Mathtrace – user guide

III. Calculus

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25. Limits

Mathtrace computes the limit of a function of one variable at a point, at infinity, and one-sided limits, and proceeds as at school: first it tries to substitute into the function. The limit of a sequence (n → ∞) works the same way – the form 1^∞ too, or a sum of powers with the fastest growing term taken out.

If substitution gives an indeterminate form (0/0, ∞/∞, …), it chooses a transformation by the form of the function – factoring and cancelling, multiplying by the conjugate, dividing by the highest power, a known limit (sin x / x) or L'Hôpital's rule, repeatedly if needed. A power with the variable in both the base and the exponent (the forms 0^0, ∞^0, 1^∞) is written as e^(g·ln f) and the limit is taken in the exponent; where the function is defined on one side of the point only (x^(ln(x+1)) for x > 0 only), it computes the limit from that side and says so. The chip “L'Hôpital's rule” chooses the rule straight away; where substitution gives no indeterminate form, the steps explain why it cannot be used.

Other letters in the problem (k in (1 + k/x)^x) are parameters – constants it works with in general; when the result holds only for positive values of the parameter, it says so. The letter can be in the point of the limit too: lim (x² − a²)/(x − a) as x → a = 2a. A limit with a parameter gets no graph.

An infinite limit, or a limit that does not exist (the one-sided limits differ), is never passed off as a number – the app says so in words and shows the one-sided limits. The result comes with a graph of the function near the point with the limit value marked.

For the limit of a sequence the graph shows the points a_1, a_2, a_3, … instead of a continuous curve.

How to enter it

`lim(x->2) (x^2-4)/(x-2)`	A typed limit: after the word lim, in brackets the variable, the arrow -> and the point, then the function.
`lim(x->oo) (3x^2+1)/(x^2-5)`	A limit at infinity: infinity is written oo (minus infinity -oo).
`lim(x->0+) 1/x`	A one-sided limit: + (from the right) or − (from the left) after the point.
`\lim_{x \to 0} \frac{\sin x}{x}`	From a photo, or in LaTeX as the camera recognizes it.
`lim(n->oo) ((n+2)/(n+1))^(2n+3)`	The limit of a sequence at infinity.

Examples

The indeterminate form 0/0 – we factor the numerator and cancel.

Input: `lim(x->2) (x^2-4)/(x-2)`

Result

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4$$

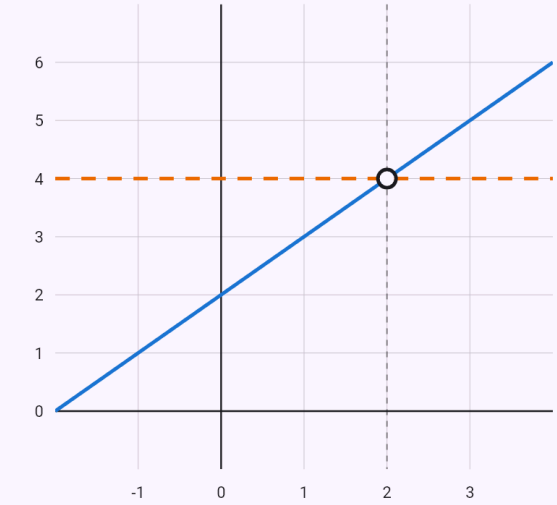
Steps

- Substituting gives the indeterminate form $\frac{0}{0}$ – the expression has to be rewritten.
- Rewrite and cancel

  $$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \to 2} (x + 2)$$

- Substitute: $\lim_{x \to 2} (x + 2) = 2 + 2 = 4$

Graph



$y = \frac{x^2 - 4}{x - 2}$

$y = 4$

Limit:

○ $(2, 4)$

Hollow dot – the value the function approaches (it need not be defined at the point itself); dashed – the limit.

A limit at infinity – numerator and denominator divided by the highest power.

Input: `lim(x->oo) (3x^2+1)/(x^2-5)`

Result

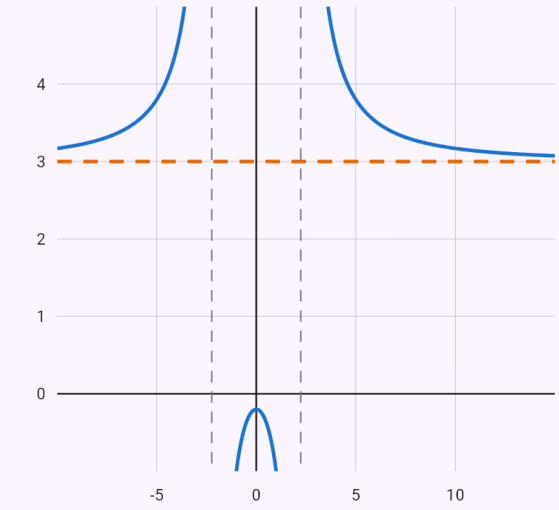
$$\lim_{x\to\infty}\frac{3x^2+1}{x^2-5}=3$$

Steps

- Substituting gives the indeterminate form $\frac{\infty}{\infty}$ – the expression has to be rewritten.
- Divide the numerator and the denominator by x^2 – terms like c/x^n tend to zero

$$\lim_{x\to\infty}\frac{3x^2+1}{x^2-5}=\lim_{x\to\infty}\frac{3+\frac{1}{x^2}}{1-\frac{5}{x^2}}=3$$

Graph



$y=\frac{3x^2+1}{x^2-5}$

$y=3$

Hollow dot – the value the function approaches (it need not be defined at the point itself); dashed – the limit.

The known limit sin x / x.

Input: `lim(x->0) sin(3x)/x`

Result

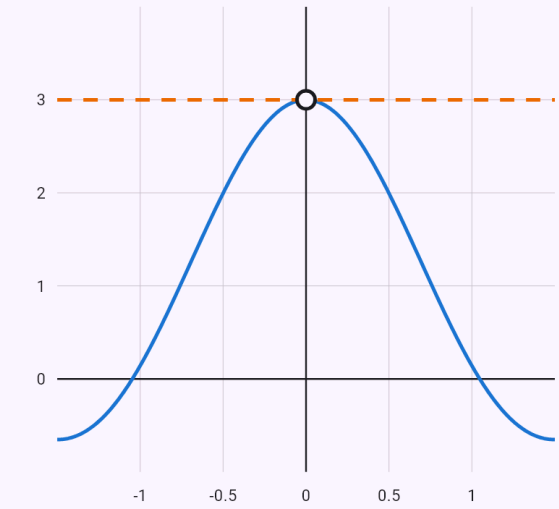
$$\lim_{x\to0}\frac{\sin(3x)}{x}=3$$

Steps

- Substituting gives the indeterminate form $\frac{0}{0}$ – the expression has to be rewritten.
- Use the known limit $\lim_{x\to0}\frac{\sin x}{x}=1$

$$\lim_{x\to0}\frac{\sin(3x)}{x}=\lim_{x\to0}\frac{\sin(3x)}{3x}\cdot3=1\cdot3=3$$

Graph



$y=\frac{\sin(3x)}{x}$

$y=3$

Limit:

○ $(0,3)$

Hollow dot – the value the function approaches (it need not be defined at the point itself); dashed – the limit.

