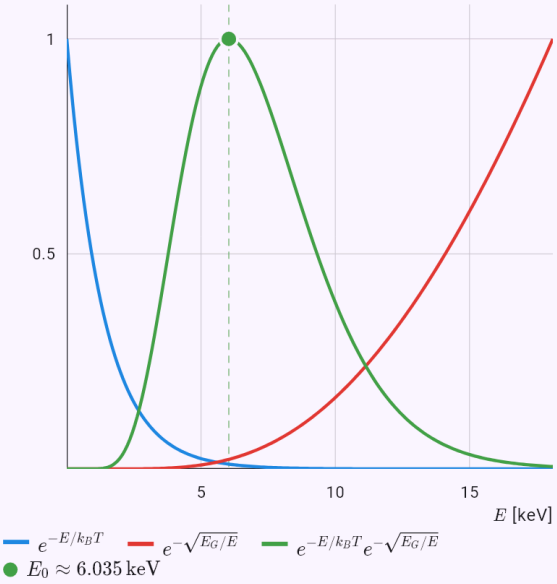
- Substituting

$$E_0 = \left(\frac{489.6 \cdot (1.34)^2}{4} \right)^{1/3} \approx 6.035\,\mathrm{keV}$$

- Width of the peak: $\Delta = \frac{4}{\sqrt{3}} \sqrt{E_0 k_B T} \approx 6.567\,\mathrm{keV}$
- Position of the peak in multiples of k_BT: $\frac{E_0}{k_B T} \approx 4.503$
- The reactions happen mostly in a narrow band of energies around E_0, several times the mean energy k_BT.

The graph shows both functions and their product, each divided by its maximum.

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

The triple product with ITER's parameters, compared with the ignition condition.

Input: `model(plasma; find=ntt[keVs_m3]; n=1e20[per_m3]; T=13[keV]; tau=3.7[s]; mode=lawson)`

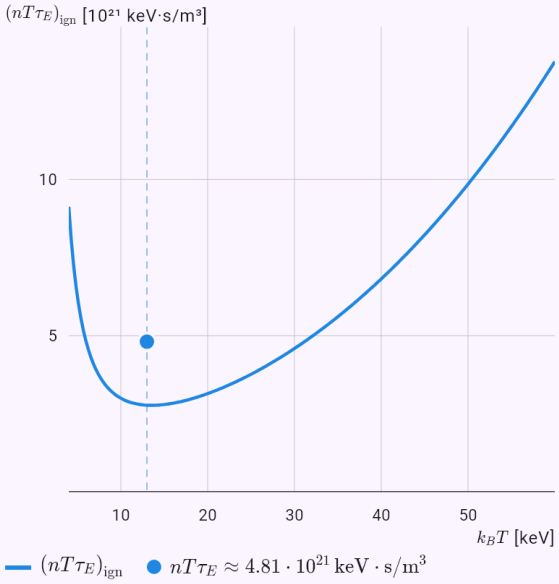
Result

$$nT\tau_E \approx 4.81 \cdot 10^{21}\,\mathrm{keV}\cdot\mathrm{s/m^3}$$

Steps

- Find $nT\tau_E$; given
  $n = 10^{20}\,\mathrm{m^{-3}}, \quad k_BT = 13\,\mathrm{keV}, \quad \tau_E = 3.7\,\mathrm{s}$
- The Lawson criterion: the plasma keeps itself hot (ignition) when the heating by the alpha particles of D-T reactions makes up for the heat losses 3nk_BT/τ_E. What decides is the triple product of the density n, the temperature T and the energy confinement time τ_E. Its minimum, about 3·10²¹ keV·s/m³, lies near 14 keV. The model's equation
  $$\frac{n^2}{4}\langle\sigma v\rangle E_\alpha \geq \frac{3nk_BT}{\tau_E} \Rightarrow nT\tau_E \geq \frac{12(k_BT)^2}{E_\alpha\langle\sigma v\rangle}$$
- D-T reaction rate (fit)
  $\langle\sigma v\rangle(13\,\mathrm{keV}) \approx 2.073 \cdot 10^{-22}\,\mathrm{m^3/s}$
- Substituting
  $$(nT\tau_E)_{\mathrm{ign}} = \frac{12\cdot(13)^2}{3520\cdot 2.073\cdot 10^{-22}} \approx 2.779\cdot 10^{21}\,\mathrm{keV}\cdot\mathrm{s},$$
- Substituting
  $nT\tau_E = 10^{20}\cdot 13\cdot 3.7 \approx 4.81\cdot 10^{21}\,\mathrm{keV}\cdot\mathrm{s/m^3}$
- The triple product is enough for ignition: the alpha heating covers the losses – $\dfrac{nT\tau_E}{(nT\tau_E)_{\mathrm{ign}}} \approx 1.731$
- The lowest triple product needed, about 3·10²¹ keV·s/m³, is near 14 keV. The bremsstrahlung losses are neglected here.

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

The D-T reaction rate at 10 keV as an integral by Simpson's rule.

Input: `model(plasma; find=sv[m3_s]; T=10[keV]; mode=reactivity; fuelmix=dt)`

Result

$$\langle \sigma v \rangle \approx 1.142 \cdot 10^{-22}\,\mathrm{m^3/s}$$

Steps

- Find $\langle \sigma v \rangle$; given: $k_B T = 10\,\mathrm{keV}$
- The reaction rate ⟨σv⟩ is the cross section σ(E) averaged over the Maxwell distribution of speeds. It is computed as an integral by Simpson's rule and compared with a published fit. The cross sections are Bosch and Hale's (Nuclear Fusion, 1992). The model's equation

$$\langle \sigma v \rangle = \sqrt{\frac{8}{\pi \mu}} (k_B T)^{-3/2} \int_0^\infty \sigma(E)\, E\, e^{-E/k_B T}\, dE$$

- Reaction: ${}^2\mathrm{H} + {}^3\mathrm{H} \to {}^4\mathrm{He} + n$
- Cross section (Bosch-Hale):

$$\sigma(E) = \frac{S(E)}{E\, e^{B_G/\sqrt{E}}}, \quad B_G \approx 34.38\,\sqrt{\mathrm{keV}}$$

- Simpson's rule: $n = 100, \quad h = \dfrac{40 k_B T}{n} = 4\,\mathrm{keV}$
- Integral

$$I = \int_0^{40 k_B T} \sigma(E)\, E\, e^{-E/k_B T}\, dE \approx \frac{h}{3}(f_0 + 4f_1 + 2f_2 +$$

- Values in SI

$$\mu \approx 2.005 \cdot 10^{-27}\,\mathrm{kg}, \quad k_B T \approx 1.602 \cdot 10^{-15}\,\mathrm{J}, \quad I \approx 2.$$

- Substituting

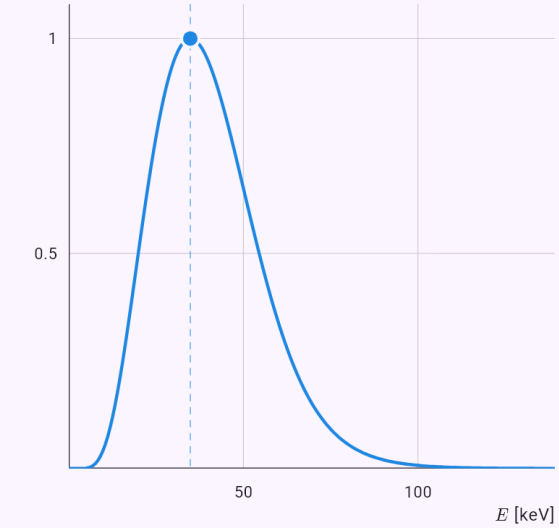
$$\langle \sigma v \rangle = \sqrt{\frac{8}{\pi \cdot 2.005 \cdot 10^{-27}}} \cdot \left(1.602 \cdot 10^{-15}\right)^{-3/2} \cdot 2.055$$

- Compared with the fit

$$\langle \sigma v \rangle_{\mathrm{BH}} \approx 1.136 \cdot 10^{-22}\,\mathrm{m^3/s}, \quad \delta \approx 0.5\,\%$$

- The integrand matters mostly around the Gamow peak, so integrating up to 40 k_BT is enough. The difference from the fit is of the order of a per cent.

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

The Larmor radius of a fusion alpha particle in 5 T.

Input: `model(plasma; find=rL[cm]; m=4.00151[u]; Z=2[one]; Ek=3520[keV]; B=5[T]; mode=larmor;`

`particle=alpha)`

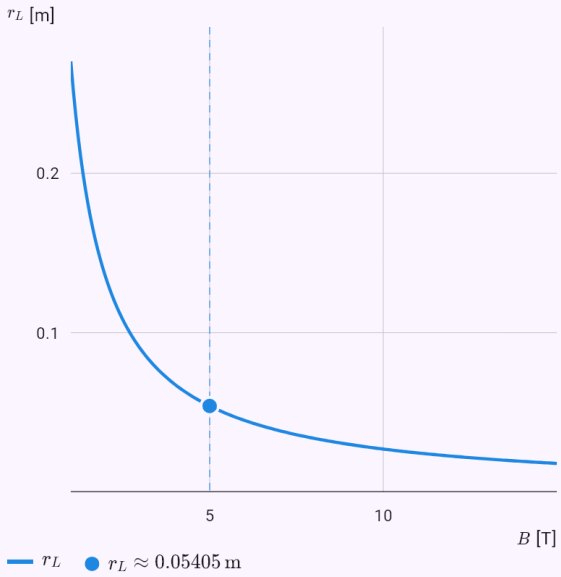
Result

$$r_L \approx 5.405\,\text{cm}$$

Steps

- Find r_L; given:
  $m = 4.002\,\text{u}, \quad Z = 2, \quad E_k = 3520\,\text{keV}, \quad B = 5\,\text{T}$
- A charged particle in a magnetic field circles around a field line (the Larmor radius), which is why a magnetic field keeps a hot plasma away from the walls. The calculation is relativistic; for slow ions γ ≈ 1. The model's equation: $r_L = \dfrac{p}{|Z|eB} = \dfrac{\gamma m v}{|Z|eB}, \quad f_c = \dfrac{|Z|eB}{2\pi\gamma m}$
- Mass: $m \approx 6.645 \cdot 10^{-27}\,\text{kg}$
- Lorentz factor and speed
  $\gamma = 1 + \dfrac{E_k}{mc^2} \approx 1.00094, \quad v \approx 1.302 \cdot 10^{7}\,\text{m/s} \approx 0.04$
- Momentum
  $p = \dfrac{\sqrt{E_k^2 + 2E_k mc^2}}{c} \approx 8.659 \cdot 10^{-20}\,\text{kg} \cdot \text{m/s}$
- Substituting
  $r_L = \dfrac{8.659 \cdot 10^{-20}}{2 \cdot 1.602 \cdot 10^{-19} \cdot 5} \approx 0.05405\,\text{m}$
- Ions in a tokamak circle on radii of millimetres to centimetres, which is why the magnetic field keeps them away from the walls.
- Converting the units: $r_L \approx 0.05405\,\text{m} = 5.405\,\text{cm}$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

The Lorentz factor of a proton in the LHC.

Input: `model(relativity; find=gamma[one]; m=938.272[MeV_c2]; Ek=6800[GeV]; mode=lorentz;`

given=kinetic; particle=proton)

Result

$$\gamma \approx 7248.37$$

Steps

- Find γ; given: $m = 938.3\,\mathrm{MeV}/c^2, \quad E_k = 6800\,\mathrm{GeV}$
- Converting the units:
  $m = 938.3\,\mathrm{MeV}/c^2 = 0.9383\,\mathrm{GeV}/c^2$
- From the speed (as a fraction of c) or the kinetic energy it computes γ, the speed, the total energy and the momentum. A proton in the LHC (6.8 TeV) is slower than light by only a few metres per second. The model's equation

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}, \quad E = \gamma mc^2, \quad E^2 = (pc)^2 + \left(mc^2\right)$$

- Rest energy: $mc^2 \approx 0.9383\,\mathrm{GeV}$
- Substituting: $\gamma = 1 + \frac{E_k}{mc^2} = 1 + \frac{6800}{0.9383} \approx 7248$
- Speed: $\beta = \sqrt{1 - \frac{1}{\gamma^2}} \approx 0.9999999905$
- How much slower than light:
  $1 - \beta \approx 9.517 \cdot 10^{-9}, \quad c - v \approx 2.853\,\mathrm{m/s}$
- An ultrarelativistic particle: almost all its energy is kinetic and its speed differs from c only slightly.

How many muons from 15 km reach the ground thanks to time dilation.

Input: model(relativity; find=surv[pct]; m=105.658[MeV_c2]; Ek=4[GeV]; tau=2.197[us];

`d=15[km]; mode=dilation; given=kinetic; unstable=muon)`

Result

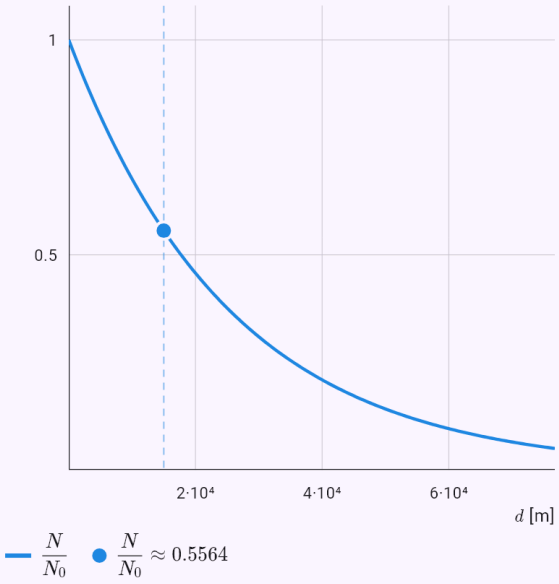
$$\frac{N}{N_0} \approx 55.64\,\%$$

Steps

- Find $\frac{N}{N_0}$; given
  $m = 105.7\,\mathrm{MeV}/c^2, \quad E_k = 4\,\mathrm{GeV}, \quad \tau_0 = 2.197\,\mu\mathrm{s},$
- Converting the units
  $m = 105.7\,\mathrm{MeV}/c^2 = 0.1057\,\mathrm{GeV}/c^2, \quad \tau_0 = 2.197\,\mu\mathrm{s}$
- An unstable particle decays with its own mean lifetime τ_0. In the laboratory its time runs γ times slower, so it gets γ times farther. Without time dilation, muons made 15 km up in the atmosphere would hardly reach the ground. The model's equation:
  $\tau = \gamma\tau_0, \quad L = \gamma\beta c\tau_0, \quad \frac{N}{N_0} = e^{-d/L}$
- Substituting: $\gamma = 1 + \frac{E_k}{mc^2} = 1 + \frac{4}{0.1057} \approx 38.86$
- Speed: $\beta = \sqrt{1 - \frac{1}{\gamma^2}} \approx 0.999668807$
- Substituting
  $L = 38.86 \cdot 0.9997 \cdot 2.998 \cdot 10^8 \cdot 2.197 \cdot 10^{-6} \approx 25\,590\,1$
- Substituting: $\frac{N}{N_0} = e^{-15\,000/25\,590} \approx 0.5564$
- Without time dilation the share getting there:
  $\frac{N}{N_0}|_{\gamma=1} = e^{-d/(\beta c\tau_0)} \approx 1.277 \cdot 10^{-10}$
- In the particle's own frame time doesn't slow down, but the distance shrinks (length contraction). Both views give the same number of decays.

- Converting the units: $\frac{N}{N_0} \approx 0.5564 = 55.64\,\%$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

The collision energy of an LHC beam on a fixed target.

Input: `model(accelerator; find=fixed[GeV]; E=6800[GeV]; m=0.938272[GeV_c2]; mode=collision;`

machine=lhc)

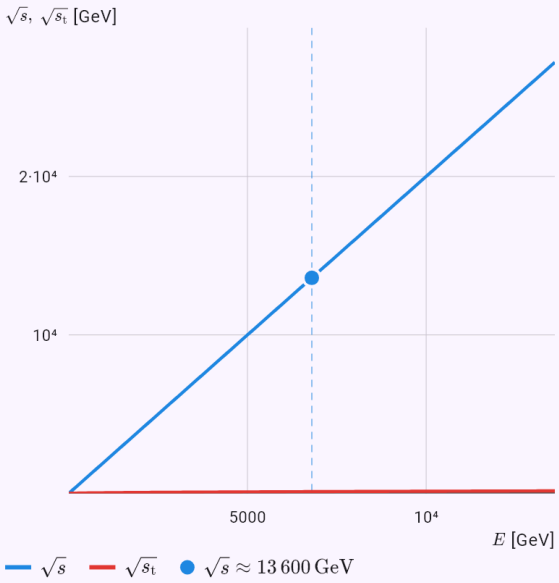
Result

$$\sqrt{s_t} \approx 113\,\mathrm{GeV}$$

Steps

- Find $\sqrt{s_t}$; given: $E = 6800\,\mathrm{GeV}, \quad m = 0.9383\,\mathrm{GeV}/c^2$
- In colliding beams the whole energy of both particles goes into the collision: √s = 2E. A beam aimed at a fixed target spends most of its energy on moving the centre of mass, and √s grows only with the square root of E. That is why colliders are built. The model's equation:
  $\sqrt{s} = 2E, \quad \sqrt{s_t} = \sqrt{2\left(mc^2\right)^2 + 2E\,mc^2}$
- √s is the system's invariant mass: the energy available for making new particles.
- Substituting
  $\sqrt{s_t} = \sqrt{2 \cdot (0.9383)^2 + 2 \cdot 6800 \cdot 0.9383} \approx 113\,\mathrm{GeV}$
- Colliding beams: $\sqrt{s} = 2E \approx 13\,600\,\mathrm{GeV}$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

An electron's energy loss per turn in LEP.

Input: `model(accelerator; find=U0[GeV]; E=104.5[GeV]; m=0.000511[GeV_c2]; R=3.096[km];`

```
mode=synchrotron; machine=lep)
```

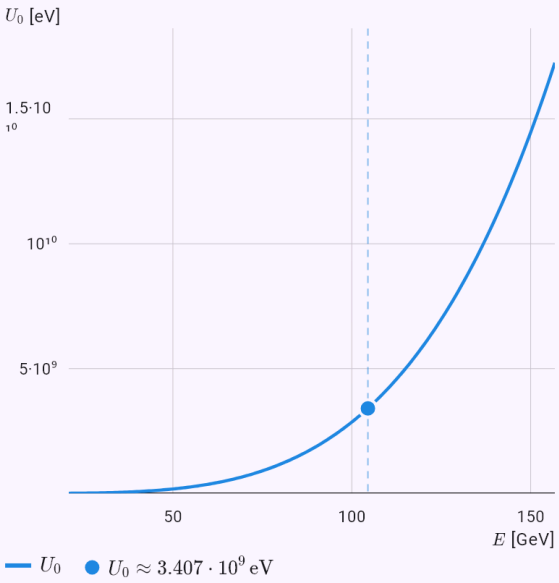
Result

$$U_0 \approx 3.407\,\text{GeV}$$

Steps

- Find U_0; given $E = 104.5\,\text{GeV}, \quad m = 5.11 \cdot 10^{-4}\,\text{GeV}/c^2, \quad R = 3.0$
- Converting the units: $R = 3.096\,\text{km} = 3096\,\text{m}$
- A charged particle on a curved path radiates (synchrotron radiation). The loss per turn grows with γ⁴, so at the same energy as 1/m⁴. An electron loses 10¹³ times more than a proton, which is why LEP stopped at 104 GeV and the LHC accelerates protons. The model's equation: $U_0 = \dfrac{e^2\beta^3\gamma^4}{3\varepsilon_0 R}, \quad P = \dfrac{U_0}{e} I$
- Lorentz factor: $\gamma = \dfrac{E}{mc^2} \approx 204\,500$
- Substituting $U_0 = \dfrac{(1.602 \cdot 10^{-19})^2 \cdot (1)^3 \cdot (204\,500)^4}{3 \cdot 8.854 \cdot 10^{-12} \cdot 3096} \approx 5.459 \cdot 10^{-1}$
- In electronvolts: $U_0 \approx 3.407 \cdot 10^{9}\,\text{eV}$
- Electrons radiate 10¹³ times more than protons of the same energy: $\dfrac{U_{0,e}}{U_{0,p}} = \left(\dfrac{m_p}{m_e}\right)^4 \approx 1.137 \cdot 10^{13}$
- The other kind of particle at the same energy and radius: $U_0' \approx 2.998 \cdot 10^{-4}\,\text{eV}$
- Converting the units: $U_0 \approx 5.459 \cdot 10^{-10}\,\text{J} = 3.407\,\text{GeV}$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

44. Statics and strength of materials

The Statics form (Applications, field Engineering) has three modes. A beam lies on the x axis from 0 to L and is supported by a pin, a roller or a fixed support; you can load it by vertical and inclined forces, moments and a distributed load - uniform, triangular or trapezoidal. Several forces go in one field separated by spaces, their positions and angles in the same order. Inner hinges make a [[gerberBeam|Gerber beam]].

The steps start with a [[freeBody|free-body diagram]]: the reactions replace the supports and the rank of the equations of equilibrium shows the [[staticDeterminacy|static determinacy]]. When the beam is statically indeterminate or improperly supported, the app says so with the degree of indeterminacy. The equations are solved one at a time - always the one with a single unknown left (a moment about a support, for a Gerber beam the part beyond a hinge first) - and an equation not used serves as a check.

The app lists the internal forces N, T and M piece by piece between the notable points and finds the greatest bending moment by [[schwedlerTheorem|Schwedler's theorem]]: where the shear force is zero or where it jumps. Under the sketch of the beam with its loads and reactions there are the diagrams of N, T and M at the same scale of length.

The mode Body in the plane computes the reactions of a body given by its supports' points and the forces' points of application - a ladder against a wall, for example. For a roller you choose the reaction's direction by the angle β.

The cross-section's properties are the area, the [[centroid|centroid]], the second moments by the [[parallelAxisTheorem|parallel axis theorem]], the [[sectionModulus|section moduli]] and for an unsymmetric section also the [[principalAxes|principal axes]] - for a rectangle, a tube, T, I, U, L profiles and a section composed of rectangles and circles with holes. The picture shows the section at true scale with its centroid and axes. The rolled profiles' fillets are neglected.

The form Tanker with a liquid computes a partly filled tanker braking, speeding up and in a curve. The surface tilts perpendicular to the resultant of gravity and the inertial force ($\tan\theta = a/g$), the liquid's centre of gravity moves and with it the whole's. The axle loads are the reactions of a beam on two supports; in a curve the app computes the [[loadTransferRatio|load transfer]] to the outer wheels and the critical speed at which the tanker tips over - compared with a rigid load, with the natural frequency of the [[sloshing|sloshing]] too. An interactive demo is at mathtrace.app under "Try it yourself".

The form Beam design compares three variants of a beam in bending: the lightest standard profile from the tables (IPE, HEB, round and rectangular tubes), the optimal constant section of a chosen shape (a rectangle, a circle, a tube with a limited wall slenderness) and a [[uniformStrengthBeam|beam of uniform strength]]. It watches the strength, the allowed deflection L/n and the shear at the free end; the steps show which condition is [[activeConstraint|active]] and the chart the section, the moment and the

deflection along the beam. The inspiration was a thesis in optimization: an arm designed by optimization came out much lighter than one from the tables.

Examples

A simply supported beam with a force and a distributed load on part of it: the reactions, the pieces and the greatest moment.

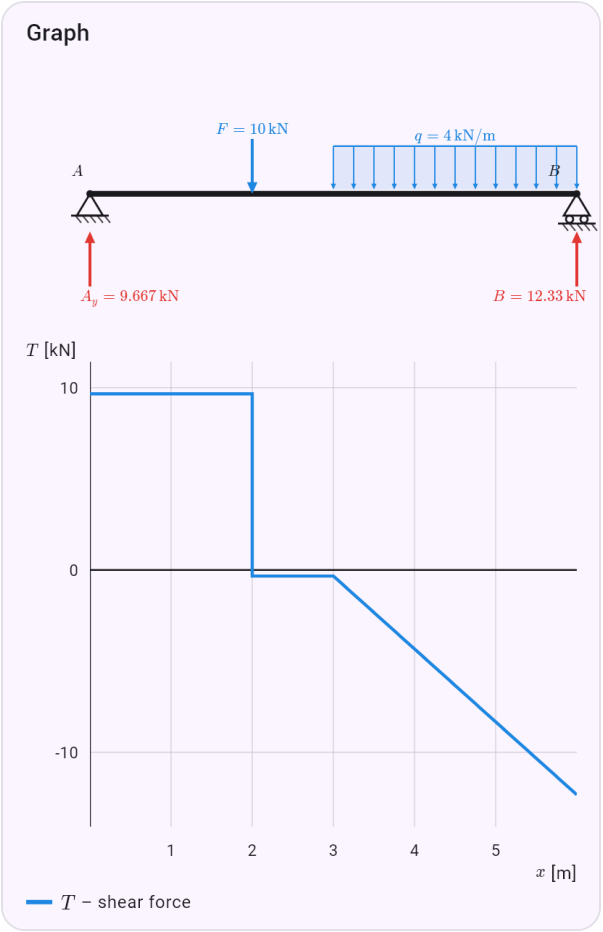
Input: `model(statics; mode=beam; find=Mmax[kNm]; L=6[m]; supA=pin; supB=roller; F=10[kN]; xF=2[m]; q=4[kN_m]; qa=3[m]; qb=6[m])`

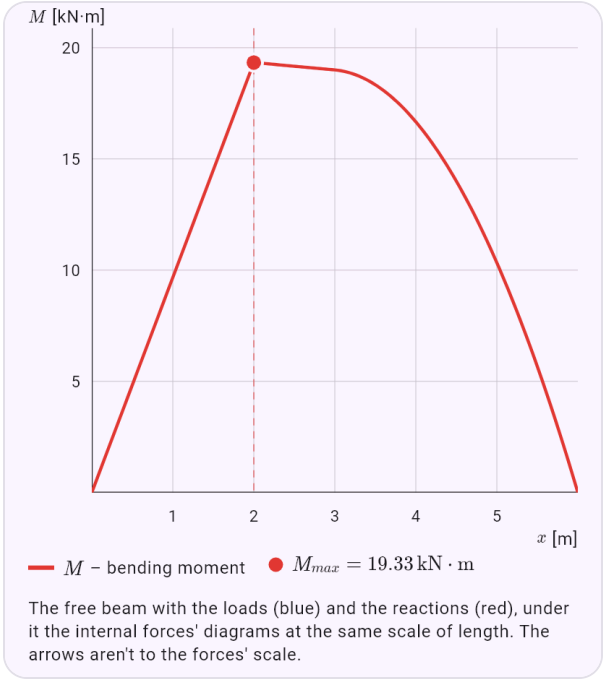
Result

$$M_{max} \approx 19.33\,\mathrm{kN\cdot m}$$
$$x \approx 2\,\mathrm{m}$$

Steps

- Find M_{max}; given
  $L = 6\,\mathrm{m}, \quad F = 10\,\mathrm{kN}, \quad x_F = 2\,\mathrm{m}, \quad q_a = 4\,\mathrm{kN/m},$
- The beam lies on the x axis from 0 to L. A pin carries a horizontal and a vertical force, a roller only the force perpendicular to its base, a fixed support a moment as well. A force makes the angle α with the beam (90° = vertically down), a moment is positive counterclockwise, a distributed load positive downward. Inner hinges make a Gerber beam: the bending moment is zero in a hinge, which gives one more equation. The diagrams of N, T and M are built piece by piece by Schwedler's theorem. The model's equation:
  $\sum F_x = 0, \quad \sum F_y = 0, \quad \sum M = 0$
- The distributed load's resultant (at the centroid of the load's area):
  $Q = q\,l = 4 \cdot 3 = 12\,\mathrm{kN}, \quad x_Q = 3 + \frac{3}{2} = 4.5\,\mathrm{m}$
- The unknown reactions: $A_x,\ A_y,\ B$
- Statically determinate: as many unknowns as independent equations (n unknowns, e equations, h the rank): $n = 3, \quad e = 3, \quad h(\mathbf{A}) = 3$
- Sum of the moments about a support - an equation with one unknown
  $\sum M_A = 0: \quad B \cdot 6 - 10 \cdot 2 - 12 \cdot 4.5 = 0 \quad \Rightarrow \quad B$
- Sum of the moments about a support - an equation with one unknown
  $\sum M_B = 0: \quad -A_y \cdot 6 + 10 \cdot 4 + 12 \cdot 1.5 = 0 \quad \Rightarrow$
- Sum of the horizontal forces:
  $\sum F_x = 0: \quad A_x = 0 \quad \Rightarrow \quad A_x = 0\,\mathrm{kN}$
- A check by an equation not used:
  $\sum F_y = 9.667 + 12.33 - 10 - 12 = 0$
- Schwedler's theorem: the bending moment's derivative is the shear force and the shear force's derivative is minus the distributed load. Where T is zero, M has an extremum.
- The internal forces in the piece (x in m, forces in kN, moments in kN·m):
  $x \in [0,\ 2]: \quad T = 9.667, \quad M = 9.667x$
- The internal forces in the piece (x in m, forces in kN, moments in kN·m):
  $x \in [2,\ 3]: \quad T = -0.3333, \quad M = 20 - 0.3333x$
- The internal forces in the piece (x in m, forces in kN, moments in kN·m):
  $x \in [3,\ 6]: \quad T = 11.67 - 4x, \quad M = 2 + 11.67x - 2x^2$
- The greatest bending moment is where the shear force jumps: $M_{max} = M(2) = 19.33\,\mathrm{kN\cdot m}$
- A positive bending moment bends the beam down (the bottom fibres in tension), a negative one up, typically over a support or at a fixed end.





The free beam with the loads (blue) and the reactions (red), under it the internal forces' diagrams at the same scale of length. The arrows aren't to the forces' scale.

A cantilever with an inclined force at its end and a uniform load: the horizontal and vertical reactions and the fixed end's moment.

Input: `model(statics; mode=beam; find=R[kN]; L=3[m]; supA=fixed; supB=none; F=8[kN];`

```
xF=3[m]; aF=60[deg]; q=2[kN_m])
```

Result

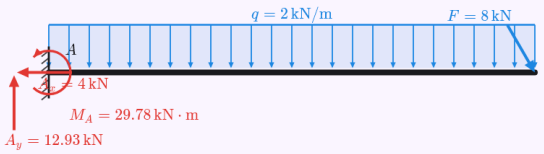
$$A_x = -4\,\text{kN}$$
$$A_y = 12.93\,\text{kN}$$
$$M_A = 29.78\,\text{kN}\cdot\text{m}$$

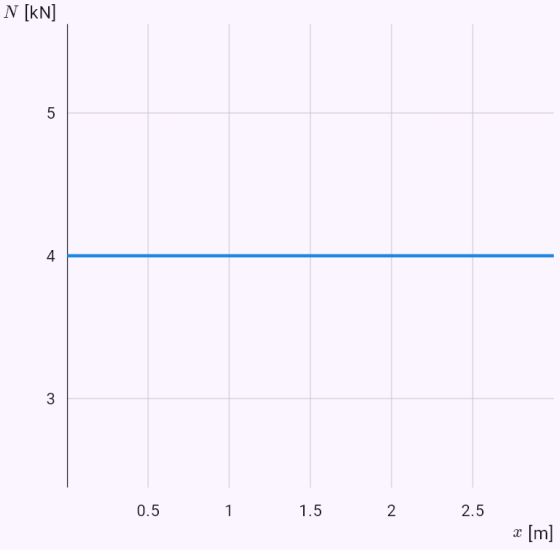
Steps

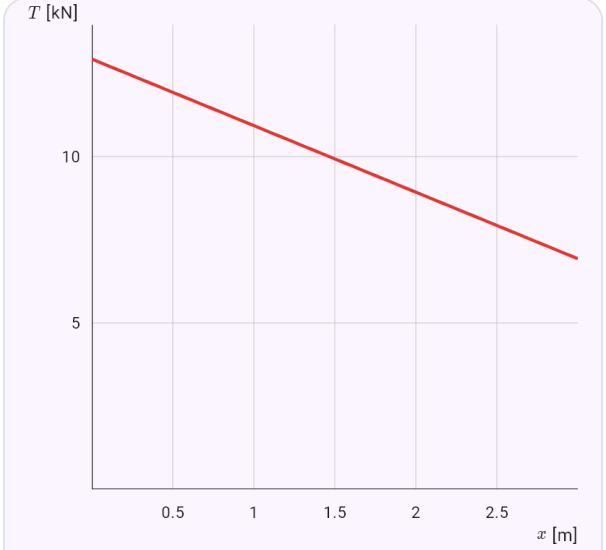
- Find R; given
  $L = 3\,\text{m}, \quad F = 8\,\text{kN}, \quad x_F = 3\,\text{m}, \quad \alpha = 60\,^\circ, \quad q_a =$
- The beam lies on the x axis from 0 to L. A pin carries a horizontal and a vertical force, a roller only the force perpendicular to its base, a fixed support a moment as well. A force makes the angle α with the beam (90° = vertically down), a moment is positive counterclockwise, a distributed load positive downward. Inner hinges make a Gerber beam: the bending moment is zero in a hinge, which gives one more equation. The diagrams of N, T and M are built piece by piece by Schwedler's theorem. The model's equation:
  $\sum F_x = 0, \quad \sum F_y = 0, \quad \sum M = 0$
- The distributed load's resultant (at the centroid of the load's area):
  $Q = q\,l = 2 \cdot 3 = 6\,\text{kN}, \quad x_Q = 0 + \frac{3}{2} = 1.5\,\text{m}$
- An inclined force's components:
  $F_x = F\cos\alpha = 4\,\text{kN}, \quad F_y = -F\sin\alpha = -6.928\,\text{kN}$
- The unknown reactions: $A_x,\ A_y,\ M_A$
- Statically determinate: as many unknowns as independent equations (n unknowns, e equations, h the rank): $n = 3, \quad e = 3, \quad h(\mathbf{A}) = 3$

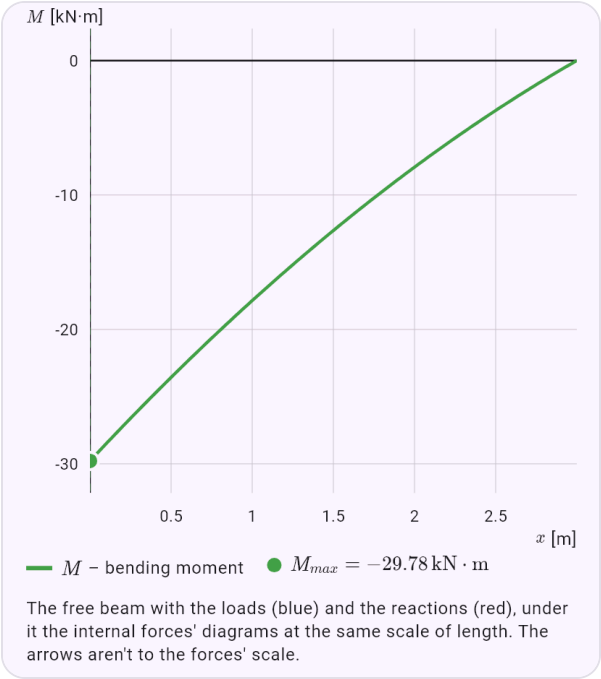
- Sum of the moments about a support - an equation with one unknown
  $\sum M_A = 0: \quad M_A - 6.928 \cdot 3 - 6 \cdot 1.5 = 0 \quad \Rightarrow \quad M$
- Sum of the horizontal forces:
  $\sum F_x = 0: \quad A_x + 4 = 0 \quad \Rightarrow \quad A_x = -4\,\text{kN}$
- Sum of the vertical forces
  $\sum F_y = 0: \quad A_y - 6.928 - 6 = 0 \quad \Rightarrow \quad A_y = 12.93$

Graph









The free beam with the loads (blue) and the reactions (red), under it the internal forces' diagrams at the same scale of length. The arrows aren't to the forces' scale.

A Gerber beam on three supports with a hinge: the calculation starts with the part beyond the hinge.

Input: `model(statics; mode=beam; find=Mmax[kNm]; L=12[m]; supA=pin; supB=roller; xB=6[m];`

```
xs=12[m]; xh=8[m]; F=12[kN]; xF=9[m]; q=2[kN_m]; qa=0[m]; qb=6[m])
```

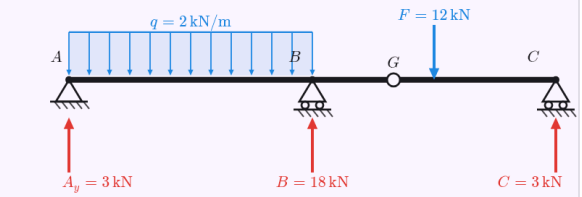
Result

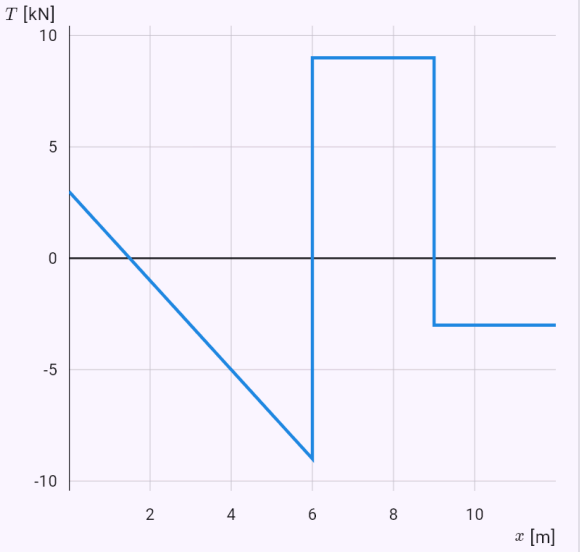
$$M_{max} \approx -18\,\mathrm{kN}\cdot\mathrm{m}$$
$$x \approx 6\,\mathrm{m}$$

Steps

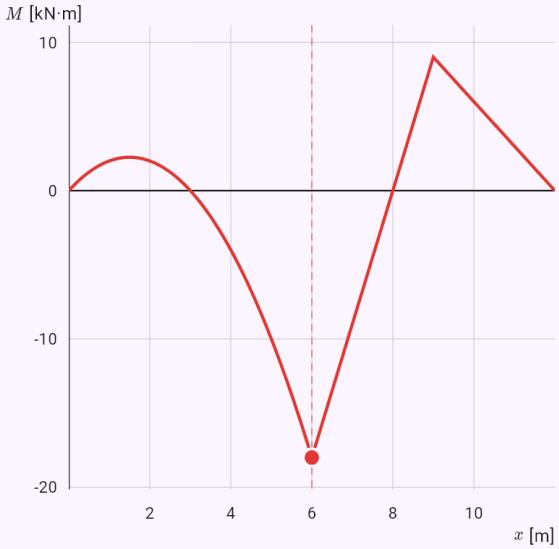
- Find M_{max}; given
  $L = 12\,\mathrm{m}, \quad x_B = 6\,\mathrm{m}, \quad x_s = 12\,\mathrm{m}, \quad x_G = 8\,\mathrm{m}, \quad F$
- The beam lies on the x axis from 0 to L. A pin carries a horizontal and a vertical force, a roller only the force perpendicular to its base, a fixed support a moment as well. A force makes the angle α with the beam (90° = vertically down), a moment is positive counterclockwise, a distributed load positive downward. Inner hinges make a Gerber beam: the bending moment is zero in a hinge, which gives one more equation. The diagrams of N, T and M are built piece by piece by Schwedler's theorem. The model's equation:
  $\sum F_x = 0, \quad \sum F_y = 0, \quad \sum M = 0$
- The distributed load's resultant (at the centroid of the load's area):
  $Q = q\,l = 2\cdot 6 = 12\,\mathrm{kN}, \quad x_Q = 0 + \frac{6}{2} = 3\,\mathrm{m}$
- The unknown reactions: $A_x,\ A_y,\ B,\ C$
- Statically determinate: as many unknowns as independent equations (n unknowns, e equations, h the rank): $n = 4, \quad e = 3 + 1 = 4, \quad h(\mathbf{A}) = 4$
- The moment of the part right of the hinge about the hinge is zero:
  $\sum M_G = 0: \quad C\cdot 4 - 12\cdot 1 = 0 \quad \Rightarrow \quad C = 3\,\mathrm{kN}$
- Sum of the moments about a support - an equation with one unknown
  $\sum M_A = 0: \quad B\cdot 6 + 3\cdot 12 - 12\cdot 9 - 12\cdot 3 = 0 \quad \Rightarrow$
- The moment of the part left of the hinge about the hinge is zero
  $\sum M_G = 0: \quad -A_y\cdot 8 - 18\cdot 2 + 12\cdot 5 = 0 \quad \Rightarrow \quad A$
- Sum of the horizontal forces:
  $\sum F_x = 0: \quad A_x = 0 \quad \Rightarrow \quad A_x = 0\,\mathrm{kN}$
- A check by an equation not used:
  $\sum F_y = 3 + 18 + 3 - 12 - 12 = 0$
- Schwedler's theorem: the bending moment's derivative is the shear force and the shear force's derivative is minus the distributed load. Where T is zero, M has an extremum.
- The internal forces in the piece (x in m, forces in kN, moments in kN·m):
  $x \in [0,\ 6]: \quad T = 3 - 2x, \quad M = 3x - x^2$
- The internal forces in the piece (x in m, forces in kN, moments in kN·m):
  $x \in [6,\ 9]: \quad T = 9, \quad M = -72 + 9x$
- The internal forces in the piece (x in m, forces in kN, moments in kN·m):
  $x \in [9,\ 12]: \quad T = -3, \quad M = 36 - 3x$
- The greatest bending moment is where the shear force jumps: $M_{max} = M(6) = -18\,\mathrm{kN}\cdot\mathrm{m}$
- A positive bending moment bends the beam down (the bottom fibres in tension), a negative one up, typically over a support or at a fixed end.

Graph





T – shear force



M – bending moment $M_{max} = -18\,\mathrm{kN \cdot m}$

The free beam with the loads (blue) and the reactions (red), under it the internal forces' diagrams at the same scale of length. The arrows aren't to the forces' scale.

A ladder against a smooth wall with the ladder's and a person's weight: the reactions on the ground and on the wall.