L'Hôpital's rule, applied twice in a row.

Input: `lim(x->0) (e^x-1-x)/x^2`

Result

$$\lim_{x\to 0}\frac{-x+e^x-1}{x^2}=\frac{1}{2}$$

Steps

- Substituting gives the indeterminate form $\frac{0}{0}$ – the expression has to be rewritten.
- L'Hôpital's rule – differentiate the numerator and the denominator

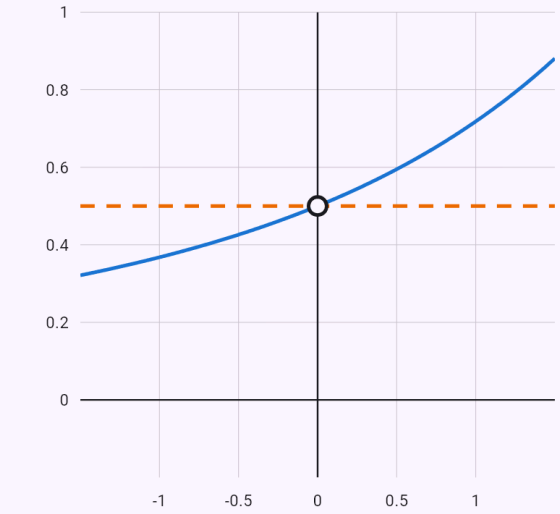
$$\lim_{x\to 0}\frac{-x+e^x-1}{x^2}=\lim_{x\to 0}\frac{(-x+e^x-1)'}{\left(x^2\right)'}=\lim_{x\to 0}\frac{e^x-1}{2x}$$

- Substituting gives the indeterminate form $\frac{0}{0}$ – the expression has to be rewritten.
- L'Hôpital's rule – differentiate the numerator and the denominator

$$\lim_{x\to 0}\frac{e^x-1}{2x}=\lim_{x\to 0}\frac{(e^x-1)'}{(2x)'}=\lim_{x\to 0}\frac{e^x}{2}$$

- Substitute: $\lim_{x\to 0}\frac{e^x}{2}=\frac{e^0}{2}=\frac{1}{2}$

Graph



$y=\frac{e^x-1-x}{x^2}$

$y=\frac{1}{2}$

Limit:

$\circ\ \left(0,\frac{1}{2}\right)$

Hollow dot – the value the function approaches (it need not be defined at the point itself); dashed – the limit.

The indeterminate form 0^0 – the power written as e^(g·ln f); the function is defined for x > 0 only, so the

limit from the right.

Input: `lim(x->0) x^(ln(x+1))`

Result

$$\lim_{x\to 0^+} x^{\ln(x+1)} = 1$$

Steps

- The function is not defined to the left of 0, so we compute the limit from the right.
- Substituting gives the indeterminate form 0^0 – the expression has to be rewritten.
- Both the base and the exponent contain the variable – write the power as e raised to the exponent times the logarithm of the base and take the limit in the exponent

$$\lim_{x\to 0^+} x^{\ln(x+1)} = \lim_{x\to 0^+} e^{\ln x\cdot\ln(x+1)}$$

- Substituting gives the indeterminate form $0\cdot\infty$ – the expression has to be rewritten.
- Rewrite the product as a quotient

$$\lim_{x\to 0^+} \ln x\cdot\ln(x+1) = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\ln(x+1)}}$$

- Substituting gives the indeterminate form $\frac{\infty}{\infty}$ – the expression has to be rewritten.
- L'Hôpital's rule – differentiate the numerator and the denominator

$$\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\ln(x+1)}} = \lim_{x\to 0^+} \frac{(\ln x)'}{\left(\frac{1}{\ln(x+1)}\right)'} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{(x+1)(\ln(x+1))^2}}$$

- Substituting gives the indeterminate form $\frac{0}{0}$ – the expression has to be rewritten.
- The factor $x+1$ has the value 1 at the point – take it out of the limit

$$\lim_{x\to 0^+} -\frac{(x+1)(\ln(x+1))^2}{x} = 1\cdot\lim_{x\to 0^+} -\frac{(\ln(x+1))^2}{x}$$

- Substituting gives the indeterminate form $\frac{0}{0}$ – the expression has to be rewritten.
- L'Hôpital's rule – differentiate the numerator and the denominator

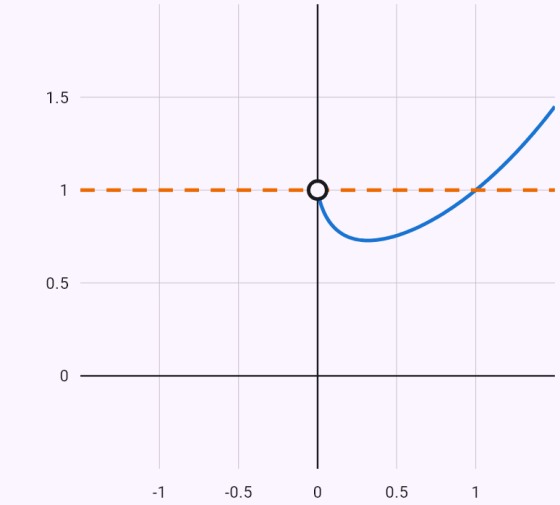
$$\lim_{x\to 0^+} \frac{-(\ln(x+1))^2}{x} = \lim_{x\to 0^+} \frac{\left(-(\ln(x+1))^2\right)'}{(x)'} = \lim_{x\to 0^+}$$

- Substitute

$$\lim_{x\to 0^+} -\frac{2\ln(x+1)}{x+1} = -\frac{2\ln(1+0)}{1+0} = 0$$

- Put the limit of the exponent back:
$\lim_{x\to 0^+} x^{\ln(x+1)} = e^0 = 1$

Graph



$y = x^{\ln(x+1)}$
$y = 1$

Limit:
○ $(0, 1)$

Hollow dot – the value the function approaches (it need not be defined at the point itself); dashed – the limit.

The one-sided limits differ – the limit does not exist.

Input: `lim(x->0) 1/x`

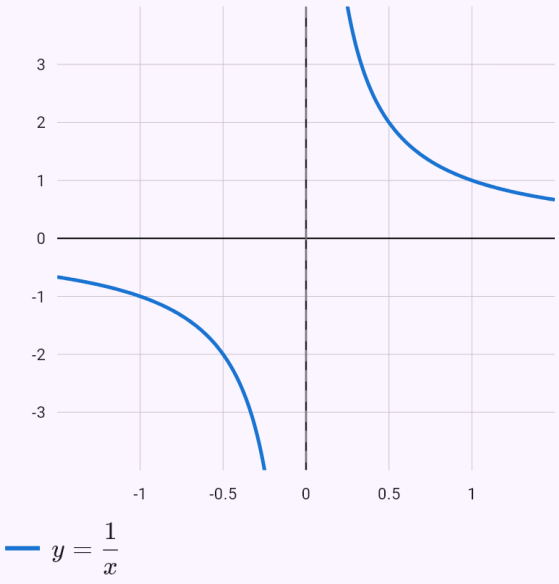
Result

The limit does not exist – the one-sided limits differ.