Input: `model(statics; mode=body; find=R[kN]; supA=pin; Ax=0[m]; Ay=0[m]; supB=roller;`

`Bx=3[m]; By=4[m]; Ba=180[deg]; P=0.2 0.8[kN]; Px=1.5 2.25[m]; Py=2 3[m])`

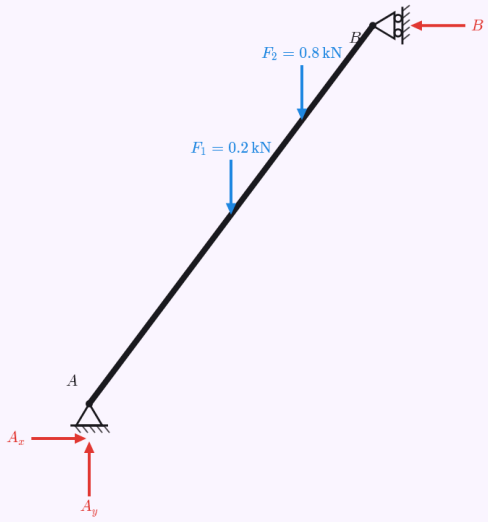
Result

$$A_x = 0.525\,\text{kN}$$
$$A_y = 1\,\text{kN}$$
$$B = 0.525\,\text{kN}$$

Steps

- Find R; given
  $x_A = 0\,\text{m}, \quad y_A = 0\,\text{m}, \quad x_B = 3\,\text{m}, \quad y_B = 4\,\text{m}, \quad \beta_B$
- A body in the plane given by its supports' points and the forces' points of application (x to the right, y up). For a roller, the angle β gives the direction of the reaction: 90° from below, 0° from a wall on the left to the right, 180° from a wall on the right to the left. Example: a ladder against a smooth wall - its foot held on the floor by friction (a pin), its top sliding along the wall (a roller). The model's equation:
  $\sum F_x = 0, \quad \sum F_y = 0, \quad \sum M = 0$
- The unknown reactions: $A_x,\ A_y,\ B$
- Statically determinate: as many unknowns as independent equations (n unknowns, e equations, h the rank): $n = 3, \quad e = 3, \quad h(\mathbf{A}) = 3$
- Sum of the moments about a support - an equation with one unknown
  $\sum M_A = 0: \quad B \cdot 4 - 0.2 \cdot 1.5 - 0.8 \cdot 2.25 = 0 \quad \Rightarrow$
- Sum of the horizontal forces:
  $\sum F_x = 0: \quad A_x - 0.525 = 0 \quad \Rightarrow \quad A_x = 0.525\,\text{kN}$
- Sum of the moments about a support - an equation with one unknown
  $\sum M_B = 0: \quad 0.525 \cdot 4 - A_y \cdot 3 + 0.2 \cdot 1.5 + 0.8 \cdot 0.7$
- A check by an equation not used:
  $\sum F_y = 1 - 0.2 - 0.8 = 0$

Graph



The free body: the loads blue, the support reactions red. The arrows aren't to the forces' scale. The reactions' values are in the result.

A fixed support with a roller is once statically indeterminate.

Input: `model(statics; mode=beam; find=R[kN]; L=5[m]; supA=fixed; supB=roller; F=10[kN];`

xF=2[m])

Result

$$s = n - h = 1$$

Steps

- Find R; given: $L = 5\,\mathrm{m}, \quad F = 10\,\mathrm{kN}, \quad x_F = 2\,\mathrm{m}$
- The beam lies on the x axis from 0 to L. A pin carries a horizontal and a vertical force, a roller only the force perpendicular to its base, a fixed support a moment as well. A force makes the angle α with the beam (90° = vertically down), a moment is positive counterclockwise, a distributed load positive downward. Inner hinges make a Gerber beam: the bending moment is zero in a hinge, which gives one more equation. The diagrams of N, T and M are built piece by piece by Schwedler's theorem. The model's equation:
  $\sum F_x = 0, \quad \sum F_y = 0, \quad \sum M = 0$
- The unknown reactions: $A_x,\ A_y,\ M_A,\ B$
- Statically indeterminate: more unknowns than independent equations:
  $n = 4, \quad e = 3, \quad h(\mathbf{A}) = 3 \quad \Rightarrow \quad s = n - h = 1$
- There are more unknown reactions than the equations of equilibrium give. The missing equations come from the deformation (e.g. zero deflection over a support) - the app doesn't solve statically indeterminate beams yet. The degree s tells how many supports are redundant.

Graph



The free beam with the loads (blue) and the reactions (red), under it the internal forces' diagrams at the same scale of length. The arrows aren't to the forces' scale.

A T section: the centroid, the second moment and the section modulus.

Input: `model(statics; mode=section; shape=T; find=Wy[mm3]; b=100[mm]; h=120[mm]; tf=20[mm];`

```
tw=20[mm])
```

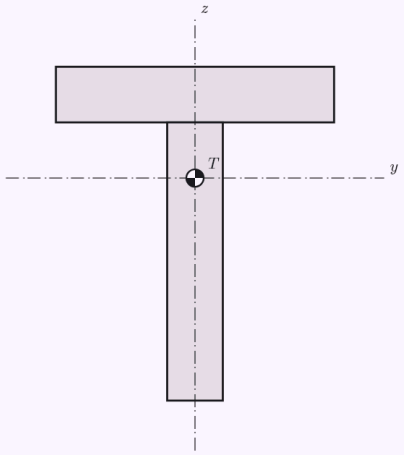
Result

$$W_y \approx 66\,670\,\mathrm{mm}^3$$

Steps

- Find W_y; given
  $b = 100\,\mathrm{mm}, \quad h = 120\,\mathrm{mm}, \quad t_f = 20\,\mathrm{mm}, \quad t_w = 20$
- We split the section into rectangles and circles and subtract the holes. The centroid is the weighted mean of the parts' centroids, the second moments about the axes through it come from the parallel axis theorem. The y axis is horizontal, z vertical, so I_y governs bending in the vertical plane. The moment and the distance of the extreme fibres give the section modulus W, for an unsymmetric section (L) also the principal axes. The rolled profiles' fillets are neglected, so the values differ slightly from the tables. The model's equation
  $A = \sum A_i, \quad y_T = \frac{\sum A_i y_i}{A}, \quad z_T = \frac{\sum A_i z_i}{A}, \quad I_y = \mathbf{\Sigma}$
- A part of the section
  $1: \quad A_1 = 100 \cdot 20 = 2000\,\mathrm{mm}^2, \quad y_1 = 50\,\mathrm{mm}, \quad z_1 =$
- A part of the section
  $2: \quad A_2 = 20 \cdot 100 = 2000\,\mathrm{mm}^2, \quad y_2 = 50\,\mathrm{mm}, \quad z_2 =$
- The cross-sectional area: $A = A_1 + A_2 = 4000\,\mathrm{mm}^2$
- The centroid as the weighted mean of the parts' centroids
  $y_T = \frac{2000 \cdot 50 + 2000 \cdot 50}{4000} = 50\,\mathrm{mm}, \quad z_T = \frac{2000 \cdot 11}{}$
- Parallel axis theorem: the part's own moment + its area × the squared distance of the axes
  $I_{y,1} = \left(\frac{100 \cdot 20^3}{12} + 2000 \cdot 30^2\right) = 1\,867\,000\,\mathrm{mm}^4, \quad I_z$
- Parallel axis theorem: the part's own moment + its area × the squared distance of the axes
  $I_{y,2} = \left(\frac{20 \cdot 100^3}{12} + 2000 \cdot (-30)^2\right) = 3\,467\,000\,\mathrm{mm}^4,$
- The whole section's second moments
  $I_y = I_{y,1} + I_{y,2} = 5\,333\,000\,\mathrm{mm}^4, \quad I_z = I_{z,1} + I_{z,2} = 1$
- The section moduli (e = the extreme fibres' distance from the centroid)
  $W_y = \frac{I_y}{e_z} = \frac{5\,333\,000}{80} = 66\,670\,\mathrm{mm}^3, \quad W_z = \frac{I_z}{e_y} = \frac{1}{}'$
- The radii of gyration
  $i_y = \sqrt{\frac{I_y}{A}} = 36.51\,\mathrm{mm}, \quad i_z = \sqrt{\frac{I_z}{A}} = 20.82\,\mathrm{mm}$

Graph



The section at true scale with its centroid T and the axes y, z through it; for an unsymmetric section also the principal axes 1, 2.

An equal-leg angle L: the product of inertia and the principal axes at 45°.

Input: `model(statics; mode=section; shape=L; find=I1[mm4]; b=100[mm]; h=100[mm]; t=10[mm])`

Result

$I_1 \approx 2\,866\,000\,\text{mm}^4$

$I_2 \approx 734\,300\,\text{mm}^4$

$\alpha \approx 45^\circ$

Steps

- Find I_1; given: $b = 100\,\text{mm}, \quad h = 100\,\text{mm}, \quad t = 10\,\text{mm}$
- We split the section into rectangles and circles and subtract the holes. The centroid is the weighted mean of the parts' centroids, the second moments about the axes through it come from the parallel axis theorem. The y axis is horizontal, z vertical, so I_y governs bending in the vertical plane. The moment and the distance of the extreme fibres give the section modulus W, for an unsymmetric section (L) also the principal axes. The rolled profiles' fillets are neglected, so the values differ slightly from the tables. The model's equation

$$A = \sum A_i, \quad y_T = \frac{\sum A_i y_i}{A}, \quad z_T = \frac{\sum A_i z_i}{A}, \quad I_y = \text{}$$

- A part of the section

$$1: \quad A_1 = 10 \cdot 100 = 1000\,\text{mm}^2, \quad y_1 = 5\,\text{mm}, \quad z_1 =$$

- A part of the section

$$2: \quad A_2 = 90 \cdot 10 = 900\,\text{mm}^2, \quad y_2 = 55\,\text{mm}, \quad z_2 =$$

- The cross-sectional area: $A = A_1 + A_2 = 1900\,\text{mm}^2$
- The centroid as the weighted mean of the parts' centroids

$$y_T = \frac{1000 \cdot 5 + 900 \cdot 55}{1900} = 28.68\,\text{mm}, \quad z_T = \frac{1000 \cdot 5}{1}$$

- Parallel axis theorem: the part's own moment + its area × the squared distance of the axes

$$I_{y,1} = \left(\frac{10 \cdot 100^3}{12} + 1000 \cdot 21.32^2\right) = 1\,288\,000\,\text{mm}^4,$$

- Parallel axis theorem: the part's own moment + its area × the squared distance of the axes

$$I_{y,2} = \left(\frac{90 \cdot 10^3}{12} + 900 \cdot (-23.68)^2\right) = 512\,300\,\text{mm}^4,$$

- The whole section's second moments

$$I_y = I_{y,1} + I_{y,2} = 1\,800\,000\,\text{mm}^4, \quad I_z = I_{z,1} + I_{z,2} = 1$$

- The product of inertia

$$D_{yz} = \sum A_i (y_i - y_T)(z_i - z_T) = -1\,066\,000\,\text{mm}^4$$

- The principal second moments

$$I_{1,2} = \frac{I_y + I_z}{2} \pm \sqrt{\left(\frac{I_y - I_z}{2}\right)^2 + D_{yz}^2}: \quad I_1 = 2\,866\,0$$

- The principal axis 1 turned from the y axis:

$$\tan 2\alpha = \frac{-2D_{yz}}{I_y - I_z} \quad \Rightarrow \quad \alpha = 45^\circ$$

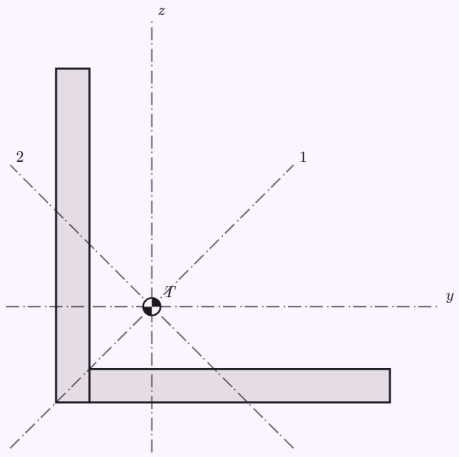
- The section moduli (e = the extreme fibres' distance from the centroid)

$$W_y = \frac{I_y}{e_z} = \frac{1\,800\,000}{71.32} = 25\,240\,\text{mm}^3, \quad W_z = \frac{I_z}{e_y} = \frac{1}{}$$

- The radii of gyration

$$i_y = \sqrt{\frac{I_y}{A}} = 30.78\,\text{mm}, \quad i_z = \sqrt{\frac{I_z}{A}} = 30.78\,\text{mm}$$

Graph



The section at true scale with its centroid T and the axes y, z through it; for an unsymmetric section also the principal axes 1, 2.

A plate with an off-centre round hole: a composite section and the shifted centroid.

Input: `model(statics; mode=section; shape=composite; find=Iy[mm4]; rb=200[mm]; rh=100[mm];`

```
ry=0[mm]; rz=0[mm]; hd=40[mm]; hcy=60[mm]; hcz=60[mm])
```

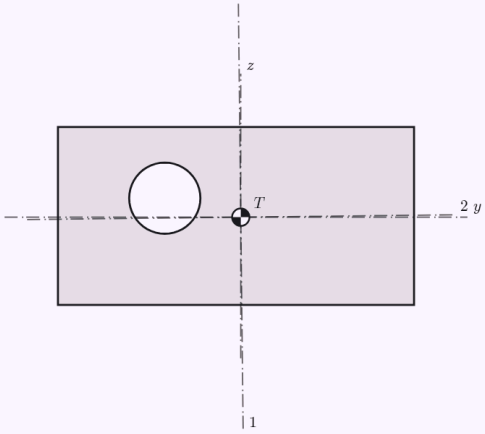
Result

$$I_y \approx 1.641 \cdot 10^7\,\mathrm{mm}^4$$

Steps

- Find I_y; given
  $b = 200\,\mathrm{mm}, \quad h = 100\,\mathrm{mm}, \quad y = 0\,\mathrm{mm}, \quad z = 0\,\mathrm{mm}$
- We split the section into rectangles and circles and subtract the holes. The centroid is the weighted mean of the parts' centroids, the second moments about the axes through it come from the parallel axis theorem. The y axis is horizontal, z vertical, so I_y governs bending in the vertical plane. The moment and the distance of the extreme fibres give the section modulus W, for an unsymmetric section (L) also the principal axes. The rolled profiles' fillets are neglected, so the values differ slightly from the tables. The model's equation
  $A = \sum A_i, \quad y_T = \frac{\sum A_i y_i}{A}, \quad z_T = \frac{\sum A_i z_i}{A}, \quad I_y = \mathbf{Y}$
- A part of the section
  $1: \quad A_1 = 200 \cdot 100 = 20\,000\,\mathrm{mm}^2, \quad y_1 = 100\,\mathrm{mm},$
- A hole (subtracted)
  $2: \quad A_2 = -\frac{\pi \cdot 40^2}{4} = -1257\,\mathrm{mm}^2, \quad y_2 = 60\,\mathrm{mm},$
- The cross-sectional area: $A = A_1 - A_2 = 18\,740\,\mathrm{mm}^2$
- The centroid as the weighted mean of the parts' centroids
  $y_T = \frac{20\,000 \cdot 100 - 1257 \cdot 60}{18\,740} = 102.7\,\mathrm{mm}, \quad z_T = \frac{20}{}$
- Parallel axis theorem: the part's own moment + its area × the squared distance of the axes
  $I_{y,1} = \left(\frac{200 \cdot 100^3}{12} + 20\,000 \cdot 0.6704^2\right) = 1.668 \cdot 10^7\,\mathrm{m}$
- Parallel axis theorem: the part's own moment + its area × the squared distance of the axes
  $I_{y,2} = -\left(\frac{\pi \cdot 40^4}{64} + 1257 \cdot 10.67^2\right) = -268\,700\,\mathrm{mm}^4,$
- The whole section's second moments
  $I_y = I_{y,1} + I_{y,2} = 1.641 \cdot 10^7\,\mathrm{mm}^4, \quad I_z = I_{z,1} + I_{z,2} =$
- The product of inertia
  $D_{yz} = \sum A_i (y_i - y_T)(z_i - z_T) = 536\,400\,\mathrm{mm}^4$
- The principal second moments
  $I_{1,2} = \frac{I_y + I_z}{2} \pm \sqrt{\left(\frac{I_y - I_z}{2}\right)^2 + D_{yz}^2}: \quad I_1 = 6.44 \cdot 1$
- The principal axis 1 turned from the y axis
  $\tan 2\alpha = \frac{-2D_{yz}}{I_y - I_z} \quad \Rightarrow \quad \alpha = -89.36^\circ$
- The section moduli (e = the extreme fibres' distance from the centroid)
  $W_y = \frac{I_y}{e_z} = \frac{1.641 \cdot 10^7}{50.67} = 323\,800\,\mathrm{mm}^3, \quad W_z = \frac{I_z}{e_y} =$
- The radii of gyration
  $i_y = \sqrt{\frac{I_y}{A}} = 29.59\,\mathrm{mm}, \quad i_z = \sqrt{\frac{I_z}{A}} = 58.61\,\mathrm{mm}$

Graph



The section at true scale with its centroid T and the axes y, z through it; for an unsymmetric section also the principal axes 1, 2.

A tanker half full of water braking at 5 m/s²: the front axle carries tens of per cent more.

Input: `model(tanker; mode=brake; find=Nf[kN]; shape=cylinder; liquid=water; f=50[pct]; a=5[m_s2]; rho=1000[kg_m3]; Lt=8[m]; W=2[m]; H=2[m]; xt=0.6[m]; zt=1.25[m]; M=14[t];`

`xM=6.4[m]; zM=1.05[m]; xr=2.65[m]; xf=9.9[m])`

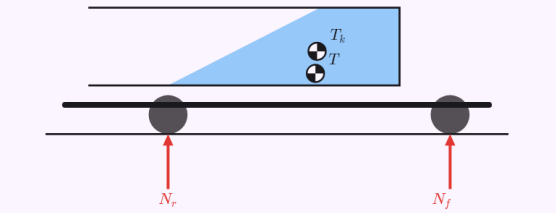
Result

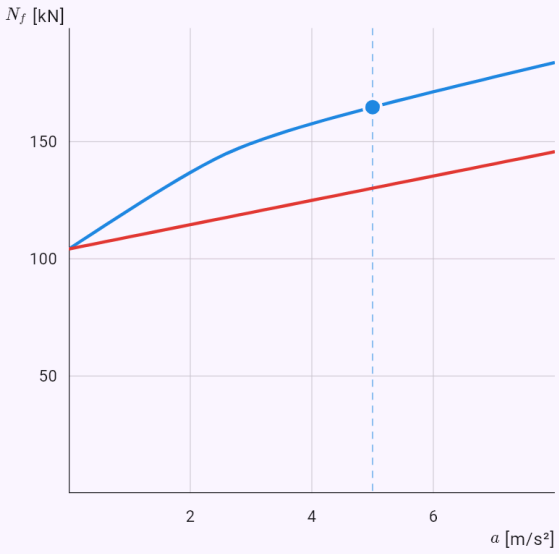
$$N_f \approx 164.7\,\text{kN}$$

Steps

- Find N_f; given
  $\varphi = 50\,\%, \quad a = 5\,\text{m/s}^2, \quad \rho = 1000\,\text{kg/m}^3, \quad L_n = 8$
- Braking with the deceleration a: the liquid surges forward, its surface tilts by the angle θ and the centre of gravity moves forward and up. The axle loads are the reactions of a beam on two supports with the inertial force m·a at the centre of gravity. The model's equation: $\tan\theta = \frac{a}{g}$
- The tank's and the liquid's volume
  $V = 25.13\,\text{m}^3, \quad V_k = \varphi V = 12.57\,\text{m}^3, \quad m_k = \rho V_k =$
- The surface's tilt - perpendicular to the resultant of gravity and the inertial force
  $\tan\theta = \frac{a}{g} = \frac{5}{9.81} = 0.5097 \quad \Rightarrow \quad \theta = 27.01^\circ$
- The liquid's centre of gravity moves forward and up
  $x_k : 4.6 \to 6.481\,\text{m}, \quad z_k : 1.825 \to 2.127\,\text{m}, \quad \Delta x_k =$
- The whole vehicle's centre of gravity
  $x_T = \frac{m_0 x_0 + m_k x_k}{m_0 + m_k} = 6.438\,\text{m}, \quad z_T = 1.56\,\text{m}$
- The axle loads from the moments about the rear axle (the inertial force at the centre of gravity)
  $\sum M_r = 0: \quad N_f \cdot 7.25 = mg(x_T - x_r) + maz_T \quad \Rightarrow$
- The front axle at rest and in motion; with a rigid load
  $N_f : 104.2 \to 164.7\,\text{kN}\ (+58.12\,\%), \quad N_{f,\,\text{rigid}} = 130.2$
- The first wave's natural frequency (sloshing)
  $\omega^2 = \frac{\pi g}{L}\tanh\frac{\pi h}{L}: \quad \omega = 1.073\,\text{s}^{-1}, \quad T = \frac{2\pi}{\omega} = 5.85$
- The quasi-static model holds for steady braking. After a hard stop the liquid oscillates with the period T and its surges against the tank's end can raise the load briefly even more; that's why tankers have baffles.

Graph





The side view with the tilted surface, the liquid's centre of gravity T_k, the whole's T and the axle loads; the chart shows the front axle's load against the deceleration for the liquid and a rigid load.