Steps

- Substituting gives a non-zero number over zero – the limit is infinite, its sign from the left and from the right: $\lim_{x\to 0} \frac{1}{x} \to \frac{1}{0}$
- One-sided limits: $\lim_{x\to 0^-} \frac{1}{x} = -\infty$, $\lim_{x\to 0^+} \frac{1}{x} = \infty$

Graph



The limit of a sequence – take out the fastest growing term 9^n.

Input: `lim(n->oo) (6^n+7^n+9^n)^(1/(2n))`

Result

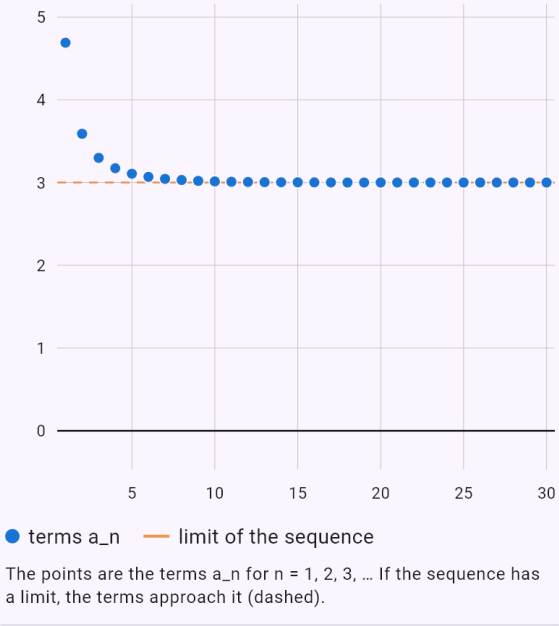
$$\lim_{n\to\infty} (6^n + 7^n + 9^n)^{\frac{1}{2n}} = 3$$

Steps

- Take out the fastest growing term 9^n – the others divided by it tend to zero

$$\lim_{n\to\infty} (6^n + 7^n + 9^n)^{\frac{1}{2n}} = \lim_{n\to\infty} \sqrt{9} \cdot \left(\frac{2^n}{3^n} + \frac{7^n}{9^n} + 1\right)^{\frac{1}{2n}} =$$

Graph



The points are the terms a_n for n = 1, 2, 3, … If the sequence has a limit, the terms approach it (dashed).

A limit with a parameter k – the known limit (1 + 1/x)^x = e gives e^k.

Input: `lim(x->oo) (1+k/x)^x`

Result

$$\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^{x}=e^{k}$$

Steps

- Letters other than the variable are parameters – fixed numbers we treat as constants: k
- Substituting gives the indeterminate form 1^{∞} – the expression has to be rewritten.
- We write the base as $1+\frac{1}{t}$ with t = $\frac{x}{k}$, and the power as $\left(1+\frac{1}{t}\right)^{t}$ raised to k. The expression in square brackets tends to e – the known limit $\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{t}=e$. The limit of the exponent remains

$$\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^{x}=\lim_{x\to\infty}\left[\left(1+\frac{1}{\frac{x}{k}}\right)^{\frac{x}{k}}\right]^{k}=e^{k}$$

26. Derivatives

The app differentiates by the rules – the sum, constant multiple, product, quotient and chain rules – and says at every step which rule it used. It also handles higher derivatives and the value of the derivative at a point. Basic functions (x^2, sin x, e^x, …) can also be differentiated from the definition, as the limit of the difference quotient. The result comes with a graph of the function and its derivative.

For a function given implicitly by an equation in x and y ($x^3 + y^3 = 3xy$) it differentiates both sides by x (y as a function of x) and expresses y′, or uses the formula y′ = −F_x / F_y; it also finds y″ and, at a given point, the tangent and the normal, and draws the curve.

At a singular point (the curve crosses itself or has a cusp) it finds the tangents from the equation's lowest-degree terms – for the folium at the origin x = 0 and y = 0.

The differential df = f′(x) dx, at a point with a step ("differential √x, x0 = 4, dx = 0.1") with the approximate value $f(x_0 + h) \approx f(x_0) + f'(x_0)\cdot h$. "approx √4.1" picks the point $x_0 = 4$, where the value is known exactly, and compares the estimate with the true value.

How to enter it

`(x^2 sin x)'`	A derivative with a prime after the bracket.
`d/dx (e^x / x)`	The d/dx notation, also from a photo (\frac{d}{dx}).
`(x^3 ln x)''`	Higher derivatives with more primes, or with the order at the "Differentiate" chip.
`(x^2 + 1)'|_{x=2}`	The derivative at a point, or fill in the point at the chip.
`x^2 sin x`	An expression with the "Differentiate" chip – there you also choose the order, the point and the method (by the rules / from the definition).
`x^3 + y^3 = 3xy`	An equation in x and y – the "Implicit derivative" chip (order 1 or 2, the method, a point $[x_0; y_0]$) or "Graph of the curve"; written diff(x^3+y^3=3xy, x).
`differential √x, x0 = 4, dx = 0.1`	The differential at a point with a step (without a point in general: differential x^2 sin x).

`approx √4.1`	An approximate value by the differential.

Examples

The product rule.

Input: `(x^2 sin x)'`

Result

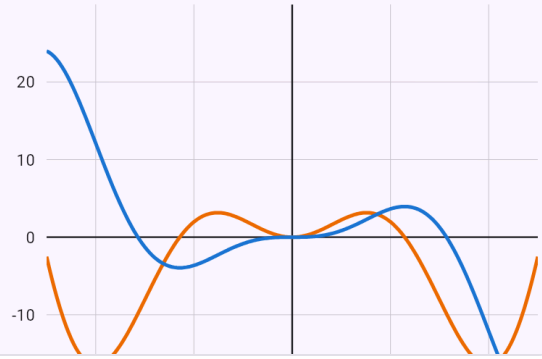
$$y' = 2x \sin x + x^2 \cos x$$

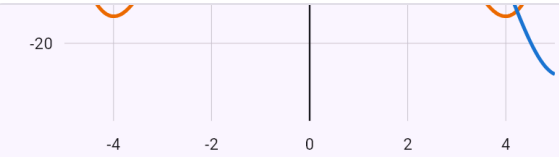
Steps

- Product rule (u·v)' = u'·v + u·v'

$$\left(x^2 \sin x\right)' = \left(x^2\right)' \cdot \sin x + x^2 \cdot (\sin x)' = 2x \sin x + x^2$$

Graph





$y = x^2 \sin x$

$y' = 2x \sin x + x^2 \cos x$

Where the derivative is positive, the function increases; where it is negative, it decreases. At the derivative's zeros the function has a horizontal tangent.

The quotient rule.

Input: `d/dx (e^x / x)`

Result

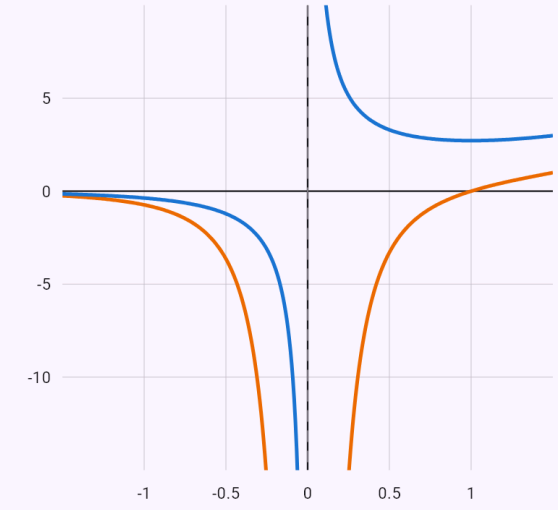
$$y' = \frac{xe^x - e^x}{x^2}$$

Steps

- Quotient rule (u/v)' = (u'·v − u·v')/v²

$$\left(\frac{e^x}{x}\right)' = \frac{(e^x)' \cdot x - e^x \cdot (x)'}{x^2} = \frac{xe^x - e^x}{x^2}$$

Graph



$y = \frac{e^x}{x}$

$y' = \frac{xe^x - e^x}{x^2}$

Where the derivative is positive, the function increases; where it is negative, it decreases. At the derivative's zeros the function has a horizontal tangent.

The second derivative.

Input: `(x^3 ln x)''`

Result

$$y'' = 5x + 6x \ln x$$

Steps

- Derivative number 1: $y' = \left(x^3 \ln x\right)'$
- Product rule (u·v)' = u'·v + u·v'

$$\left(x^3 \ln x\right)' = \left(x^3\right)' \cdot \ln x + x^3 \cdot (\ln x)' = 3x^2 \ln x + x^2$$

- Derivative number 2: $y'' = \left(x^2 + 3x^2 \ln x\right)'$
- The derivative of a sum is the sum of the derivatives:

$$\left(x^2 + 3x^2 \ln x\right)' = \left(x^2\right)' + \left(3x^2 \ln x\right)'$$

- Take the constant out of the derivative:

$$\left(3x^2 \ln x\right)' = 3 \cdot \left(x^2 \ln x\right)'$$

- Product rule (u·v)' = u'·v + u·v'

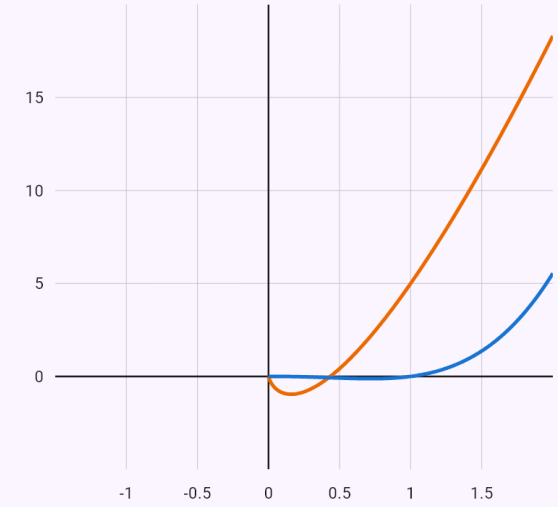
$$\left(x^2 \ln x\right)' = \left(x^2\right)' \cdot \ln x + x^2 \cdot (\ln x)' = 2x \ln x + x$$

- Substitute:

$$\left(3x^2 \ln x\right)' = 3 \cdot (x + 2x \ln x) = 3(x + 2x \ln x)$$

- Substitute: $\left(x^2 + 3x^2 \ln x\right)' = 2x + 3(x + 2x \ln x)$
- Simplify: $y'' = 5x + 6x \ln x$

Graph



$y = x^3 \ln x$

$y'' = 5x + 6x \ln x$

Where the second derivative is positive, the function is convex (bends upwards); where it is negative, it is concave (bends downwards).

The derivative from the definition, as a limit.

Input: x^2 Differentiate → From the definition

Result

$$y' = 2x$$

Steps

- The derivative from the definition

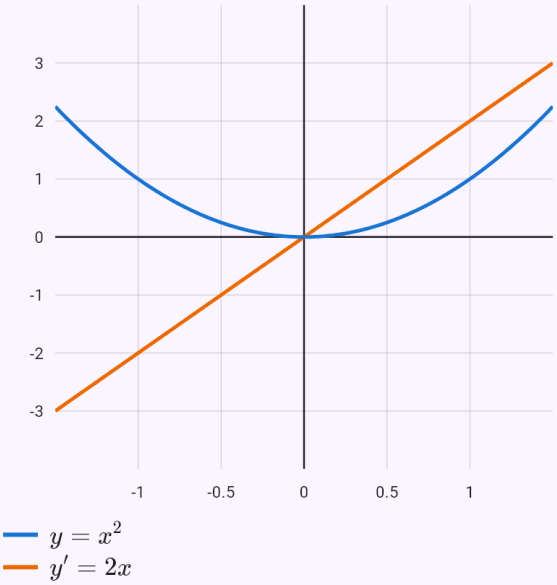
$$y' = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

- Substituting gives the indeterminate form $\frac{0}{0}$ – the expression has to be rewritten.
- Rewrite and cancel

$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{h(2x+h)}{h} = \lim_{h \to 0} (2x+h)$$

- Substitute: $\lim_{h \to 0} (2x+h) = 2x$

Graph



Where the derivative is positive, the function increases; where it is negative, it decreases. At the derivative's zeros the function has a horizontal tangent.

An implicit derivative – the folium of Descartes, the tangent and the normal at [3/2; 3/2], the second

derivative.

Input: `x^3 + y^3 = 3xy` Implicit derivative, order 2, point [3/2; 3/2]

Result

$$y' = -\frac{x^2 - y}{y^2 - x}$$

$$y'' = -\frac{2xy}{\left(y^2 - x\right)^3}$$

$$y'(\frac{3}{2}) = -1$$

t: $y = 3 - x$

n: $y = x$

$$y''(\frac{3}{2}) = -\frac{32}{3}$$

Conditions: $y^2 - x \neq 0$

Steps

- The function y = y(x) is given by the equation $x^3 + y^3 = 3xy$ – we differentiate both sides by x, taking y as a function of x (the chain rule: e.g. $(y^3)' = 3y^2 \cdot y'$).
- Differentiating both sides: $\left(x^3 + y^3\right)' = (3xy)' \Rightarrow 3y^2y' + 3x^2 = 3xy' + 3y$
- The terms with y' to the left, the rest to the right, y' factored out: $y'\left(3y^2 - 3x\right) = -3x^2 + 3y$
- Expressing y': $y' = -\frac{x^2 - y}{y^2 - x}$
- The second derivative – y' differentiated by x again (y and y' are functions of x)

  $$y'' = \left(-\frac{x^2 - y}{y^2 - x}\right)' = \frac{2x^2yy' - 2xy^2 - y^2y' + x^2 - xy'}{y^4 - 2xy^2 + x^2}$$