The same tanker in a curve of R = 60 m: the critical speed with the liquid and with a rigid load.

Input: `model(tanker; mode=curve; find=vcrit[km_h]; shape=cylinder; liquid=water; f=50[pct]; R=60[m]; v=60[km_h]; tw=2[m]; k=7[deg]; zr=0.9[m]; rho=1000[kg_m3]; Lt=8[m]; W=2[m];`

```
H=2[m]; xt=0.6[m]; zt=1.25[m]; M=14[t]; xM=6.4[m]; zM=1.05[m]; xr=2.65[m]; xf=9.9[m])
```

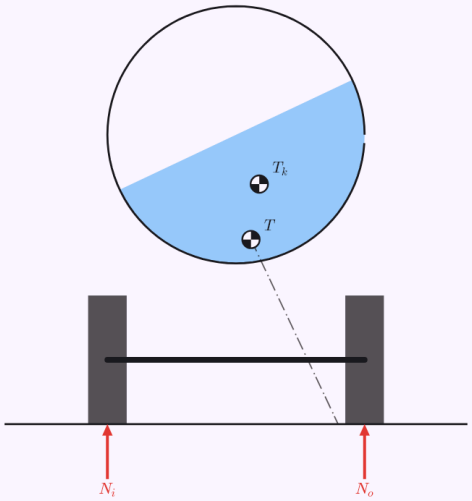
Result

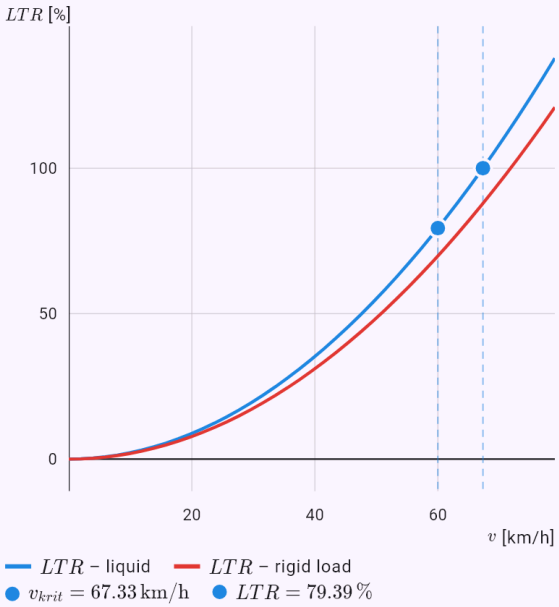
$$v_{krit} \approx 67.33\,\mathrm{km/h}$$

Steps

- Find v_{krit}; given
  $\varphi = 50\,\%, \quad R = 60\,\mathrm{m}, \quad v = 60\,\mathrm{km/h}, \quad s = 2\,\mathrm{m}, \quad k$
- A curve of the radius R at the speed v: the centripetal acceleration a = v²/R, the liquid spills outward and so does the centre of gravity. The vehicle tips over when the resultant of gravity and the centrifugal force passes through the outer wheels (LTR = 100 %). The body rolling on its springs (k_r degrees per g) moves the centre of gravity further out. The model's equation:
  $\tan\theta = \dfrac{a_y}{g}, \quad a_y = \dfrac{v^2}{R}$
- The tank's and the liquid's volume
  $V = 25.13\,\mathrm{m}^3, \quad V_k = \varphi V = 12.57\,\mathrm{m}^3, \quad m_k = \rho V_k =$
- The centripetal acceleration in the curve
  $a_y = \dfrac{v^2}{R} = \dfrac{16.67^2}{60} = 4.63\,\mathrm{m/s^2} = 0.4719\,g$
- The surface's tilt - perpendicular to the resultant of gravity and the inertial force:
  $\tan\theta = \dfrac{a_y}{g} = 0.4719 \quad \Rightarrow \quad \theta = 25.26^\circ$
- The liquid's centre in the cross-section (y out of the curve): $y_k = 0.1811\,\mathrm{m}, \quad z_k = 1.866\,\mathrm{m}$
- The body's roll moves the whole centre of gravity further out
  $\varphi_r = k_r \dfrac{a_y}{g} = 3.304^\circ, \quad y_T = 0.1165\,\mathrm{m}, \quad z_T = 1.435\,\mathrm{m}$
- The load transfer to the outer wheels:
  $LTR = \dfrac{N_o - N_i}{N_o + N_i} = \dfrac{a_y z_T/g + y_T}{s/2} = 79.39\,\%$
- The rollover limit: the resultant passes through the outer wheels
  $\dfrac{a_y}{g} z_T(a_y) + y_T(a_y) = \dfrac{s}{2} \quad \Rightarrow \quad a_{y,\mathrm{lim}} = 5.83\,\mathrm{m/s^2} = 0$
- A rigid load for comparison:
  $a_{y,\mathrm{lim}} = 0.6765\,g, \quad v_{krit} = 71.84\,\mathrm{km/h}$
- The first wave's natural frequency (sloshing)
  $\omega^2 = \dfrac{\pi g}{L}\tanh\dfrac{\pi h}{L}: \quad \omega = 3.606\,\mathrm{s^{-1}}, \quad T = \dfrac{2\pi}{\omega} = 1.74$
- Real tankers tip over already at about 0.35–0.45 g: besides the roll on the springs the tyres deflect and the liquid oscillates in the curve too. The quasi-static limit is an upper estimate.

Graph





The view from behind in the curve (outward to the right) with the tilted surface, the centre of gravity and the resultant; the chart shows the load transfer to the outer wheels against the speed - at 100 % the inner wheels lift.

A 2 m cantilever with a 10 kN force: IPE 200 from the tables, the optimal tube and a tube of uniform strength.

Input: `model(beamdesign; find=m[kg]; support=cantilever; load=force; shape=tube; material=s235; L=2[m]; F=10[kN]; sigma=160[MPa]; tau=95[MPa]; E=210[GPa]; rho=7850[kg_m3]; n=250; lambda=50)`

Result

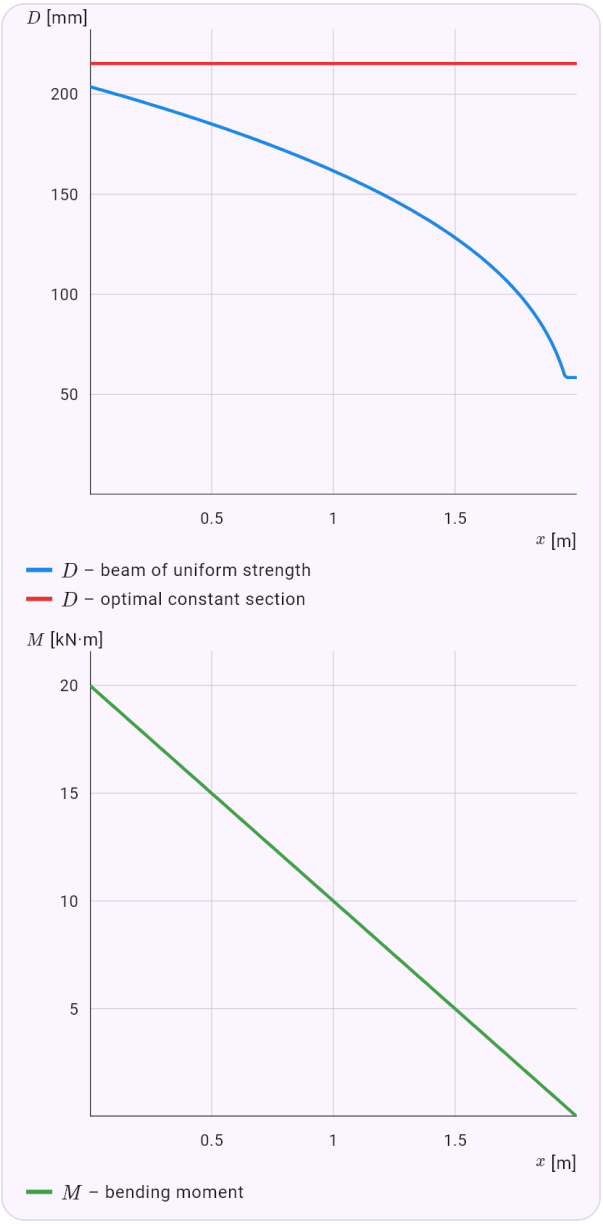
$$m_{tab} \approx 44.75\,\text{kg}\ (\text{IPE}\,200)$$
$$m_{opt} \approx 44.79\,\text{kg}$$
$$m_{st} \approx 24.1\,\text{kg}$$

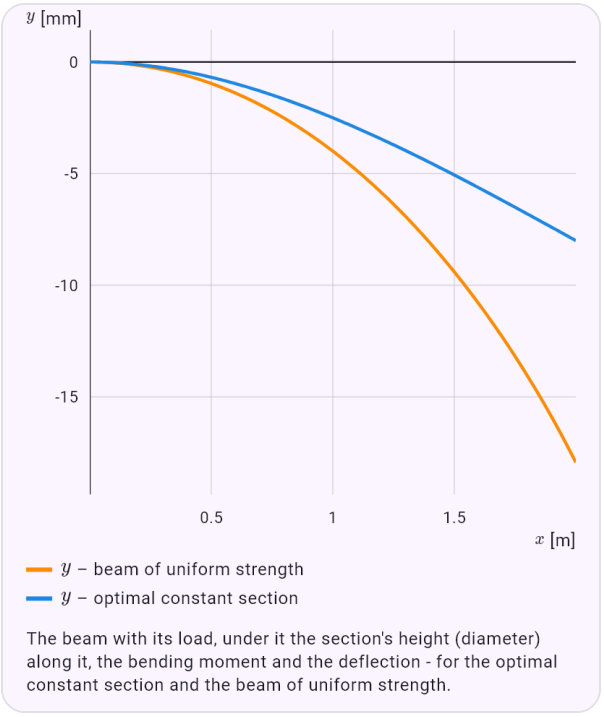
Steps

- Find m; given
  $L = 2\,\text{m}, \quad F = 10\,\text{kN}, \quad \sigma_{dov} = 160\,\text{MPa}, \quad \tau_{dov} = 95$
- A cantilever of the length L with a force F at its free end or a distributed load q along all of it. The greatest moment is at the fixed end. The model's equation
  $$\sigma = \frac{M_{max}}{W} \le \sigma_{dov}, \quad y_{max} \le \frac{L}{n}, \quad EIy'' = -M$$
- The greatest bending moment:
  $M_{max} = FL = 20\,\text{kN}\cdot\text{m}$
- The strength condition - the smallest section modulus
  $$W \ge \frac{M_{max}}{\sigma_{dov}} = \frac{20\cdot 10^3}{160\cdot 10^6} = 125\,\text{cm}^3$$
- The deflection condition (integrating E·I·y'' = −M) - the smallest second moment
  $$y_{max} = \frac{FL^3}{3EI} \le \frac{L}{n} = 8\,\text{mm} \quad \Rightarrow \quad I \ge 1587\,\text{cm}^4$$
- The lightest suitable profile of each family
  $\text{IPE}\,200: \ W = 194\,\text{cm}^3,\ I = 1943\,\text{cm}^4,\ 22.37\,\text{kg/m},$
- 1. By the tables - the lightest of all
  $\text{IPE}\,200: \quad m = \rho AL = 7850\cdot 28.5\cdot 10^{-4}\cdot 2 = 44.75$
- A thinner wall gives a smaller area for the same W, so the optimum is at the slenderness limit: $\frac{D}{t} = \lambda = 50$
- 2. The optimal constant section - the smallest size meeting both conditions
  $$D_\sigma = \sqrt[3]{\frac{W}{c_W}} = 203.7\,\text{mm}, \quad D_y = \sqrt[4]{\frac{I}{c_I}} = 215.2\,\text{mm}$$
- The deflection condition is active (the stress comes out below the allowable one): $y_{max} = \frac{L}{n}$
- The optimal section's area and mass:
  $A = 28.53\,\text{cm}^2, \quad m = 44.79\,\text{kg}$
- 3. A beam of uniform strength - σ(x) = σ_allow everywhere, the smallest section at the free end for the shear
  $$D(x) = \sqrt[3]{\frac{M(x)}{c_W\sigma_{dov}}}, \quad D_{min} = \sqrt{\frac{2T}{c_A\tau_{dov}}} = 58.47\,\text{mm}$$
- The mass and the deflection of the beam of uniform strength
  $$m = \rho\int_0^L A(x)dx = 24.1\,\text{kg}, \quad y_{max} = 17.94\,\text{mm}$$
- A beam of uniform strength watches only the stress, so it is more flexible and exceeds the allowed deflection. In practice its section grows, or it is used where the strength decides.
- The variants' masses against the profile from the tables:
  $\frac{m_{opt}}{m_{tab}} = 100.1\,\%, \quad \frac{m_{st}}{m_{tab}} = 53.86\,\%$
- The beam's own weight is neglected; for long and heavy beams it adds to the distributed load.

Graph







The beam with its load, under it the section's height (diameter) along it, the bending moment and the deflection - for the optimal constant section and the beam of uniform strength.

A rectangle of uniform strength of constant width: its height a parabola along the beam, a third less material.

Input: `model(beamdesign; find=m[kg]; support=cantilever; load=force; shape=rect; material=s235; L=2[m]; F=10[kN]; sigma=160[MPa]; tau=95[MPa]; E=210[GPa]; rho=7850[kg_m3];`

```
n=50; k=2)
```

Result

$$m_{tab} \approx 37.52\,\text{kg}\ (\text{IPE}\,180)$$

$$m_{opt} \approx 102.9\,\text{kg}$$

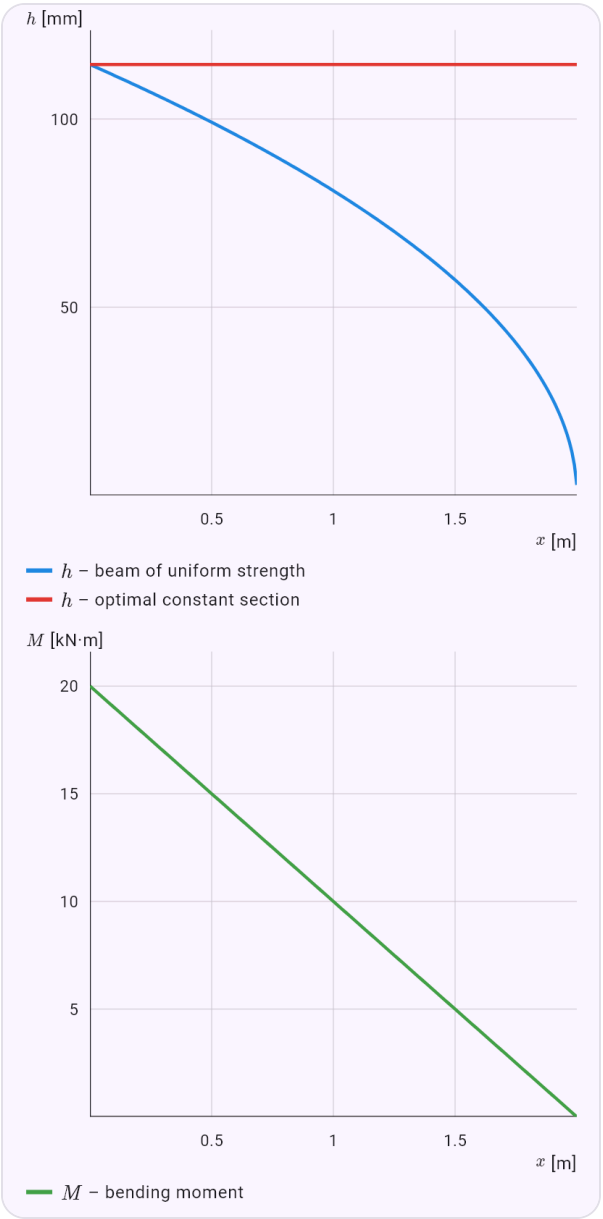
$$m_{st} \approx 68.58\,\text{kg}$$

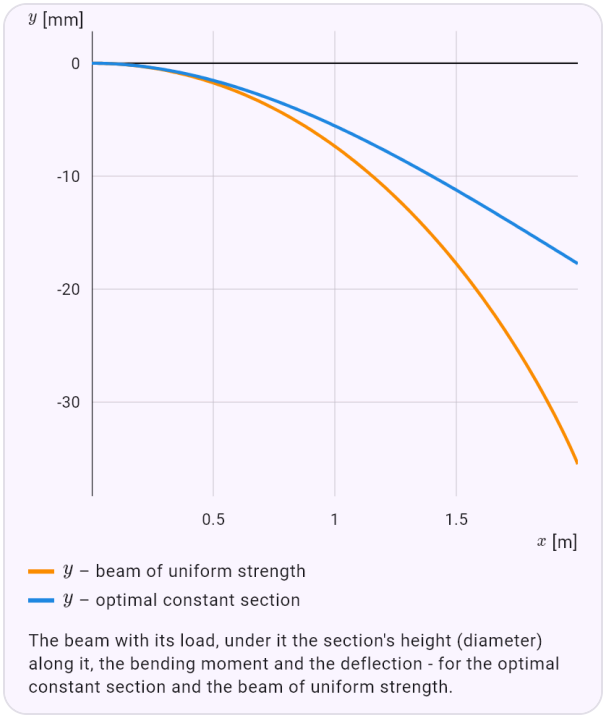
Steps

- Find m; given
  $L = 2\,\text{m}, \quad F = 10\,\text{kN}, \quad \sigma_{dov} = 160\,\text{MPa}, \quad \tau_{dov} = 95$
- A cantilever of the length L with a force F at its free end or a distributed load q along all of it. The greatest moment is at the fixed end. The model's equation
  $$\sigma = \frac{M_{max}}{W} \le \sigma_{dov}, \quad y_{max} \le \frac{L}{n}, \quad EIy'' = -M$$
- The greatest bending moment:
  $M_{max} = FL = 20\,\text{kN}\cdot\text{m}$
- The strength condition - the smallest section modulus
  $$W \ge \frac{M_{max}}{\sigma_{dov}} = \frac{20\cdot 10^3}{160\cdot 10^6} = 125\,\text{cm}^3$$
- The deflection condition (integrating E·I·y" = −M) - the smallest second moment
  $$y_{max} = \frac{FL^3}{3EI} \le \frac{L}{n} = 40\,\text{mm} \quad \Rightarrow \quad I \ge 317.5\,\text{cm}^4$$
- The lightest suitable profile of each family
  $\text{IPE}\,180:\ W = 146\,\text{cm}^3,\ I = 1317\,\text{cm}^4,\ 18.76\,\text{kg/m},$
- 1. By the tables - the lightest of all
  $\text{IPE}\,180: \quad m = \rho AL = 7850\cdot 23.9\cdot 10^{-4}\cdot 2 = 37.52$
- 2. The optimal constant section - the smallest size meeting both conditions
  $b_\sigma = \sqrt[3]{\frac{W}{c_W}} = 57.24\,\text{mm}, \quad b_y = \sqrt[4]{\frac{I}{c_I}} = 46.71\,\text{mm}$ =
- The strength condition is active (the deflection comes out below the allowed one): $\sigma = \sigma_{dov}$
- The optimal section's area and mass:
  $A = 65.52\,\text{cm}^2, \quad m = 102.9\,\text{kg}$
- 3. A beam of uniform strength - σ(x) = σ_allow everywhere, the smallest section at the free end for the shear
  $$h(x) = \sqrt{\frac{6M(x)}{b\sigma_{dov}}}, \quad h_{min} = \frac{1.5T}{b\tau_{dov}} = 2.759\,\text{mm}$$
- The mass and the deflection of the beam of uniform strength
  $$m = \rho\int_0^L A(x)dx = 68.58\,\text{kg}, \quad y_{max} = 35.5\,\text{mm}$$
- The variants' masses against the profile from the tables:
  $\frac{m_{opt}}{m_{tab}} = 274.1\,\%, \quad \frac{m_{st}}{m_{tab}} = 182.8\,\%$
- The beam's own weight is neglected; for long and heavy beams it adds to the distributed load.

Graph







The beam with its load, under it the section's height (diameter) along it, the bending moment and the deflection - for the optimal constant section and the beam of uniform strength.

A 6 m simply supported beam with a distributed load: the deflection decides.