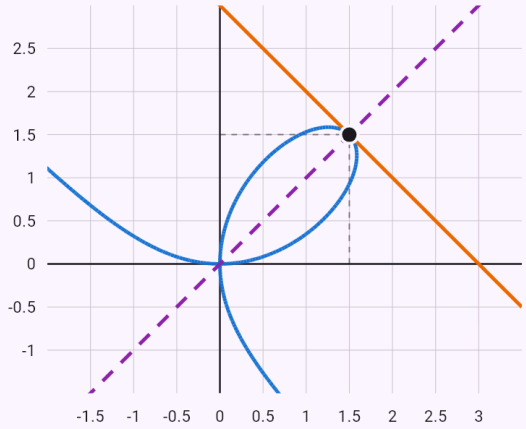
- Substituting y'

  $$y'' = \frac{2xy(x + y + 1)(x^2 - xy + y^2 - x - y + 1)}{\left(-y^2 + x\right)^3}$$

- On the curve $x^3 + y^3 - 3xy = 0$ holds – which simplifies it: $y'' = -\frac{2xy}{\left(y^2 - x\right)^3}$

- The point $[\frac{3}{2};\ \frac{3}{2}]$ lies on the curve: $x^3 + y^3 - 3xy|_{x=\frac{3}{2},\ y=\frac{3}{2}} = 0$
- The derivative at the point $[\frac{3}{2};\ \frac{3}{2}]$ – the tangent's slope: $y'(\frac{3}{2}) = -1$
- The tangent: $y - \frac{3}{2} = -\left(x - \frac{3}{2}\right) \Rightarrow y = 3 - x$
- The normal is perpendicular to the tangent, its slope is 1: $y - \frac{3}{2} = \left(x - \frac{3}{2}\right) \Rightarrow y = x$
- The second derivative at the point: $y''(\frac{3}{2}) = -\frac{32}{3}$

Graph



$x^3 + y^3 = 3xy$
t: $y = 3 - x$
n: $y = x$

The curve given by the equation. At the given point, the tangent (solid) and the normal (dashed).

The differential of √x at 4 with the step 0.1.

Input: `differential √x, x0 = 4, dx = 0.1`

Result

$$f(4.1) \approx \frac{81}{40} = 2.025$$

Steps

- The differential – the linear part of the function's increment: $\mathrm{d}f = f'(x)\mathrm{d}x = \frac{1}{2\sqrt{x}}\,\mathrm{d}x$
- The differential at x_0 for the step h
  $\mathrm{d}f(4) = f'(4)\cdot h = \frac{1}{4}\cdot 0.1 = \frac{1}{40} = 0.025$
- We replace the increment by the differential. The exact value is 2.0248457, the estimate's error 0.00015
  $f(4.1) \approx f(4) + \mathrm{d}f(4) = 2 + \frac{1}{40} = \frac{81}{40} = 2.025$

An approximate value of ∛27.1 by the differential.

Input: `approx \sqrt[3]{27.1}`

Result

$$f(27.1) \approx \frac{811}{270} = 3.0037$$

Steps

- We write the number as a value of the function $f(x) = \sqrt[3]{x}$ near $x_0 = 27$, where we know it exactly; the step is $h = 27.1 - 27 = 0.1$
- The differential – the linear part of the function's increment: $\mathrm{d}f = f'(x)\mathrm{d}x = \frac{1}{3\sqrt[3]{x^2}}\,\mathrm{d}x$
- The differential at x_0 for the step h
  $\mathrm{d}f(27) = f'(27)\cdot h = \frac{1}{27}\cdot 0.1 = \frac{1}{270} = 0.0037$
- We replace the increment by the differential. The exact value is 3.0036991, the estimate's error 0.0000046
  $f(27.1) \approx f(27) + \mathrm{d}f(27) = 3 + \frac{1}{270} = \frac{811}{270} = 3.0037$

27. Function analysis

The app analyzes a function in the usual order: domain, even or odd, intercepts, the sign of the function, limits at the ends of the domain and asymptotes, the first derivative – monotonicity and extrema (also by higher derivatives), the second derivative – concavity and inflection points, and finally the range. It sums everything up in an overview and draws a graph with the points marked.

For a trigonometric function the app finds the period, examines the function on one period and writes the results with the period (a maximum at x = π/2 + 2kπ); if only the derivative is periodic (x + 2 sin x), the extrema and monotonicity repeat with the period. Absolute values are removed by the signs on the intervals and the one-sided derivatives compared at the corners; where the derivative is infinite, a vertical tangent is shown. Nested absolute values and sums of absolute values are analyzed by pieces; where the function is constant on an interval, this is stated with its value (a minimum at every point of the interval).

Absolute (global) extrema on a closed interval: “extrema of x^3 - 3x on [-2, 3]” – the app finds the stationary points and the points without a derivative inside the interval and compares the values there and at the end points. Without an interval, “extrema …” analyzes the function. The inverse function (“inverse of f(x) = …”) comes from solving y = f(x) for x, checked by composing; the graph shows f, f^{-1} and the line y = x.

A parabola’s vertex (“vertex of the parabola y = x^2 - 4x + 7”, a bare y = … is enough) by completing the square – with the axis, the range and the shift of the graph of $y = ax^2$. Write a composite function with its definitions: “f(x) = x^2 + 1, g(x) = 2x - 3, f(g(x))”.

A graph by shifts, stretches and reflections: “graph transformations y = 2sin(3(x - 1)) + 1” – the app starts from the basic function (sin, cos, tan, cot, e^x, a^x, ln, $\log_a$, a root, a power, 1/x, |x|) and step by step compresses or stretches it along the x axis, shifts it right or left, stretches it along the y axis, reflects it and shifts it up or down; |f(x)| and f(|x|) too. The graph shows the basic and the resulting function.

How to enter it

`analysis y = x^3 - 3x`	With the word “analysis” before the function.
`x^3 - 3x`	A function with the “Analyze function” chip.
`extrema of x^3 - 3x on [-2, 3]`	Absolute extrema on a closed interval (also “max and min …, x ∈ [a, b]”).
`inverse of f(x) = (2x+1)/(x-3)`	The inverse function (also “f^{-1}, f(x) = …”).

Examples

A polynomial – extrema and an inflection point.

Input: `x^3 - 3x` Analyze function

Result

Domain: $D(f) = \mathbb{R}$

Range: $R(f) = \mathbb{R}$

Parity: odd

x-intercepts: $\left[-\sqrt{3};\ 0\right]$, $[0;\ 0]$, $\left[\sqrt{3};\ 0\right]$

Increasing: $(-\infty,\ -1]$, $[1,\ \infty)$

Decreasing: $[-1,\ 1]$

Local maximum: $[-1;\ 2]$

Local minimum: $[1;\ -2]$

Convex: $[0,\ \infty)$

Concave: $(-\infty,\ 0]$

Points of inflection: $[0;\ 0]$

Steps

- Domain: $D(f) = \mathbb{R}$
- $f(-x) = (-x)^3 - 3(-x) = -x^3 + 3x = -f(x)$ – the function is odd (its graph is symmetric about the origin).
- Intersections with the x-axis:
  $f(x) = 0 \ \Rightarrow\ x = -\sqrt{3},\ x = 0,\ x = \sqrt{3}$, i.e.
  $\left[-\sqrt{3};\ 0\right]$, $[0;\ 0]$, $\left[\sqrt{3};\ 0\right]$
- Intersection with the y-axis: $f(0) = 0$, i.e. $[0;\ 0]$
- The sign of the function between its zeros and the points outside the domain:

	$(-\infty,\ -\sqrt{3})$	$(-\sqrt{3},\ 0)$	$(0,\ \sqrt{3})$	$(\sqrt{3},\ \infty)$
$f(x)$	$-$	$+$	$-$	$+$

  The function is positive for $x \in \left(-\sqrt{3},\ 0\right) \cup \left(\sqrt{3},\ \infty\right)$ and negative for $x \in \left(-\infty,\ -\sqrt{3}\right) \cup \left(0,\ \sqrt{3}\right)$
- The limits at the ends of the domain
  $\lim\limits_{x \to -\infty} f(x) = -\infty$
  $\lim\limits_{x \to \infty} f(x) = \infty$
- The function has no vertical asymptotes.
- No asymptote at ±∞: $k = \lim\limits_{x \to \pm\infty} \dfrac{f(x)}{x} = \infty$
- The first derivative: $f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$
- Stationary points: $f'(x) = 0 \ \Rightarrow\ x = -1,\ x = 1$
- The sign of the first derivative – where it's positive the function increases, where negative it decreases:

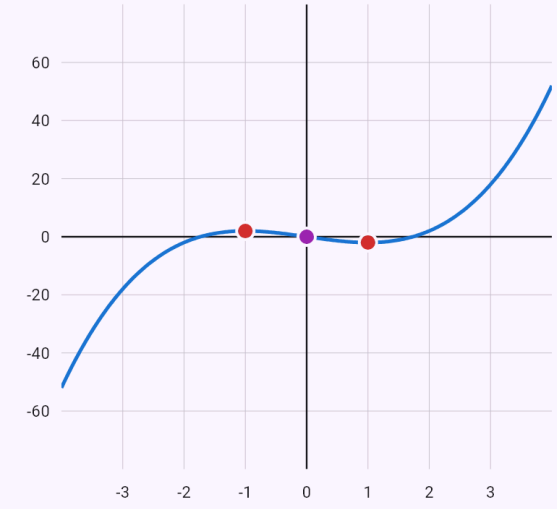
	$(-\infty,\ -1)$	$(-1,\ 1)$	$(1,\ \infty)$
$f'(x)$	$+$	$-$	$+$
$f(x)$	↗	↘	↗

  The function is increasing on $(-\infty,\ -1]$, $[1,\ \infty)$ and decreasing on $[-1,\ 1]$
- At $x = -1$: $f''(-1) = -6 < 0$ – a local maximum.
- At $x = 1$: $f''(1) = 6 > 0$ – a local minimum.
- Local maximum $[-1;\ 2]$, local minimum $[1;\ -2]$
- The second derivative: $f''(x) = 6x$
- Candidates for points of inflection: $f''(x) = 0 \ \Rightarrow\ x = 0$
- The sign of the second derivative – where it's positive the function is convex, where negative concave:

	$(-\infty,\ 0)$	$(0,\ \infty)$
$f''(x)$	$-$	$+$
$f(x)$	∩	∪

  The function is convex on $[0,\ \infty)$ and concave on $(-\infty,\ 0]$
- At $x = 0$: $f'''(0) = 6 > 0$ – the first non-zero derivative (from the third on) is of an odd order: a point of inflection.
- Points of inflection: $[0;\ 0]$
- Range – from the monotonicity, extrema and limits: $R(f) = \mathbb{R}$

Graph



$y = x^3 - 3x$

Notable points:

- maximum $(-1, 2)$
- inflection $(0, 0)$
- minimum $(1, -2)$

Local extrema in red, points of inflection in purple, a hollow circle – a point where the function isn't defined; asymptotes dashed.

A rational function – asymptotes.

Input: `x^2/(x - 1)` Analyze function

Result

Domain: $D(f) = \mathbb{R} \setminus \{1\}$

Range: $R(f) = (-\infty,\, 0] \cup [4,\, \infty)$

Parity: neither even nor odd

x-intercepts: $[0;\ 0]$

Increasing: $(-\infty,\, 0],\ [2,\, \infty)$

Decreasing: $[0,\, 1),\ (1,\, 2]$

Local maximum: $[0;\ 0]$

Local minimum: $[2;\ 4]$

Convex: $(1,\, \infty)$

Concave: $(-\infty,\, 1)$

Asymptotes: $x = 1,\ y = x + 1$

Steps

- Domain – conditions $x \neq 1$: $D(f) = \mathbb{R} \setminus \{1\}$
- The domain $D(f) = \mathbb{R} \setminus \{1\}$ isn't symmetric about zero – the function is neither even nor odd.
- Intersections with the x-axis: $f(x) = 0 \ \Rightarrow\ x = 0$, i.e. $[0;\ 0]$
- Intersection with the y-axis: $f(0) = 0$, i.e. $[0;\ 0]$
- The sign of the function between its zeros and the points outside the domain:

	$(-\infty,\, 0)$	$(0,\, 1)$	$(1,\, \infty)$
$f(x)$	$-$	$-$	$+$

  The function is positive for $x \in (1,\, \infty)$ and negative for $x \in (-\infty,\, 0) \cup (0,\, 1)$

- The limits at the ends of the domain

$$\lim_{x \to -\infty} f(x) = -\infty$$
$$\lim_{x \to 1^-} f(x) = -\infty$$
$$\lim_{x \to 1^-} f(x) = \infty$$
$$\lim_{x \to \infty} f(x) = \infty$$

- Vertical asymptotes (an infinite limit there): $x = 1$
- Asymptote at ±∞: $k = \lim_{x \to \pm\infty} \frac{f(x)}{x} = 1$, $q = \lim_{x \to \pm\infty} (f(x) - x) = 1$ – the line $y = x + 1$
- The first derivative

$$f'(x) = \frac{x^2 - 2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}$$

- Stationary points: $f'(x) = 0 \ \Rightarrow\ x = 0,\ x = 2$
- The sign of the first derivative – where it's positive the function increases, where negative it decreases:

	$(-\infty,\, 0)$	$(0,\, 1)$	$(1,\, 2)$	$(2,\, \infty)$
$f'(x)$	$+$	$-$	$-$	$+$
$f(x)$	↗	↘	↘	↗

  The function is increasing on $(-\infty,\, 0],\ [2,\, \infty)$ and decreasing on $[0,\, 1),\ (1,\, 2]$