Input: `model(beamdesign; find=m[kg]; support=simple; load=uniform; shape=tube; material=s235; L=6[m]; q=8[kN_m]; sigma=160[MPa]; tau=95[MPa]; E=210[GPa]; rho=7850[kg_m3];`

n=300; lambda=40)

Result

$$m_{tab} \approx 184.2\,\text{kg}\ (\text{IPE}\,240)$$

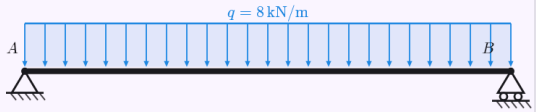
$$m_{opt} \approx 214.3\,\text{kg}$$

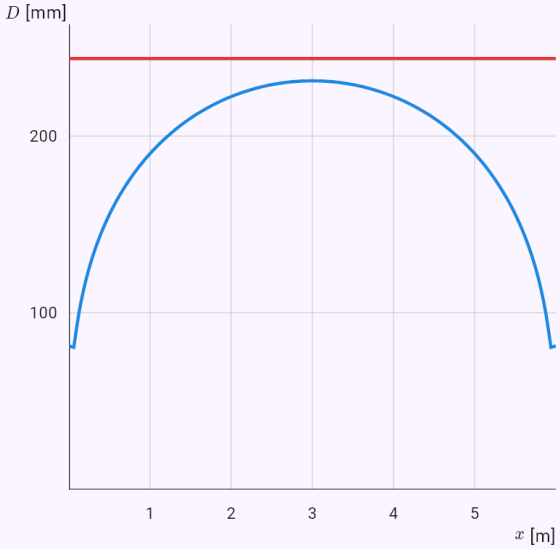
$$m_{st} \approx 142.7\,\text{kg}$$

Steps

- Find m; given
  $L = 6\,\text{m}, \quad q = 8\,\text{kN/m}, \quad \sigma_{dov} = 160\,\text{MPa}, \quad \tau_{dov} = 9$
- A simply supported beam of the length L with a force F in the middle or a distributed load q along all of it. The greatest moment is in the middle. The model's equation
  $\sigma = \dfrac{M_{max}}{W} \le \sigma_{dov}, \quad y_{max} \le \dfrac{L}{n}, \quad EIy'' = -M$
- The greatest bending moment:
  $M_{max} = \dfrac{qL^2}{8} = 36\,\text{kN}\cdot\text{m}$
- The strength condition - the smallest section modulus
  $W \ge \dfrac{M_{max}}{\sigma_{dov}} = \dfrac{36\cdot 10^3}{160\cdot 10^6} = 225\,\text{cm}^3$
- The deflection condition (integrating E·I·y" = −M) - the smallest second moment
  $y_{max} = \dfrac{5qL^4}{384EI} \le \dfrac{L}{n} = 20\,\text{mm} \quad \Rightarrow \quad I \ge 3214\,\text{cm}^4$
- The lightest suitable profile of each family
  $\text{IPE}\,240:\ W = 324\,\text{cm}^3,\ I = 3892\,\text{cm}^4,\ 30.69\,\text{kg/m},$
- 1. By the tables - the lightest of all
  $\text{IPE}\,240: \quad m = \rho AL = 7850\cdot 39.1\cdot 10^{-4}\cdot 6 = 184.2$
- A thinner wall gives a smaller area for the same W, so the optimum is at the slenderness limit: $\dfrac{D}{t} = \lambda = 40$
- 2. The optimal constant section - the smallest size meeting both conditions
  $D_\sigma = \sqrt[3]{\dfrac{W}{c_W}} = 231.2\,\text{mm}, \quad D_y = \sqrt[4]{\dfrac{I}{c_I}} = 243.8\,\text{mm}$
- The deflection condition is active (the stress comes out below the allowable one): $y_{max} = \dfrac{L}{n}$
- The optimal section's area and mass:
  $A = 45.5\,\text{cm}^2, \quad m = 214.3\,\text{kg}$
- 3. A beam of uniform strength - σ(x) = σ_allow everywhere, the smallest section at the free end for the shear
  $D(x) = \sqrt[3]{\dfrac{M(x)}{c_W\sigma_{dov}}}, \quad D_{min} = \sqrt{\dfrac{2T}{c_A\tau_{dov}}} = 81.23\,\text{mm}$
- The mass and the deflection of the beam of uniform strength
  $m = \rho\displaystyle\int_0^L A(x)dx = 142.7\,\text{kg}, \quad y_{max} = 32.22\,\text{mm}$
- A beam of uniform strength watches only the stress, so it is more flexible and exceeds the allowed deflection. In practice its section grows, or it is used where the strength decides.
- The variants' masses against the profile from the tables:
  $\dfrac{m_{opt}}{m_{tab}} = 116.4\,\%, \quad \dfrac{m_{st}}{m_{tab}} = 77.48\,\%$
- The beam's own weight is neglected; for long and heavy beams it adds to the distributed load.

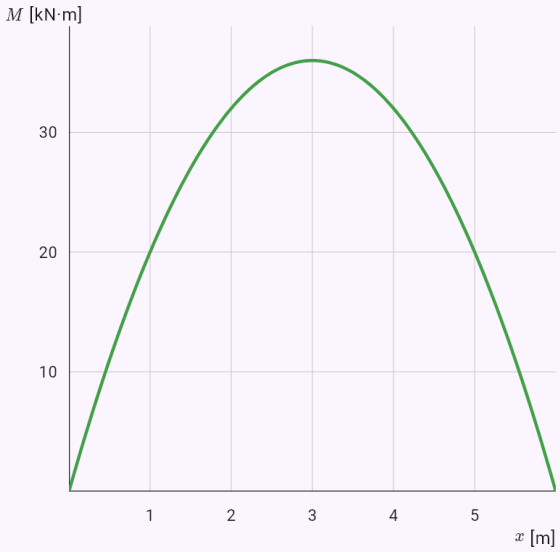
Graph



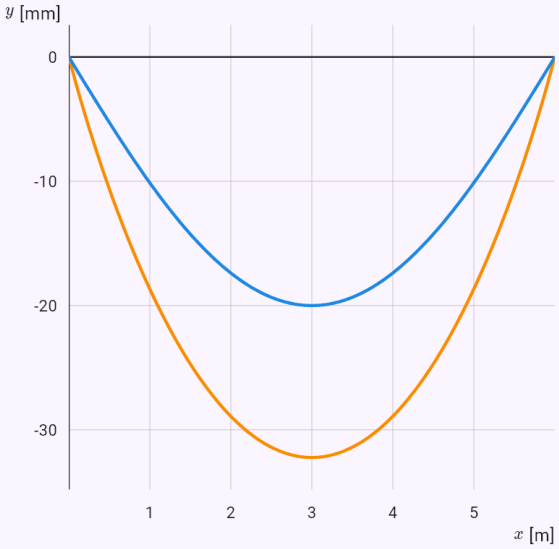


D – beam of uniform strength

D – optimal constant section



M – bending moment



y – beam of uniform strength

y – optimal constant section

The beam with its load, under it the section's height (diameter) along it, the bending moment and the deflection - for the optimal constant section and the beam of uniform strength.

45. Astronomy and astrophysics

The field Astronomy and astrophysics is in the Applications section. Besides Kepler's laws, the bodies' influence on each other and the three-body problem (described in the Physics chapter), it has the forms Black holes, Neutron stars, Gravitational waves, Magnitudes and distances, Radiation of stars, Redshift and Hubble's law, and Exoplanets. The formulas are ready and the steps show the substitution. The app doesn't solve the equations of general relativity; it uses only their results for a non-rotating black hole.

The Black holes form computes the [[schwarzschildRadius|Schwarzschild radius]], the photon sphere and the innermost stable orbit, a black hole's "density" with a graph over the mass on a logarithmic scale, the escape velocity, the gravitational time dilation and redshift (also for two clocks, such as GPS), the tidal force, Hawking's temperature with the evaporation time and the mass of Sgr A* from the orbit of the star S2. The menu of objects fills in the Sun, the Earth, a neutron star, a stellar black hole, Sgr A* or M87*.

Neutron stars have the density, the gravity at the surface, the spin-up in the collapse by the conservation of angular momentum and the shortest possible rotation period. Gravitational waves give the frequency from an orbiting pair's separation (and back) and the energy radiated in a merger such as GW150914.

The basics of an introductory astronomy course: Pogson's equation, the distance modulus with the [[absoluteMagnitude|absolute magnitude]] and the parallax; Wien's and the Stefan-Boltzmann law of a [[blackBody|black body]], the solar constant and a planet's equilibrium temperature; the Doppler [[redshift|redshift]] and Hubble's law with the Hubble time; for exoplanets the transit depth, the radial velocity method and the [[habitableZone|habitable zone]].

Kepler's laws gained the Hohmann transfer between two circular orbits (Δv and the flight time), and the bodies' influence on each other the [[rocheLimit|Roche limit]] and the Hill sphere.

Examples

How small a ball the Earth would have to be squeezed into to become a black hole.

Input: `model(blackhole; find=rs[mm]; M=3.0035e-6[Msun]; mode=horizon; astro=earth)`

Result

$$r_s \approx 8.87\,\mathrm{mm}$$

Steps

- Find r_s; given: $M = 3.004 \cdot 10^{-6}\,M_\odot$
- Converting the units:
  $M = 3.004 \cdot 10^{-6}\,M_\odot = 5.972 \cdot 10^{24}\,\mathrm{kg}$
- The Schwarzschild radius r_s = 2GM/c² is the radius of the event horizon. The Earth would have to be squeezed into a ball about 18 mm across, the Sun into a ball of 3 km radius. Light can orbit on the photon sphere at 1.5 r_s, and the innermost stable orbit of matter (ISCO) is at 3 r_s. The model's equation
  $$r_s = \frac{2GM}{c^2}, \quad r_{\mathrm{ph}} = \frac{3}{2} r_s, \quad r_{\mathrm{ISCO}} = 3 r_s$$
- Substituting
  $$r_s = \frac{2 \cdot 6.674 \cdot 10^{-11} \cdot 5.972 \cdot 10^{24}}{\left(2.998 \cdot 10^{8}\right)^2} \approx 0.00887\,\mathrm{m}$$
- For a non-rotating black hole. Around a rotating (Kerr) one, the ISCO and the photon sphere lie closer in.
- Converting the units: $r_s \approx 0.00887\,\mathrm{m} = 8.87\,\mathrm{mm}$

The "density" of the black hole M87* – thinner than air.

Input: `model(blackhole; find=rho[kg_m3]; M=6.5e9[Msun]; mode=bhdensity; astro=m87)`

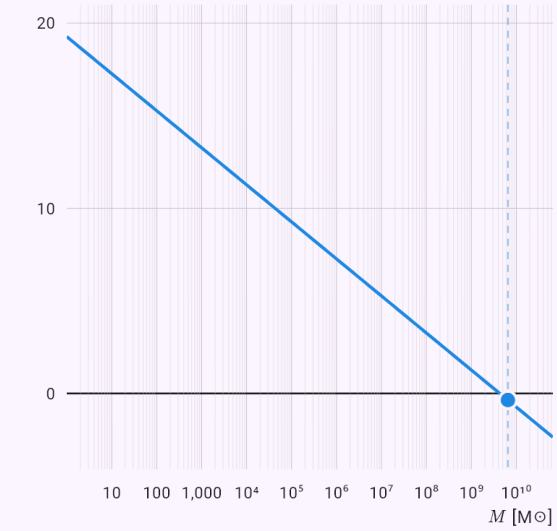
Result

$$\rho \approx 0.4362\,\mathrm{kg/m^3}$$

Steps

- Find ρ; given: $M = 6.5 \cdot 10^9\, M_\odot$
- Converting the units:
  $M = 6.5 \cdot 10^9\, M_\odot = 1.293 \cdot 10^{40}\,\mathrm{kg}$
- The mean "density" inside the horizon: the mass divided by the volume of a ball of radius r_s. It falls as 1/M². A stellar black hole is denser than an atomic nucleus, one of 10^8 solar masses has the density of water, and M87* is thinner than air. The graph shows the mass on a logarithmic scale. The model's equation:
  $\rho = \dfrac{M}{\frac{4}{3}\pi r_s^3} = \dfrac{3c^6}{32\pi G^3 M^2}$
- Substituting
  $\rho = \dfrac{3 \cdot (2.998 \cdot 10^8)^6}{32\pi \cdot (6.674 \cdot 10^{-11})^3 \cdot (1.293 \cdot 10^{40})^2} \approx 0.4362\,\mathrm{kg}$
- For comparison: an atomic nucleus, water and air (about 1.2 kg/m³)
  $\rho_{\mathrm{nucl}} \approx 2.3 \cdot 10^{17}\,\mathrm{kg/m^3}, \quad \rho_{\mathrm{H_2O}} = 1000\,\mathrm{kg/m^3}$
- Thinner than air – the "density" is only an average over the horizon's volume.

Graph



The horizontal axis is logarithmic – each mark is ten times the previous one. The dot is the given value or the answer.

How much the GPS satellites' clocks gain each day because of gravity.

Input: `model(blackhole; find=dt[us]; M=3.0035e-6[Msun]; r=6371[km]; r2=26571[km]; t=1[d];`

`mode=gravdilation; astro=earth)`

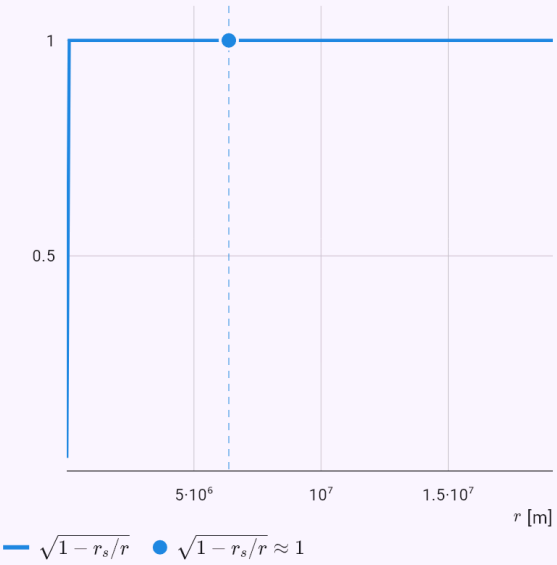
Result

$$\Delta t \approx 45.7257\,\mu\text{s}$$

Steps

- Find Δt; given
  $M = 3.004 \cdot 10^{-6}\,M_\odot, \quad r = 6371\,\text{km}, \quad r_2 = 26\,570\,\text{k}$
- Converting the units
  $M = 3.004 \cdot 10^{-6}\,M_\odot = 5.972 \cdot 10^{24}\,\text{kg}, \quad r = 6371\,\text{km}$
- Clocks near a massive body run slower: dτ/dt = √(1 − r_s/r). Light from there arrives shifted to the red. With a second distance r_2 the mode compares two clocks. The GPS satellites (r_2 = 26 571 km) gain about 45.7 µs a day on the ground. The model's equation
  $$\frac{d\tau_1}{d\tau_2} = \frac{\sqrt{1 - r_s/r}}{\sqrt{1 - r_s/r_2}}, \quad 1 + z = \frac{d\tau_2}{d\tau_1}$$
- Substituting
  $$r_s = \frac{2 \cdot 6.674 \cdot 10^{-11} \cdot 5.972 \cdot 10^{24}}{\left(2.998 \cdot 10^{8}\right)^2} \approx 0.00887\,\text{m}$$
- Lag per unit of time: $1 - \frac{d\tau}{dt} \approx 5.292 \cdot 10^{-10}$
- Substituting
  $$\Delta t = \left(1 - \frac{d\tau}{dt}\right) \cdot 86\,400 \approx 4.573 \cdot 10^{-5}\,\text{s}$$
- Two clocks at different distances compared; a positive difference means that the nearer clock runs slower.
- Converting the units: $\Delta t \approx 4.573 \cdot 10^{-5}\,\text{s} = 45.73\,\mu\text{s}$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

The tidal force on a person at a stellar black hole's horizon.

Input: `model(blackhole; find=dFh[N]; M=10[Msun]; m=70[kg]; L=2[m]; mode=tidal;`

`astro=stellar)`

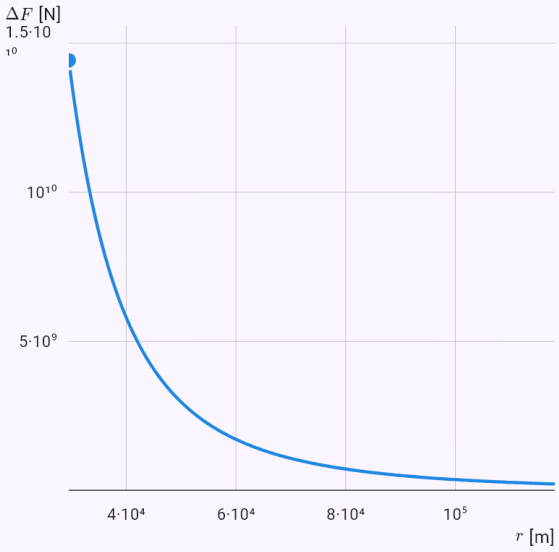
Result

$$\Delta F_s \approx 1.443 \cdot 10^{10}\ \mathrm{N}$$

Steps

- Find ΔF_s; given: $M = 10\,M_\odot,\quad m = 70\,\mathrm{kg},\quad L = 2\,\mathrm{m}$
- Converting the units: $M = 10\,M_\odot = 1.988 \cdot 10^{31}\,\mathrm{kg}$
- The tidal force is the difference between the gravity at the head and at the feet. A stellar black hole tears a person apart long before the horizon ("spaghettification"); at the horizon of a big black hole (Sgr A*) the force is tiny. The model's equation: $\Delta F \approx \dfrac{2GMm\,L}{r^3}$
- Substituting $r_s = \dfrac{2 \cdot 6.674 \cdot 10^{-11} \cdot 1.988 \cdot 10^{31}}{\left(2.998 \cdot 10^{8}\right)^2} \approx 29\,530\,\mathrm{m}$
- Expressing the sought quantity: $\Delta F_s = \dfrac{2GMm\,L}{r_s^3} = \dfrac{mLc^6}{4G^2M^2}$
- Substituting $\Delta F_s = \dfrac{2 \cdot 6.674 \cdot 10^{-11} \cdot 1.988 \cdot 10^{31} \cdot 70 \cdot 2}{(29\,530)^3} \approx 1.443 \cdot 1$
- In multiples of the body's weight: $\dfrac{\Delta F}{mg} \approx 2.101 \cdot 10^{7}$
- The tidal force is far more than a body can bear – "spaghettification".

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

The mass of Sgr A* from the orbit of the star S2.

Input: `model(blackhole; find=M[Msun]; a=1019[au]; T=16.05[yr]; mode=s2)`

Result

$$M \approx 4\,107\,000\,M_\odot$$

Steps

- Find M; given: $a = 1019\,\mathrm{au},\quad T = 16.05\,\mathrm{yr}$
- Converting the units $a = 1019\,\mathrm{au} = 1.524 \cdot 10^{14}\,\mathrm{m},\quad T = 16.05\,\mathrm{yr} = 5.065$
- The star S2 orbits the centre of the Galaxy on an ellipse with a semi-major axis of about 1 019 au in 16.05 years. Kepler's third law gives the mass of Sgr A*: about 4 million Suns in a region smaller than the Solar System. This measurement won the 2020 Nobel Prize in Physics. The model's equation: $\dfrac{T^2}{a^3} = \dfrac{4\pi^2}{GM} \Rightarrow M = \dfrac{4\pi^2 a^3}{GT^2}$
- Substituting $M = \dfrac{4\pi^2 \cdot (1.524 \cdot 10^{14})^3}{6.674 \cdot 10^{-11} \cdot \left(5.065 \cdot 10^{8}\right)^2} \approx 8.168 \cdot 10^{36}\,\mathrm{kg}$
- In solar masses: $M \approx 4\,107\,000\,M_\odot$
- Horizon of Sgr A* $r_s = \dfrac{2GM}{c^2} \approx 1.213 \cdot 10^{10}\,\mathrm{m},\quad \dfrac{a}{r_s} \approx 12\,570$
- The 2020 Nobel Prize in Physics (Genzel, Ghez) for discovering a supermassive compact object at the centre of the Galaxy.
- Converting the units: $M \approx 8.168 \cdot 10^{36}\,\mathrm{kg} = 4\,107\,000\,M_\odot$

The core's rotation period after the collapse into a neutron star.

Input: `model(neutronstar; find=T2[ms]; T1=25.4[d]; R1=696000[km]; R2=12[km]; mode=collapse)`

Result

$$T_2 \approx 0.6524\,\text{ms}$$

Steps

- Find T_2; given:
  $T_1 = 25.4\,\text{d}, \quad R_1 = 696\,000\,\text{km}, \quad R_2 = 12\,\text{km}$
- Converting the units
  $T_1 = 25.4\,\text{d} = 2\,195\,000\,\text{s}, \quad R_1 = 696\,000\,\text{km} = 6.96 \cdot$
- Conservation of angular momentum: when the core shrinks, it spins faster, and the period falls with the square of the radius (like a figure skater pulling in her arms). That is how pulsars with periods of milliseconds to seconds are born. The model's equation
  $I_1\omega_1 = I_2\omega_2, \quad I = \frac{2}{5}MR^2, \quad \omega = \frac{2\pi}{T} \Rightarrow T_2 = T_1\left(\frac{R_2}{R_1}\right.$
- The star is taken as a uniform ball that loses no mass.
- Substituting
  $T_2 = 2\,195\,000 \cdot \left(\frac{12\,000}{6.96 \cdot 10^8}\right)^2 \approx 6.524 \cdot 10^{-4}\,\text{s}$
- How many times faster it spins:
  $\frac{T_1}{T_2} = \left(\frac{R_1}{R_2}\right)^2 \approx 3.364 \cdot 10^9$
- Real pulsars spin slower (milliseconds to seconds): the core spins slower before the collapse, and the ejected envelope takes angular momentum away.
- Converting the units: $T_2 \approx 6.524 \cdot 10^{-4}\,\text{s} = 0.6524\,\text{ms}$

The energy radiated as gravitational waves in the merger GW150914.