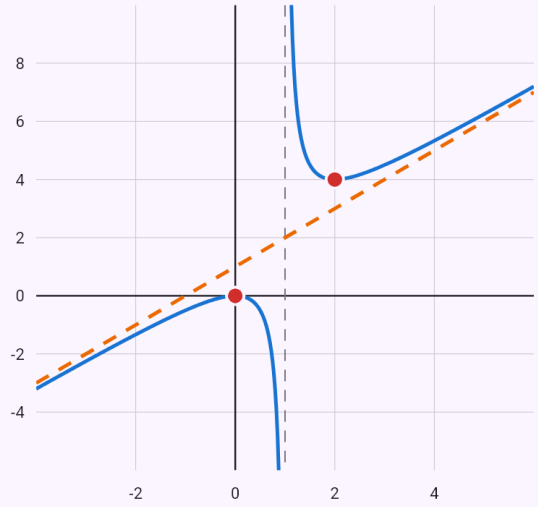
- At $x = 0$: $f''(0) = -2 < 0$ – a local maximum.
- At $x = 2$: $f''(2) = 2 > 0$ – a local minimum.
- Local maximum $[0;\ 0]$, local minimum $[2;\ 4]$
- The second derivative: $f''(x) = \frac{2}{(x-1)^3}$
- The equation $f''(x) = 0$ has no solution.
- The sign of the second derivative – where it's positive the function is convex, where negative concave:

	$(-\infty,\, 1)$	$(1,\, \infty)$
$f''(x)$	$-$	$+$
$f(x)$	∩	∪

  The function is convex on $(1,\, \infty)$ and concave on $(-\infty,\, 1)$

- The function has no points of inflection.
- Range – from the monotonicity, extrema and limits: $R(f) = (-\infty,\, 0] \cup [4,\, \infty)$

Graph



$y = \frac{x^2}{x-1}$

$y = x + 1$

Notable points:

● maximum $(0, 0)$ ● minimum $(2, 4)$

Local extrema in red, points of inflection in purple, a hollow circle – a point where the function isn't defined; asymptotes dashed.

A periodic function – everything on one period, the results with the period.

Input: `sin 2x + 2cos x` Analyze function

Result

Domain: $D(f) = \mathbb{R}$

Range: $R(f) = \left[-\frac{3\sqrt{3}}{2},\ \frac{3\sqrt{3}}{2}\right]$

Parity: neither even nor odd

Period: $p = 2\pi$

x-intercepts: $\left[\frac{\pi}{2} + k\pi;\ 0\right]$

Increasing: $\left[-\frac{7\pi}{6} + 2k\pi,\ \frac{\pi}{6} + 2k\pi\right]$

Decreasing: $\left[\frac{\pi}{6} + 2k\pi,\ \frac{5\pi}{6} + 2k\pi\right]$

Local maximum: $\left[\frac{\pi}{6} + 2k\pi;\ \frac{3\sqrt{3}}{2}\right]$

Local minimum: $\left[\frac{5\pi}{6} + 2k\pi;\ -\frac{3\sqrt{3}}{2}\right]$

Convex:
$\left[\frac{\pi}{2} + 2k\pi,\ \pi + \arcsin\left(\frac{1}{4}\right) + 2k\pi\right], \left[\frac{3\pi}{2} + 2k\pi,\right.$

Concave:
$\left[\pi + \arcsin\left(\frac{1}{4}\right) + 2k\pi,\ \frac{3\pi}{2} + 2k\pi\right], \left[-\arcsin\left(\right.\right.$

Points of inflection:
$\left[\frac{\pi}{2} + k\pi;\ 0\right], \left[\pi + \arcsin\left(\frac{1}{4}\right) + 2k\pi;\ -\frac{3\sqrt{15}}{8}\right],$

Steps

- Domain: $D(f) = \mathbb{R}$
- The function is periodic with the period 2π – we examine it on one period (from 0 to 2π) and repeat the results with the period, k ∈ $\mathbb{Z}$.
- $f(-x) = 2\cos x - \sin(2x)$ – neither f(x) nor -f(x): the function is neither even nor odd.
- Intersections with the x-axis: $f(x) = 0 \Rightarrow x = \frac{\pi}{2} + k\pi$, i.e. $\left[\frac{\pi}{2} + k\pi;\ 0\right]$
- Intersection with the y-axis: $f(0) = 2$, i.e. $[0;\ 2]$
- The sign of the function between its zeros and the points outside the domain:

$f(x)$	$(0,\ \frac{\pi}{2})$	$(\frac{\pi}{2},\ \frac{3\pi}{2})$	$(\frac{3\pi}{2},\ 2\pi)$
	+	−	+

The function is positive for $x \in \left(-\frac{\pi}{2} + 2k\pi,\ \frac{\pi}{2} + 2k\pi\right)$ and negative for $x \in \left(\frac{\pi}{2} + 2k\pi,\ \frac{3\pi}{2} + 2k\pi\right)$

- A periodic function (that isn't constant) has no limit at ±∞.
- The first derivative: $f'(x) = 2\cos(2x) - 2\sin x$
- Stationary points

$f'(x) = 0 \Rightarrow x = \frac{\pi}{6} + 2k\pi,\ x = \frac{5\pi}{6} + 2k\pi,\ x = \frac{3\pi}{2}$

- The sign of the first derivative – where it's positive the function increases, where negative it decreases:

$f'(x)$	$(0,\ \frac{\pi}{6})$	$(\frac{\pi}{6},\ \frac{5\pi}{6})$	$(\frac{5\pi}{6},\ \frac{3\pi}{2})$	$(\frac{3\pi}{2},\ 2\pi)$
	+	−	+	+
$f(x)$	↗	↘	↗	↗

The function is increasing on $\left[-\frac{7\pi}{6} + 2k\pi,\ \frac{\pi}{6} + 2k\pi\right]$ and decreasing on $\left[\frac{\pi}{6} + 2k\pi,\ \frac{5\pi}{6} + 2k\pi\right]$

- Local maximum $\left[\frac{\pi}{6} + 2k\pi;\ \frac{3\sqrt{3}}{2}\right]$, local minimum $\left[\frac{5\pi}{6} + 2k\pi;\ -\frac{3\sqrt{3}}{2}\right]$
- The second derivative: $f''(x) = -2\cos x - 4\sin(2x)$
- Candidates for points of inflection

$f''(x) = 0 \Rightarrow x = \frac{\pi}{2} + k\pi,\ x = \pi + \arcsin\left(\frac{1}{4}\right) + 2k$

- The sign of the second derivative – where it's positive

the function is convex, where negative concave:

	$(0,\ \frac{\pi}{2})$	$(\frac{\pi}{2},\ \pi+\arcsin\left(\frac{1}{4}\right))$	$(\pi+\arcsin\left(\frac{1}{4}\right),\ \frac{3}{2}$
$f''(x)$	$-$	$+$	$-$
$f(x)$	$\cap$	$\cup$	$\cap$