Input: `model(gwave; find=E[J]; M1=36[Msun]; M2=29[Msun]; Mf=62[Msun]; mode=gwenergy)`

Result

$$E \approx 5.361 \cdot 10^{47}\,\text{J}$$

Steps

- Find E; given:
  $M_1 = 36\,M_\odot, \quad M_2 = 29\,M_\odot, \quad M_f = 62\,M_\odot$
- Converting the units
  $M_1 = 36\,M_\odot = 7.158 \cdot 10^{31}\,\text{kg}, \quad M_2 = 29\,M_\odot = 5.767$
- The merger makes a black hole lighter than the two together. The difference in mass is carried away by the gravitational waves as the energy E = Δm·c². The model's equation: $\Delta m = M_1 + M_2 - M_f, \quad E = \Delta m\,c^2$
- Substituting
  $\Delta m = 7.158 \cdot 10^{31} + 5.767 \cdot 10^{31} - 1.233 \cdot 10^{32} \approx 5.96$
- Share of the mass radiated:
  $\frac{\Delta m}{M_1 + M_2} \approx 4.62\,\%, \quad \Delta m \approx 3\,M_\odot$
- Substituting
  $E = 5.965 \cdot 10^{30} \cdot \left(2.998 \cdot 10^8\right)^2 \approx 5.361 \cdot 10^{47}\,\text{J}$
- Compared with the Sun's whole life: $\frac{E}{L_\odot \cdot 10^{10}\,\text{yr}} \approx 4438$
- LIGO caught the waves of GW150914 from 1.3 billion light years away; they moved its mirrors by a thousandth of a proton's diameter.

The distance of Sirius from the distance modulus.

Input: `model(magnitude; find=d[pc]; mapp=-1.46[one]; Mabs=1.42[one]; mode=distmod; star=sirius)`

Result

$$d \approx 2.655\ \text{pc}$$

Steps

- Find d; given: $m = -1.46, \quad M = 1.42$
- The absolute magnitude M is the one the star would have from 10 pc. The distance modulus m − M gives the distance when the star's true luminosity is known (Cepheids, type Ia supernovae). The model's equation: $m - M = 5 \log \dfrac{d}{10\ \text{pc}}$
- Expressing the sought quantity: $d = 10\ \text{pc} \cdot 10^{(m-M)/5}$
- Substituting: $d = 10 \cdot 10^{(-1.46-1.42)/5} \approx 2.655\ \text{pc}$
- In light years: $d \approx 8.658\ \text{ly}$
- The absorption of light by interstellar dust is neglected; with it, a star looks farther away.

The peak wavelength of the Sun's radiation.

Input: `model(starlight; find=lambda[nm]; T=5772[K]; mode=wien; star=sun)`

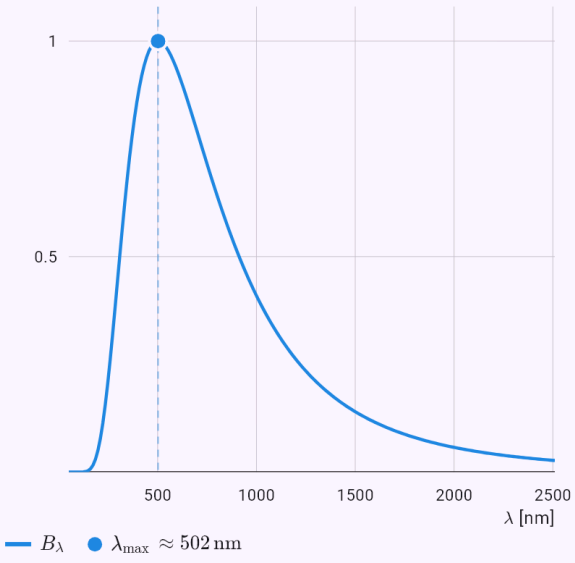
Result

$$\lambda_{\max} \approx 502\ \text{nm}$$

Steps

- Find $\lambda_{\max}$; given: $T = 5772\ \text{K}$
- Wien's displacement law: the wavelength at which a body radiates most is inversely proportional to its temperature. The Sun peaks in the green (about 500 nm), cool Betelgeuse in the infrared, hot Vega in the blue. The model's equation: $\lambda_{\max} T = b, \quad b \approx 2.898 \cdot 10^{-3}\ \text{m} \cdot \text{K}$
- Wien's constant in nm·K: $b \approx 2.898 \cdot 10^{6}\ \text{nm} \cdot \text{K}$
- Substituting: $\lambda_{\max} = \dfrac{2\,898\,000}{5772} \approx 502\ \text{nm}$
- The peak is in visible light.

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

The Earth's equilibrium temperature without the greenhouse effect.

Input: `model(starlight; find=Teq[K]; L=1[Lsun]; a=1[au]; A=0.306[one]; mode=equilibrium;`

`planet=earth)`

Result

$$T_{\text{eq}} \approx 254\,\text{K}$$

Steps

- Find T_{eq}; given: $L = 1\,L_{\odot}, \quad a = 1\,\text{au}, \quad A = 0.306$
- Converting the units
  $L = 1\,L_{\odot} = 3.828 \cdot 10^{26}\,\text{W}, \quad a = 1\,\text{au} = 1.496 \cdot 10^{11}\,\text{m}$
- The solar constant S is the power falling on 1 m² at the distance a from the star. A planet absorbs S(1 − A) over its cross-section πR² and radiates σT⁴ from its whole surface 4πR². The balance gives its temperature without the greenhouse effect. The model's equation:
  $S = \dfrac{L}{4\pi a^2}, \quad \pi R^2 S(1-A) = 4\pi R^2 \sigma T_{\text{eq}}^4$
- Substituting
  $S = \dfrac{3.828 \cdot 10^{26}}{4\pi \cdot \left(1.496 \cdot 10^{11}\right)^2} \approx 1361\,\text{W/m}^2$
- Expressing the sought quantity: $T_{\text{eq}} = \sqrt[4]{\dfrac{S(1-A)}{4\sigma}}$
- Substituting
  $T_{\text{eq}} = \sqrt[4]{\dfrac{1361 \cdot (1-0.306)}{4 \cdot 5.67 \cdot 10^{-8}}} \approx 254\,\text{K}$
- Without the greenhouse effect. The Earth's real mean temperature is higher: $T_{\oplus} \approx 288\,\text{K}$

The Hubble time as an estimate of the age of the universe.

Input: `model(redshift; find=age[Gyr]; H0=70[km_s_Mpc]; mode=hubble)`

Result

$$t_H \approx 13.97\,\text{Gyr}$$

Steps

- Find t_H; given: $H_0 = 70\,\text{km}/(\text{s} \cdot \text{Mpc})$
- Hubble's law v = H_0·d. The reciprocal of the Hubble constant (the Hubble time) estimates the age of the universe at about 14 billion years; the exact age of 13.8 billion years comes from a model of the expansion. The model's equation: $v = H_0 d, \quad t_H = \dfrac{1}{H_0}$
- Hubble constant in s⁻¹
  $H_0 = \dfrac{70}{3.086 \cdot 10^{19}}\,\text{s}^{-1} \approx 2.269 \cdot 10^{-18}\,\text{s}^{-1}$
- Substituting: $t_H = \dfrac{1}{2.269 \cdot 10^{-18}} \approx 4.408 \cdot 10^{17}\,\text{s}$
- In billions of years: $t_H \approx 13.97\,\text{Gyr}$
- The Hubble time is only an estimate of the age; the expansion slowed down and is now speeding up.
- Converting the units: $t_H \approx 4.408 \cdot 10^{17}\,\text{s} = 13.97\,\text{Gyr}$

The transit depth of the Earth in front of the Sun.

Input: `model(exoplanet; find=depth[ppm]; Rp=1[Rearth]; Rs=1[Rsun]; mode=transit; exo=earth)`

Result

$$\frac{\Delta F}{F} \approx 83.86\,\text{ppm}$$

Steps

- Find $\frac{\Delta F}{F}$; given: $R_p = 1\,R_\oplus, \quad R_* = 1\,R_\odot$
- Converting the units:
  $R_p = 1\,R_\oplus = 6371\,\text{km}, \quad R_* = 1\,R_\odot = 695\,700\,\text{km}$
- When a planet passes in front of its star, it covers part of the star's disc and the brightness drops by the ratio of the areas (R_p/R_*)². Jupiter would dim the Sun by 1 %, the Earth by only 84 millionths (ppm). The model's equation: $\frac{\Delta F}{F} = \frac{\pi R_p^2}{\pi R_*^2} = \left(\frac{R_p}{R_*}\right)^2$
- Substituting: $\frac{\Delta F}{F} = \left(\frac{6371}{695\,700}\right)^2 \approx 8.386 \cdot 10^{-5}$
- In millionths (ppm): $\frac{\Delta F}{F} \approx 83.86\,\text{ppm}$
- A transit is seen only when we look at the planet's orbit almost edge-on.
- Converting the units: $\frac{\Delta F}{F} \approx 8.386 \cdot 10^{-5} = 83.86\,\text{ppm}$

The time of a Hohmann transfer from the Earth to Mars.

Input: `model(orbit; find=tt[d]; M=1.989e30[kg]; r1=1[au]; r2=1.524[au]; mode=hohmann;`

`central=sun)`

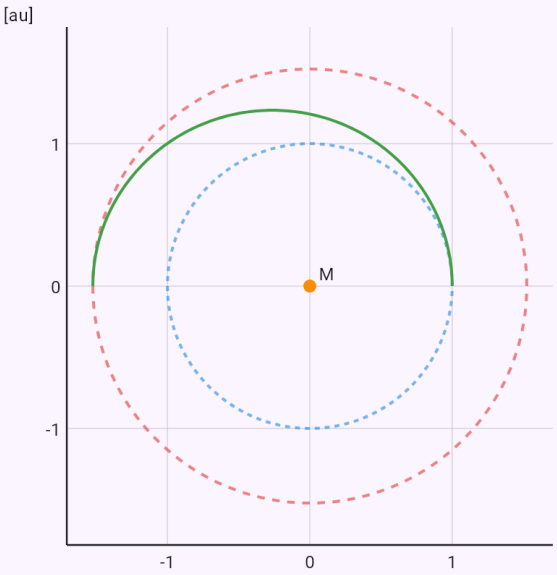
Result

$t \approx 258.9\,\mathrm{d}$

Steps

- Find t; given
  $M = 1.989 \cdot 10^{30}\,\mathrm{kg}, \quad r_1 = 1\,\mathrm{au}, \quad r_2 = 1.524\,\mathrm{au}$
- Converting the units
  $r_1 = 1\,\mathrm{au} = 1.496 \cdot 10^{11}\,\mathrm{m}, \quad r_2 = 1.524\,\mathrm{au} = 2.28 \cdot 10$
- A Hohmann transfer orbit is half an ellipse touching both the initial and the target circular orbit. Two burns (Δv) move the craft from one to the other with the least fuel. A flight from the Earth to Mars this way takes about 259 days. The model's equation
  $a_t = \frac{r_1 + r_2}{2}, \quad \Delta v_1 = \sqrt{\frac{GM}{r_1}}\left(\sqrt{\frac{2r_2}{r_1 + r_2}} - 1\right), \quad \Delta v$
- Substituting
  $a_t = \frac{1.496 \cdot 10^{11} + 2.28 \cdot 10^{11}}{2} \approx 1.888 \cdot 10^{11}\,\mathrm{m}$
- Substituting
  $t = \pi\sqrt{\frac{\left(1.888 \cdot 10^{11}\right)^3}{6.674 \cdot 10^{-11} \cdot 1.989 \cdot 10^{30}}} \approx 2.237 \cdot 10^{7}\,\mathrm{s}$
- In days: $t \approx 258.9\,\mathrm{d}$
- Outward: speed up along the flight, then speed up again on the target orbit.
- Converting the units: $t \approx 2.237 \cdot 10^{7}\,\mathrm{s} = 258.9\,\mathrm{d}$

Graph



The bodies and the notable points at true scale.

Saturn's Roche limit for an icy moon – why the rings are there.

Input: `model(twobody; find=dR[km]; R=58232[km]; rhoM=0.687[g_cm3]; rhom=0.917[g_cm3];`

`mode=roche; satellite=fluid)`

Result

$$d_R \approx 129\,000\,\text{km}$$

Steps

- Find d_R; given
  $R = 58\,230\,\text{km}, \quad \rho_M = 0.687\,\text{g/cm}^3, \quad \rho_m = 0.917\,\text{g/}$
- Converting the units
  $R = 58\,230\,\text{km} = 5.823 \cdot 10^7\,\text{m}, \quad \rho_M = 0.687\,\text{g/cm}^3 =$
- The Roche limit: closer to the planet, tidal forces tear apart a moon held together only by its own gravity. That is why Saturn's rings lie inside its Roche limit. A rigid moon survives closer in than a fluid one. The model's equation: $d_R \approx 2.44\,R\sqrt[3]{\frac{\rho_M}{\rho_m}}$
- Substituting
  $d_R = 2.44 \cdot 5.823 \cdot 10^7 \cdot \sqrt[3]{\frac{687}{917}} \approx 1.29 \cdot 10^8\,\text{m}$
- In planet radii: $\frac{d_R}{R} \approx 2.216$
- No large moon forms inside the Roche limit; rings are there instead (Saturn).
- Converting the units: $d_R \approx 1.29 \cdot 10^8\,\text{m} = 129\,000\,\text{km}$

46. Applications of complex analysis

Forms in Applications (the fields Physics, Electrical engineering, Engineering, Geometry and Calculus) use the functions of a complex variable from the previous chapter. Each has steps with a complex potential, a map or poles and a graph in the Gauss plane.

Ideal flow and an airfoil: the [[complexPotential|complex potential]] w(z), [[streamline|streamlines]] as the level lines of Im w and the pressure from Bernoulli's equation. Flow past a cylinder, with circulation too (stagnation points, the Magnus effect, lift $L' = \rho U \Gamma$), Rankine's body, flow into a corner and a vortex. Joukowski's airfoil comes from the conformal map $z = \zeta + b^2/\zeta$, the [[kuttaCondition|Kutta condition]] fixes the circulation and the Kutta–Joukowski theorem the lift; the graph shows the streamlines around the airfoil and C_L by the angle of attack.

Fields by conformal maps: a coaxial cable ($w = \ln z$ turns it into a parallel-plate capacitor - capacitance, inductance and characteristic impedance), a two-wire line by a Möbius map, the potential in a wedge between plates and a steady temperature in a half-plane; field lines and equipotentials in the graph. The stress around a circular hole in a plate by Kirsch with the Kolosov–Muskhelishvili potentials gives a [[stressConcentration|stress concentration]] of 3 in tension, 2 in equal tension both ways and 4 in shear. Mercator's map is the complex logarithm of the stereographic projection: the scale $1/\cos\varphi$, Tissot's circles and the [[loxodrome|rhumb line]] as a straight line against the great circle.

Control and signals: the stability of a loop $K \cdot e^{(-s\tau)}/(s^q(1 + T_1 s)(1 + T_2 s)(1 + T_3 s))$ by the [[nyquistCriterion|Nyquist criterion]] (the argument principle, gain and phase margins, Bode plot) and by the [[rootLocus|root locus]] (asymptotes, breakaway point, critical gain). The design of a [[butterworthFilter|Butterworth]] and a Chebyshev filter from poles on a circle and an ellipse, a second-order digital filter with poles and zeros in the z plane, the impulse response by residues and the frequency response on the unit circle. A mechanical oscillator's or an RLC circuit's resonance curve from the distances of $i\omega$ to the poles, the [[qualityFactor|quality factor]] and the bandwidth. AC circuits with phasors are in the Electrical engineering chapter.

Response and lines: Lorentz's oscillator gives a medium's refractive index and absorption; poles in the lower half-plane mean causality and the [[kramersKronig|Kramers–Kronig relations]], which the app checks by a numerical principal-value integral. A spectral line's natural width follows from the Fourier transform of a damped wave. The [[smithChart|Smith chart]]: the reflection coefficient as a Möbius transformation, the standing wave ratio, a line's input impedance, a quarter-wave transformer and stub matching.

Toys on mathtrace.app under “Try it yourself”: the Mandelbrot and Julia sets and Newton's basins - an iteration in the complex plane you can pan and zoom.

Examples

A spinning cylinder: the circulation moves the stagnation points below the horizontal axis.

Input: `model(potentialflow; mode=flowcylinder; find=stag[deg]; U=10[m_s]; R=0.5[m]; G=20[m2_s])`

Result

$$\theta_s \approx -18.56\,^\circ$$

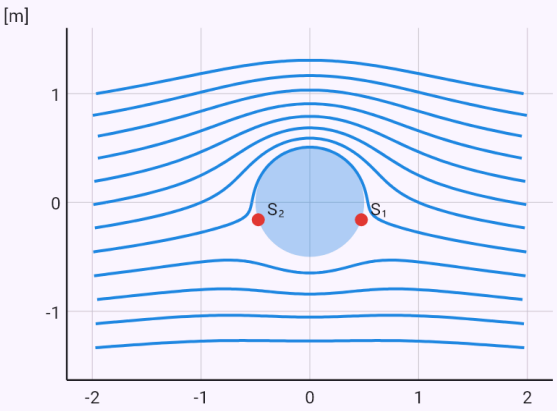
Steps

- Find θ_s; given
  $U = 10\,\mathrm{m/s}, \quad R = 0.5\,\mathrm{m}, \quad \Gamma = 20\,\mathrm{m^2/s}$
- Flow past a cylinder: the stream U(z + R²/z), optionally with a vortex of circulation Γ (a spinning cylinder). On the surface the speed is 2U·sin θ, twice the stream's at the sides. The circulation moves the stagnation points down and creates lift by the Kutta–Joukowski theorem, L' = ρUΓ per metre of length. The model's equation:
  $w(z) = U\left(z + \frac{R^2}{z}\right) + \frac{i\Gamma}{2\pi}\ln z$
- The conjugate velocity is the potential's derivative
  $\overline{\mathbf{v}} = w'(z) = U\left(1 - \frac{R^2}{z^2}\right) + \frac{i\Gamma}{2\pi z}$
- On the cylinder's surface the velocity is tangential
  $z = Re^{i\theta}: \quad v_\theta = -\left(2U\sin\theta + \frac{\Gamma}{2\pi R}\right), \quad v_r = 0$
- Largest speed on the surface:
  $v_{max} = 2U + \frac{|\Gamma|}{2\pi R} = 26.37\,\mathrm{m/s}$
- Stagnation points (v = 0)
  $\sin\theta_s = -\frac{\Gamma}{4\pi U R} = -0.3183 \quad \Rightarrow \quad \theta_s = -18.56^\circ,\ 19$
- Pressure coefficient from Bernoulli's equation
  $c_p = \frac{p - p_\infty}{\frac{1}{2}\rho U^2} = 1 - \frac{v^2}{U^2} = 1 - \left(2\sin\theta + \frac{\Gamma}{2\pi U R}\right)^2$

- Lift per metre of length (Kutta–Joukowski theorem):
  $L' = \rho U\Gamma = 1.225 \cdot 10 \cdot 20 = 245\,\mathrm{N/m}$
- A spinning cylinder or ball with circulation gets lift perpendicular to the stream - the Magnus effect that curves a spun ball. Flettner ships use it for propulsion.

Graph

Streamlines | Chart



Solid lines are streamlines, dashed ones equipotentials (only without circulation). Where the streamlines are denser, the fluid flows faster.

A 12 % Joukowski airfoil with 2 % camber at a 4° angle of attack.

Input: `model(potentialflow; mode=joukowski; find=CL[one]; U=50[m_s]; ch=1.5[m]; tc=12[pct];`

```
fc=2[pct]; aoa=4[deg])
```

Result

$$C_L \approx 0.7243$$

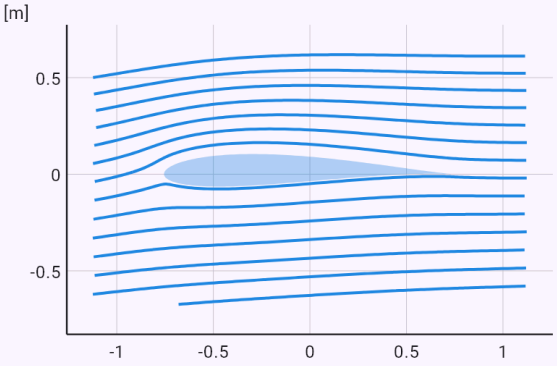
Steps

- Find C_L; given
  $U = 50\,\mathrm{m/s}, \quad c = 1.5\,\mathrm{m}, \quad t/c = 12\,\%, \quad f/c = 2\,\%,$
- Joukowski's airfoil comes from the conformal map z = ζ + b²/ζ of a circle through the point ζ = b. Moving the centre left gives thickness, moving it up gives camber. The flow around the circle carries over to the airfoil, the circulation is fixed by the Kutta condition (the flow leaves the sharp trailing edge smoothly) and the lift by the Kutta–Joukowski theorem. The familiar C_L ≈ 2π(α + β) comes out, as for a thin airfoil. A whole aeroplane's performance is in the Aeroplane form. The model's equation
  $z = \zeta + \frac{b^2}{\zeta}, \quad w(\zeta) = U\left((\zeta - \mu)e^{-i\alpha} + \frac{a^2 e^{i\alpha}}{\zeta - \mu}\right) + \frac{i\Gamma}{2\pi}$
- The circle in the ζ plane the airfoil comes from
  $b = 0.3723\,\mathrm{m}, \quad \mu = -0.03439 + 0.01489i, \quad a = |b -$
- Circulation from the Kutta condition at the trailing edge
  $|w'(b)| < \infty: \quad \Gamma = 4\pi a U \sin(\alpha + \beta) = 27.16\,\mathrm{m^2/s}$
- Lift per metre of length (Kutta–Joukowski theorem):
  $L' = \rho U \Gamma = 1664\,\mathrm{N/m}$
- Lift coefficient, thin-airfoil theory in brackets
  $C_L = \frac{L'}{\frac{1}{2}\rho U^2 c} = \frac{8\pi a}{c}\sin(\alpha + \beta) = 0.7243 \quad (2\pi(\alpha + \beta$

- Ideal fluid: valid for small angles of attack without flow separation (roughly up to 10–15° for common airfoils). A real wing of finite span has less lift due to induced drag (the Aeroplane form).

Graph

Streamlines | Chart



Streamlines around the airfoil at true scale. The trailing edge is on the right - the flow leaves it smoothly (Kutta condition); above the airfoil the streamlines are denser and the pressure lower.

Rankine's body - the width far downstream is Q/U.

Input: `model(potentialflow; mode=rankine; find=width[m]; U=5[m_s]; Q=10[m2_s])`

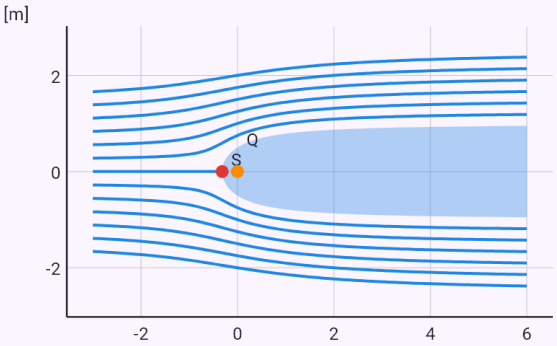
Result

$$d \approx 2\,\mathrm{m}$$

Steps

- Find d; given: $U = 5\,\mathrm{m/s}, \quad Q = 10\,\mathrm{m^2/s}$
- A source of strength Q (m²/s per metre of depth) in a stream of speed U. The dividing streamline ψ = Q/2 wraps Rankine's half-body, shaped like the front of a bridge pier or a bow. There is a stagnation point in front; far downstream the body is Q/U wide. The model's equation: $w(z) = Uz + \frac{Q}{2\pi}\ln z$
- The conjugate velocity is the potential's derivative: $w'(z) = U + \frac{Q}{2\pi z}$
- Stagnation point in front of the source: $w'(z) = 0: \quad z_s = -\frac{Q}{2\pi U} = -0.3183\,\mathrm{m}$
- Dividing streamline - the body's outline: $\psi = Uy + \frac{Q}{2\pi}\theta = \frac{Q}{2}$
- Body's width far downstream: $y \to \pm\frac{Q}{2U}: \quad d = \frac{Q}{U} = 2\,\mathrm{m}$
- Largest speed on the body's surface (numerically): $v_{max} \approx 1.26\,U = 6.298\,\mathrm{m/s} \quad (\theta \approx 63^\circ)$
- Rankine's body is a first estimate of the flow around a rounded front (a pier, a hull, a leading edge). In an ideal fluid there is no wake behind the body.

Graph



Solid lines are streamlines, dashed ones equipotentials (only without circulation). Where the streamlines are denser, the fluid flows faster.

RG-58 coaxial cable: a characteristic impedance of about 50 Ω.

Input: `model(conformal; mode=coax; find=Z[ohm]; a=0.45[mm]; b=1.475[mm]; er=2.25[one])`

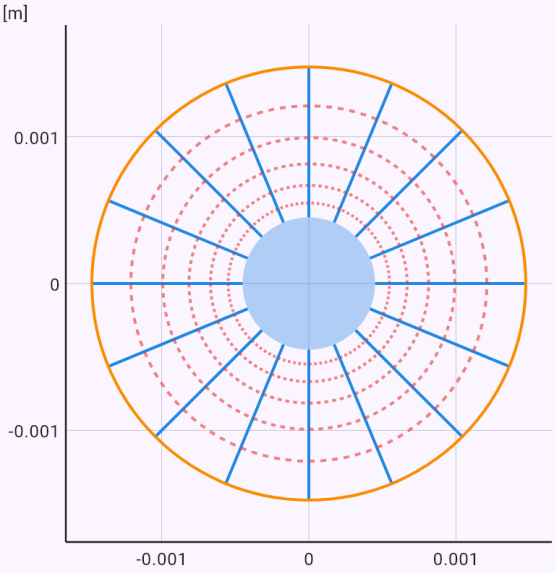
Result

$$Z_0 \approx 47.45\,\Omega$$

Steps

- Find Z_0; given:
  $a = 0.45\,\text{mm}, \quad b = 1.475\,\text{mm}, \quad \varepsilon_r = 2.25$
- Converting the units
  $a = 0.45\,\text{mm} = 4.5 \cdot 10^{-4}\,\text{m}, \quad b = 1.475\,\text{mm} = 0.001$
- A coaxial cable: the logarithm w = ln z unrolls the annulus a < |z| < b into a rectangle 2π wide and ln(b/a) high. The cable becomes a parallel-plate capacitor and the capacitance, inductance and characteristic impedance follow at once. The model's equation:
  $w = \ln z = \ln r + i\theta$
- The annulus's image in the plane w = ln z - a rectangle
  $a < |z| < b \;\mapsto\; \ln a < u < \ln b, \quad 0 \le v < 2\pi : \quad \ln \frac{b}{a}$
- The potential is linear in Re w
- Capacitance as of a parallel-plate capacitor:
  $C' = \varepsilon \frac{2\pi}{\ln(b/a)} = 105.4\,\text{pF/m}$
- The line's inductance and characteristic impedance
  $L' = \frac{\mu_0}{2\pi} \ln \frac{b}{a} = 0.2374\,\mu\text{H/m}, \quad Z_0 = \sqrt{\frac{L'}{C'}} = \frac{60\,\Omega}{\sqrt{\varepsilon_r}} \ln$
- RG-58 cable (0.9 mm conductor, 2.95 mm polyethylene) has about 50 Ω and 100 pF/m. The characteristic impedance depends only on the ratio b/a, so thin and thick cables alike have 50 or 75 Ω.

Graph



Solid lines are field lines, dashed ones equipotentials; they cross at right angles. Conductors are filled or thick.

A two-wire line's capacitance by a Möbius map.

Input: `model(conformal; mode=twowire; find=C[pF_m]; a=1[mm]; d=10[mm])`

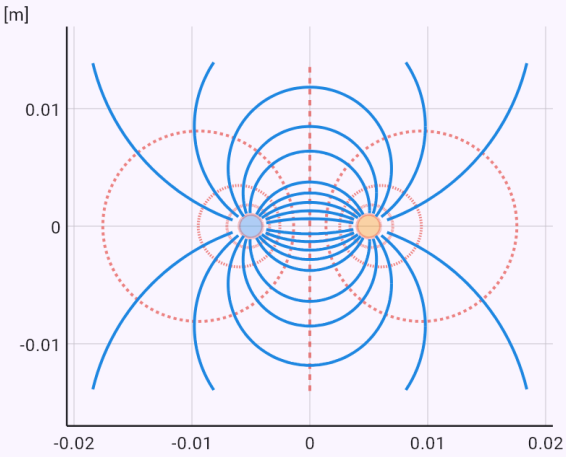
Result

$$C' \approx 12.13\,\mathrm{pF/m}$$

Steps

- Find C'; given: $a = 1\,\mathrm{mm}, \quad d = 10\,\mathrm{mm}$
- Converting the units:
  $a = 1\,\mathrm{mm} = 0.001\,\mathrm{m}, \quad d = 10\,\mathrm{mm} = 0.01\,\mathrm{m}$
- A two-wire line (two parallel wires): the Möbius map w = (z − h)/(z + h) turns the wires' circles into concentric circles - a coaxial cable. The equipotentials are Apollonius circles, the field lines circles through the points ±h. The model's equation
  $$w = \frac{z-h}{z+h}, \quad h = \sqrt{\left(\frac{d}{2}\right)^2 - a^2} = 0.004899\,\mathrm{m}$$
- The wires' images are concentric circles
  $$|w| = \left|\frac{z-h}{z+h}\right| = k: \quad \ln\frac{|w|_2}{|w|_1} = 2\,\mathrm{arcosh}\,\frac{d}{2a} = 4.585$$
- Capacitance of the two-wire line
  $$C' = \frac{\pi\varepsilon}{\mathrm{arcosh}\,(d/2a)} = 12.13\,\mathrm{pF/m}$$
- The line's inductance and characteristic impedance
  $$L' = \frac{\mu_0}{\pi}\,\mathrm{arcosh}\,\frac{d}{2a} = 0.917\,\mu\mathrm{H/m}, \quad Z_0 = \sqrt{\frac{L'}{C'}} = \frac{12}{\sqrt{}}$$
- Flat twin-lead antenna line used to have 300 Ω, a twisted pair in data cables about 100 Ω. The formula holds for wires far from other conductors.

Graph



Solid lines are field lines, dashed ones equipotentials; they cross at right angles. Conductors are filled or thick.

Steady temperature in a half-plane with a hot and a cold part of the boundary.

Input: `model(conformal; mode=halfplane; find=T[degC]; T1=100[degC]; T2=20[degC]; x=0.5[m];`

y=1[m])

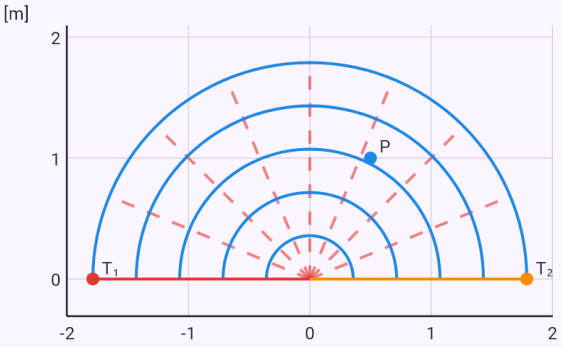
Result

$$T \approx 48.19\,^\circ\mathrm{C}$$

Steps

- Find T; given
  $T_1 = 100\,^\circ\mathrm{C}, \quad T_2 = 20\,^\circ\mathrm{C}, \quad x = 0.5\,\mathrm{m}, \quad y = 1\,\mathrm{m}$
- Steady heat conduction in the half-plane y > 0 whose boundary is at T_1 on the left and T_2 on the right. The temperature is harmonic, like the potential in a 180° wedge: $T = T_2 + (T_1 - T_2)\cdot\theta/\pi$. The isotherms are rays from the origin, the heat flows along semicircles. The model's equation
  $$\Delta T = 0, \quad T(x,0) = \begin{cases} T_1 & x < 0 \\ T_2 & x > 0 \end{cases}$$
- A harmonic function with the given boundary values
- Point's angle: $\theta = \arg(0.5 + i) = 63.43^\circ$
- Temperature at the point:
  $$T = 20 + \frac{80}{\pi} \cdot 1.107 = 48.19\,^\circ\mathrm{C}$$
- For example the ground under the border of two surfaces at different temperatures. At the origin the boundary temperature jumps from T_1 to T_2, so the flux grows without bound there.

Graph



Dashed rays are isotherms, solid semicircles heat flow lines.

A circular hole in a plate under tension: a stress concentration factor of 3.

Input: `model(hole; mode=uniaxial; find=Kt[one]; sx=100[MPa]; a=10[mm])`

Result

$$K_t \approx 3$$

Steps

- Find K_t; given: $\sigma_x = 100\,\mathrm{MPa}, \quad a = 10\,\mathrm{mm}$
- Tension σ_x in one direction: at the hole's edge across the load 3σ_x, along the load −σ_x (compression). The model's equation

$$\sigma_{rr} + \sigma_{\theta\theta} = 4\,\mathrm{Re}\,\Phi(z), \quad \sigma_{\theta\theta} - \sigma_{rr} + 2i\tau_{r\theta} = 2e^{2i\theta}(\bar{z}\Phi'$$

- Kolosov–Muskhelishvili complex potentials for the tension σ

$$\varphi(z) = \frac{\sigma}{4}\left(z + \frac{2a^2}{z}\right), \quad \psi(z) = -\frac{\sigma}{2}\left(z + \frac{a^2}{z} - \frac{a^4}{z^3}\right)$$

- At the hole's edge the radial stress is zero, the hoop stress

$$r = a: \quad \sigma_{rr} = 0, \quad \sigma_{\theta\theta} = \sigma_x + \sigma_y - 2(\sigma_x - \sigma_y)\cos 2\theta$$

- Hoop stress around the hole (Kirsch)

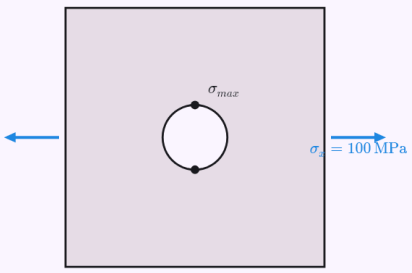
$$\sigma_{\theta\theta} = \frac{\sigma}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{\sigma}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta$$

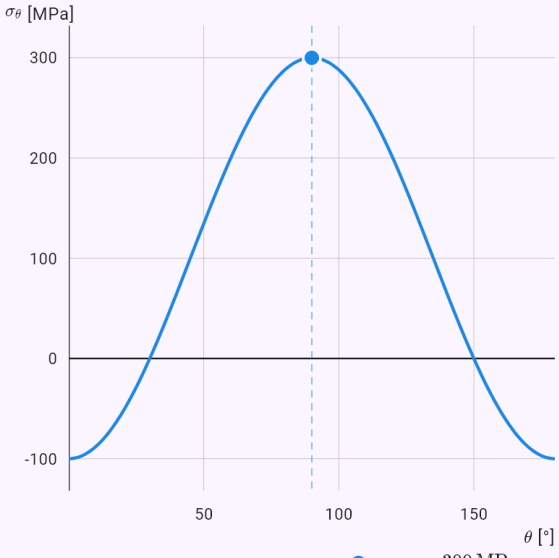
- Largest stress and the concentration factor

$$\sigma_{max} = \sigma_{\theta\theta}(a, 90^\circ) = 300\,\mathrm{MPa}, \quad K_t = \frac{|\sigma_{max}|}{100} = 3$$

- Two radii from the centre the stress is only 22 % above the plate's - the concentration is local.
- Valid for an elastic material and a hole small against the plate (width at least five times the diameter). In a narrow strip K_t is smaller but the stress in the weakened section larger.

Graph





The plate with the hole and the stress loading it (the arrows are not to scale). The chart shows the hoop stress at the hole's edge by the angle θ from the x axis; the dots on the hole are where the stress is largest.

At 60° latitude lengths on the map are doubled.

Input: `model(mercator; mode=point; find=k[one]; lat=60[deg])`

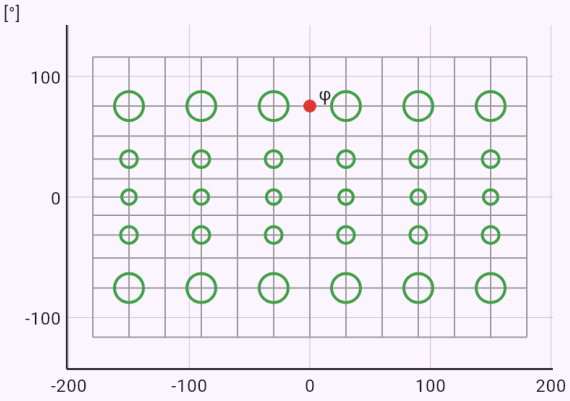
Result

$$k \approx 2$$

Steps

- Find k; given: $\varphi = 60\,^\circ$
- The parallel φ's distance from the equator on a map W wide and the scale k = 1/cos φ. Tissot's circles in the graph are equal on the globe. The model's equation
$$x + iy = R\left(\lambda + i \ln \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)\right) = -iR \ln \zeta, \quad \zeta =$$
- The map is conformal, lengths grow equally in all directions: $\left|\frac{dw}{d\zeta}\right| \neq 0: \quad k = \frac{1}{\cos \varphi}$
- Scale on the parallel φ: $k = \frac{1}{\cos 60^\circ} = 2, \quad k^2 = 4$
- Areas grow as k², four times at 60°. The poles are not on the map, they lie at infinity.

Graph



Meridians every 30° and parallels every 15°. Tissot's circles are equal on the globe and grow as 1/cos φ on the map - the shape is kept, the area is not.

Prague – New York: the great circle and the longer rhumb line.

Input: `model(mercator; mode=rhumb; find=d[km]; lat1=50.08[deg]; lon1=14.42[deg]; lat2=40.71[deg]; lon2=-74.01[deg])`

Result

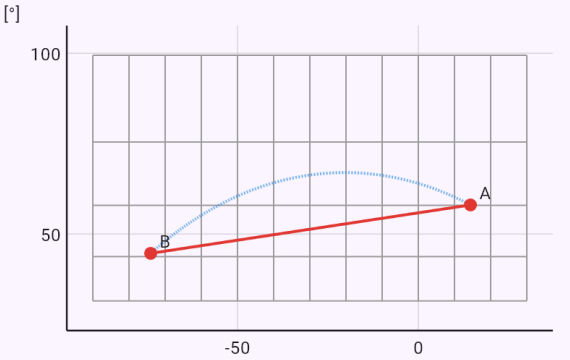
$$d \approx 6572\,\mathrm{km}$$

Steps

- Find d; given
$$\varphi_1 = 50.08\,^\circ, \quad \lambda_1 = 14.42\,^\circ, \quad \varphi_2 = 40.71\,^\circ, \quad \lambda_2 = -$$
- The rhumb line between two places (latitude and longitude, west negative): its course, length and a comparison with the great circle - the shortest way. The model's equation
$$x + iy = R\left(\lambda + i \ln \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)\right) = -iR \ln \zeta, \quad \zeta =$$
- The map is conformal, lengths grow equally in all directions: $\left|\frac{dw}{d\zeta}\right| \neq 0: \quad k = \frac{1}{\cos \varphi}$
- On the map the rhumb line is straight - its course from the coordinates' differences
$$\Delta\lambda = -88.43^\circ, \quad \Delta\psi = \ln \frac{\tan\left(\frac{\pi}{4} + \frac{\varphi_2}{2}\right)}{\tan\left(\frac{\pi}{4} + \frac{\varphi_1}{2}\right)} = -0.2337,$$
- Rhumb line's length: $L = \frac{R|\Delta\varphi|}{|\cos \alpha|} = 6960\,\mathrm{km}$
- Great circle (the shortest way on the sphere)
$$d = 2R \arcsin \sqrt{\sin^2 \frac{\Delta\varphi}{2} + \cos\varphi_1 \cos\varphi_2 \sin^2 \frac{\Delta\lambda}{2}} = 65$$
- The rhumb line is longer than the great circle but can be

sailed on a constant course. Long-haul aircraft fly great circles and change course continuously.

Graph



The solid line is the rhumb line (constant course), the dashed curve the great circle (the shortest way), both in Mercator's projection.

The phase margin of a loop with three lags.

Input: `model(control; mode=nyquist; integrators=0; find=PM[deg]; K=2[one]; T1=1[s]; T2=1[s]; T3=1[s])`

Result

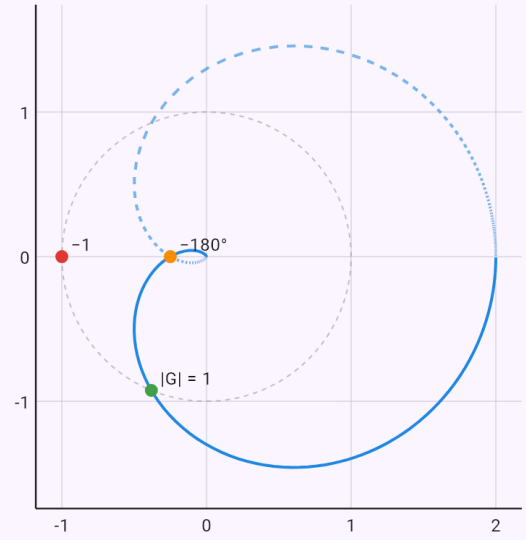
$$\varphi_m \approx 67.6\,^\circ$$

Steps

- Find φ_m; given:
  $K = 2, \quad T_1 = 1\,\text{s}, \quad T_2 = 1\,\text{s}, \quad T_3 = 1\,\text{s}$
- The Nyquist criterion follows from the argument principle: the number of times the curve G(iω) encircles −1 is the difference of the unstable poles of the closed and open loop. For a stable open loop the curve must not encircle −1. How far it stays shows the gain and phase margins. The model's equation
  $$G(s) = \frac{2}{(1+s)(1+s)(1+s)}, \quad F(s) = 1 + G(s)$$
- Argument principle: N encirclements of −1, Z unstable poles of the closed loop, P of the open one:
  $N = Z - P, \quad P = 0 \Rightarrow Z = N$
- Frequency where the phase reaches −180°
  $\arg G(i\omega_\pi) = -180^\circ: \quad \omega_\pi = 1.732\,\text{rad/s}, \quad |G(i\omega_\pi)|$
- Gain margin and the critical gain
  $$A_m = -20\log|G(i\omega_\pi)| = 12.04\,\text{dB}, \quad K_{krit} = \frac{K}{|G(i\omega_\pi)|}$$
- Frequency where |G| = 1:
  $|G(i\omega_c)| = 1: \quad \omega_c = 0.7664\,\text{rad/s}$
- Phase margin: $\varphi_m = 180^\circ + \arg G(i\omega_c) = 67.6^\circ$
- The curve doesn't encircle −1 - the closed loop is stable:
  $N = 0, \quad Z = N + P = 0$
- A usual requirement: a gain margin of at least 6 dB and a phase margin of 30–60°. A dead time keeps taking phase, which is why a delayed plant is harder to control.

Graph

Nyquist plot | Chart



The solid curve is G(iω) for ω > 0, the dashed one its mirror image for negative ω, the dashed circle |G| = 1. The closed loop is stable when the curve doesn't encircle −1.

The root locus with an integrator: a critical gain of 3.

Input: `model(control; mode=rootlocus; integrators=1; find=Kcrit[one]; K=1[one]; T1=1[s];`

`T2=0.5[s])`

Result

$$K_{krit} \approx 3$$

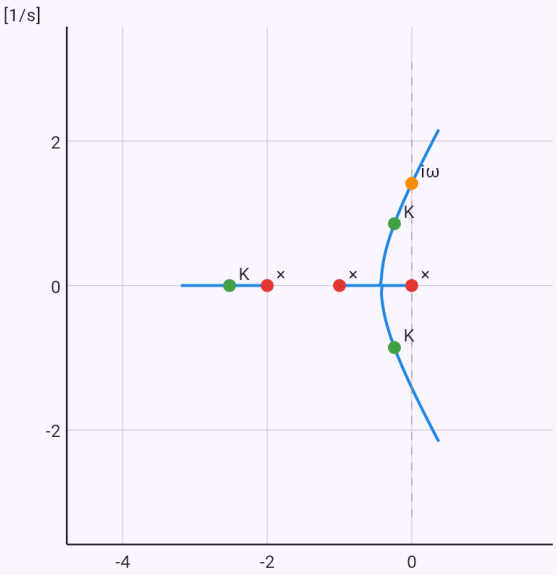
Steps

- Find K_{krit}; given: $K = 1, \quad T_1 = 1\,\text{s}, \quad T_2 = 0.5\,\text{s}$
- The root locus: how the closed loop's poles move as the gain K grows from zero. The branches start at the open loop's poles, leave along asymptotes, and on the imaginary axis the system becomes unstable (the critical gain from the Routh–Hurwitz condition). The model's equation: $1 + G(s) = 0: \quad 0.5s^3 + 1.5s^2 + s + K = 0$
- Open loop's poles (the branches' starts): $s_p = -2,\ -1,\ 0$
- The branches' asymptotes: centroid and angles

$$\sigma_a = \frac{\sum s_p}{n} = -1, \quad \theta_a = 60^\circ,\ 180^\circ,\ 300^\circ$$

- Breakaway point on the real axis: $\frac{dK}{ds} = -D'(s) = 0: \quad \sigma_b = -0.4226, \quad K = 0.1925$
- Crossing of the imaginary axis - the stability limit $s = i\omega: \quad \text{Im}\, D(i\omega) = 0 \Rightarrow \omega_{krit} = 1.414\,\text{rad/s}, \quad K_{kr}$
- The given gain is below the critical one - stable: $K = 1 < K_{krit}$
- Closed loop's poles for the given K: $K = 1: \quad s = -2.521,\ -0.2393 - 0.8579i,\ -0.2393 + 0.8579i$

Graph



The branches show the closed loop's poles for a growing K: they start at the open loop's poles (×), the points K are the poles for the given gain, iω the crossing of the imaginary axis.

The Chebyshev filter order needed for 40 dB an octave higher.

Input: `model(filterdesign; type=chebyshev; find=n[one]; fc=1000[Hz]; Ap=1[dB]; fs=2000[Hz];`

```
As=40[dB])
```

Result

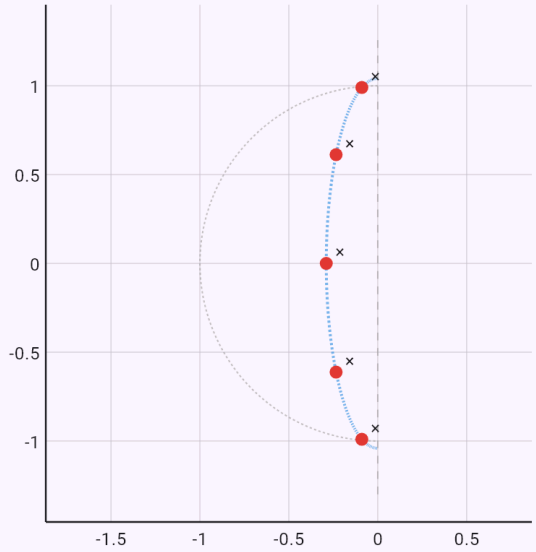
$$n \approx 5$$

Steps

- Find n; given
  $f_c = 1000\,\text{Hz}, \quad A_p = 1\,\text{dB}, \quad f_s = 2000\,\text{Hz}, \quad A_s = 4($
- Chebyshev: a ripple A_p in the passband, in return a steeper transition to the stopband than a Butterworth filter of the same order. The model's equation
  $$|H(i\omega)|^2 = \frac{1}{1 + \varepsilon^2 T_n^2(\omega/\omega_c)}$$
- Ripple parameter: $\varepsilon = \sqrt{10^{A_p/10} - 1} = 0.5088$
- Required order (rounded up)
- Poles on a half-ellipse
  $$s_k = \omega_c(-\sinh v \sin\theta_k + i \cosh v \cos\theta_k), \quad v = \frac{\operatorname{arsinh}}{n}$$
- Poles (relative to ω_c)
  $$\frac{s_k}{\omega_c} = -0.08946 + 0.9901i,\ -0.2342 + 0.6119i,\ -0.28($$
- Transfer function - all poles on the left, the filter is stable
  $$H(s) = \frac{H_0}{\prod_{k=1}^{5}(s - s_k)}, \quad \operatorname{Re} s_k < 0$$
- Chebyshev pays for steepness with ripple and a worse group delay - it keeps the shape of pulses less well.

Graph

Poles | Chart



Poles in the complex plane s (×); a stable system has them left of the imaginary axis.

A digital filter's impulse response with complex poles by residues.

Input: `model(zfilter; find=h[one]; b0=1[one]; a1=-1.6[one]; a2=0.81[one]; n=10[one])`

Result

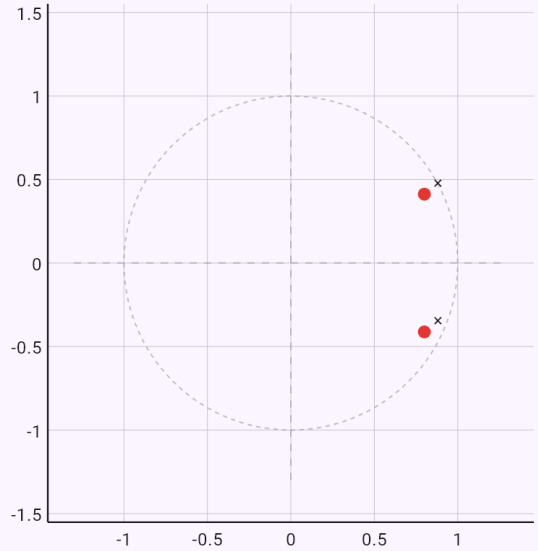
$$h[n] \approx -0.6596$$

Steps

- Find $h[n]$; given:
  $b_0 = 1, \quad a_1 = -1.6, \quad a_2 = 0.81, \quad n = 10$
- A digital filter computes the output from the last few samples: y[n] = b_0x[n] + b_1x[n−1] + b_2x[n−2] − a_1y[n−1] − a_2y[n−2]. Its transfer function is a rational function H(z) (the Z-transform). The filter is stable when all poles lie inside the unit circle. The impulse response comes from the inverse Z-transform by residues: h[n] = Σ Res[H(z)z^{n-1}]. The frequency response is H on the unit circle, z = e^(iΩ). The model's equation:
  $$H(z) = \frac{1}{1 - 1.6z^{-1} + 0.81z^{-2}} = \frac{z^2}{z^2 - 1.6z + 0.81}$$
- The filter's difference equation:
  $y[n] = x[n] + 1.6y[n-1] - 0.81y[n-2]$
- Transfer function's poles
  $z^2 - 1.6z + 0.81 = 0: \quad p = 0.8 - 0.4123i,\ 0.8 + 0.41$
- Poles inside the unit circle - the filter is stable:
  $|p|_{max} = 0.9 < 1$
- Inverse Z-transform by residues:
  $h[n] = \sum \mathrm{Res}\left[H(z)z^{n-1}\right]$
- Complex conjugate poles give a damped oscillation
  $n \geq 1: \quad h[n] = 2|R|\rho^n \cos(n\vartheta + \arg R), \quad \rho = 0.9,\ \vartheta$
- Value of the response: $h[10] = -0.6596$

Graph

Poles and zeros | Chart



The transfer function's poles (×) and zeros (○) in the z plane with the unit circle. The filter is stable when all poles are inside.

An oscillator's quality factor from its poles' position.

Input: `model(resonancecurve; system=mechanical; find=Q[one]; m=1[kg]; k=100[N_m];`

```
b=2[kg_s])
```

Result

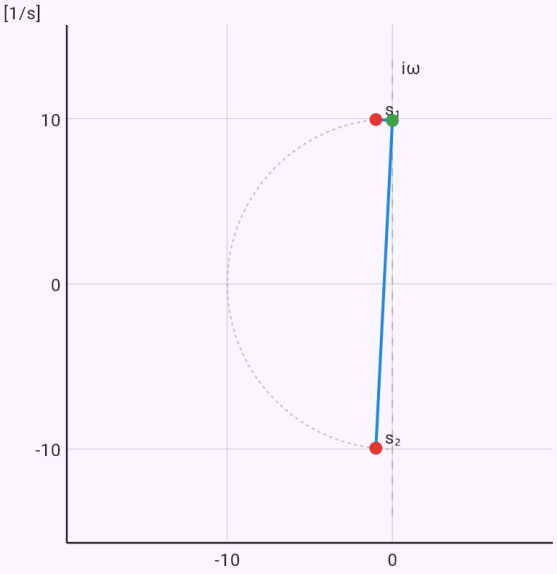
$$Q \approx 5$$

Steps

- Find Q; given: $m = 1\,\mathrm{kg}, \quad k = 100\,\mathrm{N/m}, \quad b = 2\,\mathrm{kg/s}$
- A mass on a damped spring driven by a harmonic force. X/X$_0$ is the amplitude against the static deflection under the same force. The model's equation:
  $$H(s) = \frac{1}{ms^2 + bs + k}$$
- Transfer function's poles
  $$s_{1,2} = -\delta \pm i\sqrt{\omega_0^2 - \delta^2}, \quad \omega_0 = 10\,\mathrm{rad/s}, \quad \delta = 1\,\mathrm{s}^{-1}$$
- Amplitude from the distances to the poles
  $$|H(i\omega)| = \frac{1}{m|i\omega - s_1||i\omega - s_2|}$$
- Resonance curve's peak
  $$\omega_r = \sqrt{\omega_0^2 - 2\delta^2} = 9.899\,\mathrm{rad/s}, \quad f_r = 1.576\,\mathrm{Hz}$$
- Quality factor and bandwidth
  $$Q = \frac{\omega_0}{2\delta} = 5, \quad \Delta f = \frac{2\delta}{2\pi} = \frac{f_0}{Q} = 0.3183\,\mathrm{Hz}$$
- Near the resonance the curve has a Lorentzian shape
  $$\omega \approx \omega_0: \quad |H|^2 \approx \frac{1}{4m^2\omega_0^2} \cdot \frac{1}{(\omega - \omega_0)^2 + \delta^2}$$
- That's why machines must not run long at their frame's natural frequency and bridges are checked for excitation by pedestrians and wind.

Graph

Poles | Chart



Poles in the complex plane s (×); a stable system has them left of the imaginary axis.

The refractive index below a Lorentz oscillator's resonance, checked by the Kramers–Kronig relation.

Input: `model(lorentz; mode=medium; find=n[one]; E0=2[eV]; G=0.1[eV]; Ep=1[eV]; E=1.5[eV])`

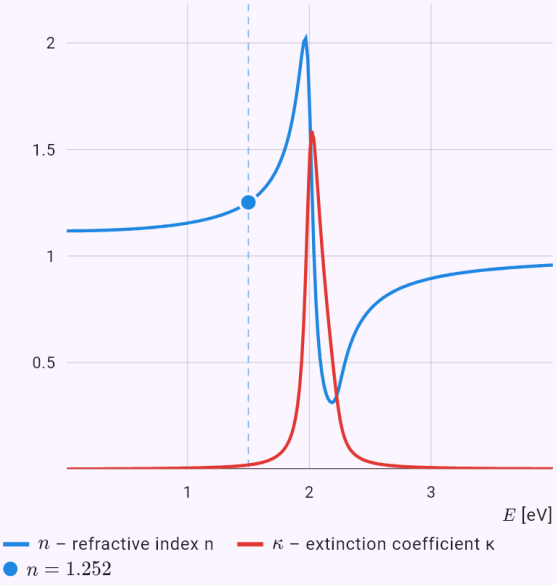
Result

$$n \approx 1.252$$

Steps

- Find n; given
  $E_0 = 2\,\mathrm{eV}, \quad \Gamma = 0.1\,\mathrm{eV}, \quad E_p = 1\,\mathrm{eV}, \quad E = 1.5\,\mathrm{eV}$
- The refractive index n and the extinction coefficient κ from n + iκ = √(1 + χ). The photon energy E, resonance E_0, width Γ and oscillator strength E_p in electronvolts.
  The model's equation: $\chi(E) = \dfrac{E_p^2}{E_0^2 - E^2 - i\Gamma E}$
- The susceptibility's poles lie in the lower half-plane
  $$E_{1,2} = \pm\sqrt{E_0^2 - \frac{\Gamma^2}{4}} - \frac{i\Gamma}{2} = \pm 1.999 - 0.05i$$
- Causality: the Kramers–Kronig relation
  $$\mathrm{Im}\, E_{1,2} < 0 \Rightarrow \chi'(E) = \frac{2}{\pi}\,\mathrm{P}\int_0^\infty \frac{E'\chi''(E')}{E'^2 - E^2}\,dE'$$
- Susceptibility at the given energy:
  $\chi = 0.5673 + 0.04862i$
- Check: χ' from the integral over χ'' (numerically) and directly: $\chi'_{KK} = 0.5673, \quad \chi' = 0.5673$
- Complex refractive index:
  $n + i\kappa = \sqrt{1+\chi} = 1.252 + 0.01942i$
- Wavelength and absorption coefficient
  $$\lambda = \frac{hc}{E} = 826.6\,\mathrm{nm}, \quad \alpha = \frac{4\pi\kappa}{\lambda} = 295\,200\,\mathrm{m}^{-1}$$
- Below the resonance the refractive index grows with energy - normal dispersion, by which a prism splits light into colours.

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

The natural width of the sodium D line - about 10 MHz.

Input: `model(lorentz; mode=spectralline; find=dnu[MHz]; tau=16[ns])`

Result

$$\Delta\nu \approx 9.947\,\text{MHz}$$

Steps

- Find $\Delta\nu$; given: $\tau = 16\,\text{ns}$
- An excited atom radiates a wave decaying exponentially with the lifetime τ. Its Fourier transform (a residue at one pole) gives the Lorentzian line shape with the full width Δν = 1/(2πτ) - a spectral line's natural width. The model's equation: $E(t) = E_0 e^{-\frac{t}{2\tau}} e^{i\omega_0 t}\ (t \geq 0)$
- Fourier transform of the decaying wave

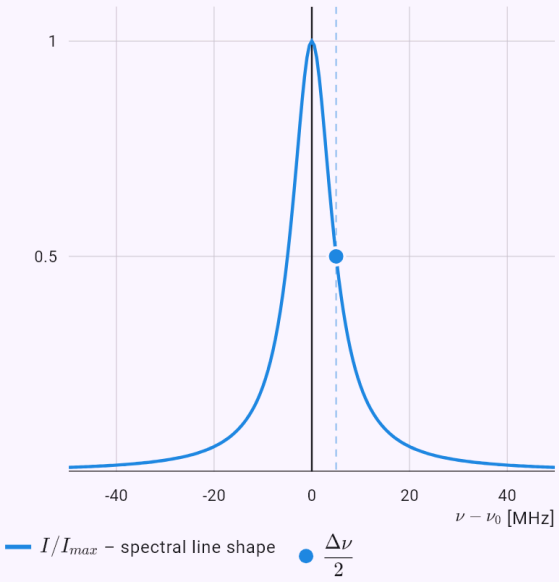
$$\hat{E}(\omega) = \int_0^\infty E(t) e^{-i\omega t} dt = \frac{E_0}{\frac{1}{2\tau} + i(\omega - \omega_0)}$$

- Lorentzian line shape and its width

$$I(\omega) \propto \left|\hat{E}\right|^2 = \frac{E_0^2}{(\omega - \omega_0)^2 + \frac{1}{4\tau^2}}: \quad \Delta\omega = \frac{1}{\tau},\ \Delta\nu = \frac{1}{2\pi}$$

- Natural line width: $\Delta\nu = \frac{1}{2\pi \cdot 16\,\text{ns}} = 9.947\,\text{MHz}$
- The sodium D line (τ ≈ 16 ns) has a natural width of about 10 MHz. Real lines in a gas are further broadened by the Doppler effect and collisions.

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

Matching a 100 Ω load to a 50 Ω line by a short-circuited stub.

Input: `model(smith; find=stub[m]; Z0=50[ohm]; R=100[ohm]; X=0[ohm]; f=100[MHz])`

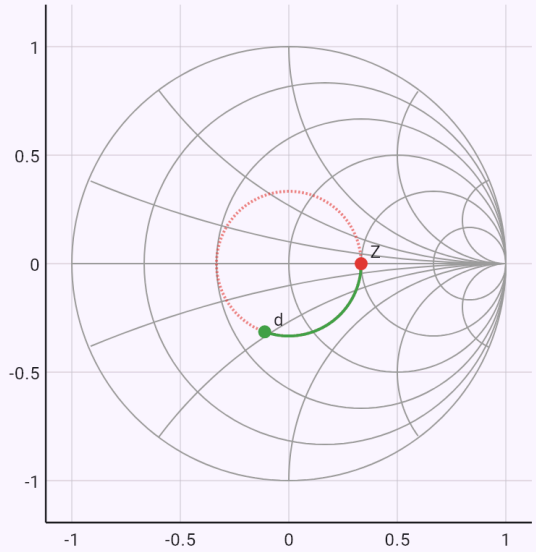
Result

$d \approx 0.4558\,\text{m}$
$l_s \approx 0.4558\,\text{m}$

Steps

- Find d; given
  $Z_0 = 50\,\Omega, \quad R_L = 100\,\Omega, \quad X_L = 0\,\Omega, \quad f = 100\,\text{MH}$
- The reflection coefficient Γ = (Z − Z_0)/(Z + Z_0) on a line of characteristic impedance Z_0 is a Möbius transformation. It maps the right half-plane of impedances onto the unit disc - the Smith chart. Moving along the line towards the generator is a clockwise turn along the circle |Γ| = const, half a wavelength is a full turn. Hence the standing wave ratio, a line's input impedance, a quarter-wave transformer and stub matching. The model's equation
  $$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{z-1}{z+1}, \quad z = \frac{Z_L}{Z_0}$$
- Load's normalized impedance: $z = \frac{100}{50} = 2$
- Reflection coefficient:
  $\Gamma = 0.3333, \quad |\Gamma| = 0.3333, \quad \arg\Gamma = 0^\circ$
- Standing wave ratio and return loss
  $$PSV = \frac{1+|\Gamma|}{1-|\Gamma|} = 2, \quad RL = -20\log|\Gamma| = 9.542\,\text{dB}$$
- Wavelength on the line: $\lambda = \frac{v_f c}{f} = 2.998\,\text{m}$
- Load's normalized admittance: $y = \frac{1}{z} = 0.5$
- The place where the admittance's real part is 1
  $\text{Re}\, y(d) = 1: \quad d = 0.152\lambda = 0.4558\,\text{m}, \quad y(d) = 1 +$
- Length of the short-circuited stub cancelling the susceptance
  $y_s = -j\cot\beta l_s = -j\,0.7071: \quad l_s = \frac{\lambda}{2\pi}\,\text{arccot}\,(0.707$
- A stub is a piece of line ended by a short circuit, connected in parallel. At the chosen frequency the line before it then sees no reflection.

Graph



Circles of constant resistance and arcs of constant reactance (r, x = 0.2; 0.5; 1; 2; 5). The dashed circle has the radius |Γ| (constant SWR), the coloured arc is the move along the line towards the generator.

Known limitations

- The flow is ideal: no viscosity, separation or wake. The model gives neither the drag nor the behaviour at large angles of attack.
- The conformal maps are for the listed shapes only; a general shape of conductors or an airfoil (the Schwarz–Christoffel transformation) isn't supported yet.
- The loop's transfer function has a fixed form with three lags and at most two integrators; zeros in the numerator and an unstable open loop aren't supported yet.
- The digital filter is of second order, the analog filters low-pass only.

Generated from the Mathtrace app