The function is convex on and concave on

$$\left[\frac{\pi}{2}+2k\pi,\ \pi+\arcsin\left(\frac{1}{4}\right)+2k\pi\right],\ \left[\frac{3\pi}{2}+2k\pi,\ 2\pi-\right.$$

$$\left[\pi+\arcsin\left(\frac{1}{4}\right)+2k\pi,\ \frac{3\pi}{2}+2k\pi\right],\ \left[-\arcsin\left(\frac{1}{4}\right)\right.$$

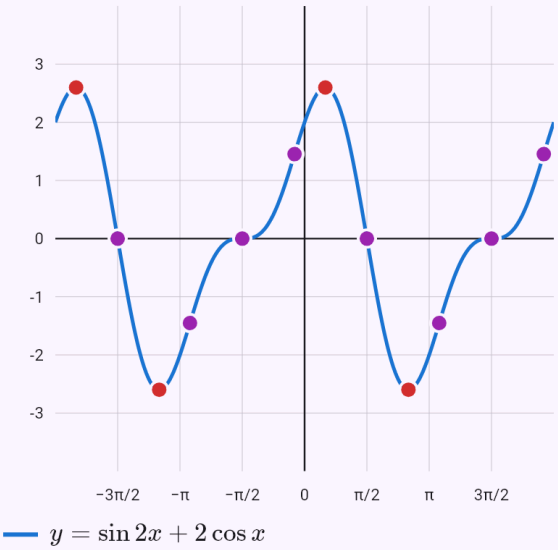
- Points of inflection

$$\left[\frac{\pi}{2}+k\pi;\ 0\right],\ \left[\pi+\arcsin\left(\frac{1}{4}\right)+2k\pi;\ -\frac{3\sqrt{15}}{8}\right],\ [2\pi$$

- Range – from the monotonicity, extrema and limits:

$$R(f)=\left[-\frac{3\sqrt{3}}{2},\ \frac{3\sqrt{3}}{2}\right]$$

Graph



$y=\sin 2x+2\cos x$

Notable points:

- maximum $\left(-\frac{11\pi}{6},\ \frac{3\sqrt{3}}{2}\right)$
- inflection $\left(-\frac{3\pi}{2},\ 0\right)$
- minimum $\left(-\frac{7\pi}{6},\ -\frac{3\sqrt{3}}{2}\right)$
- inflection $\left(-\pi+\arcsin\left(\frac{1}{4}\right),\ -\frac{3\sqrt{15}}{8}\right)$
- inflection $\left(-\frac{\pi}{2},\ 0\right)$
- inflection $\left(-\arcsin\left(\frac{1}{4}\right),\ \frac{3\sqrt{15}}{8}\right)$
- maximum $\left(\frac{\pi}{6},\ \frac{3\sqrt{3}}{2}\right)$
- inflection $\left(\frac{\pi}{2},\ 0\right)$
- minimum $\left(\frac{5\pi}{6},\ -\frac{3\sqrt{3}}{2}\right)$
- inflection $\left(\pi+\arcsin\left(\frac{1}{4}\right),\ -\frac{3\sqrt{15}}{8}\right)$
- inflection $\left(\frac{3\pi}{2},\ 0\right)$
- inflection $\left(2\pi-\arcsin\left(\frac{1}{4}\right),\ \frac{3\sqrt{15}}{8}\right)$

Local extrema in red, points of inflection in purple, a hollow circle – a point where the function isn't defined; asymptotes dashed.

Tangent – vertical asymptotes, range $\mathbb{R}$.

Input: `tg x` Analyze function

Result

Domain: $D(f) = \mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi \ : \ k \in \mathbb{Z} \right\}$

Range: $R(f) = \mathbb{R}$

Parity: odd

Period: $p = \pi$

x-intercepts: $[k\pi;\ 0]$

Increasing: $\left(-\frac{\pi}{2} + k\pi,\ \frac{\pi}{2} + k\pi \right)$

Convex: $\left[k\pi,\ \frac{\pi}{2} + k\pi \right)$

Concave: $\left(\frac{\pi}{2} + k\pi,\ \pi + k\pi \right]$

Points of inflection: $[k\pi;\ 0]$

Asymptotes: $x = \frac{\pi}{2} + k\pi$

Steps

- Domain – conditions $x \neq \frac{\pi}{2} + k\pi$:
  $D(f) = \mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi \ : \ k \in \mathbb{Z} \right\}$
- The function is periodic with the period π – we examine it on one period (from 0 to π) and repeat the results with the period, k ∈ $\mathbb{Z}$.
- $f(-x) = -\tan x = -f(x)$ – the function is odd (its graph is symmetric about the origin).
- Intersections with the x-axis: $f(x) = 0 \ \Rightarrow \ x = k\pi$, i.e. $[k\pi;\ 0]$
- Intersection with the y-axis: $f(0) = 0$, i.e. $[0;\ 0]$
- The sign of the function between its zeros and the points outside the domain:

	$(0,\ \frac{\pi}{2})$	$(\frac{\pi}{2},\ \pi)$
$f(x)$	$+$	$-$

  The function is positive for $x \in \left(k\pi,\ \frac{\pi}{2} + k\pi \right)$ and negative for $x \in \left(\frac{\pi}{2} + k\pi,\ \pi + k\pi \right)$
- A periodic function (that isn't constant) has no limit at ±∞.
- The limits at the ends of the domain
  $\lim\limits_{x \to \frac{\pi}{2}^-} f(x) = \infty$
  $\lim\limits_{x \to \frac{\pi}{2}^+} f(x) = -\infty$
- Vertical asymptotes (an infinite limit there): $x = \frac{\pi}{2} + k\pi$
- The first derivative: $f'(x) = \frac{1}{\cos^2 x}$
- The equation $f'(x) = 0$ has no solution – there are no stationary points.
- The sign of the first derivative – where it's positive the function increases, where negative it decreases:

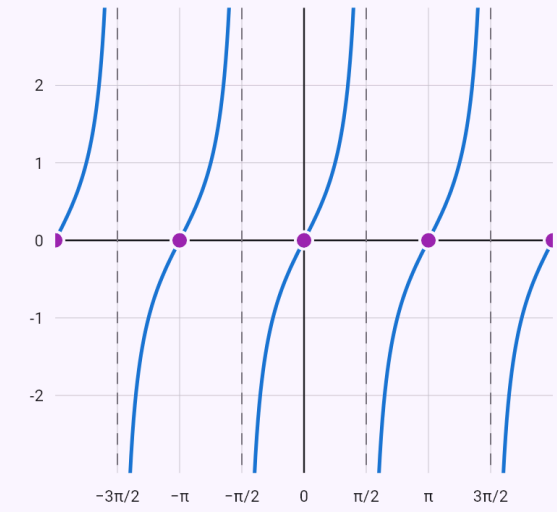
	$(0,\ \frac{\pi}{2})$	$(\frac{\pi}{2},\ \pi)$
$f'(x)$	$+$	$+$
$f(x)$	↗	↗

  The function is increasing on $\left(-\frac{\pi}{2} + k\pi,\ \frac{\pi}{2} + k\pi \right)$
- The function has no local extrema.
- The second derivative: $f''(x) = \frac{2\tan x}{\cos^2 x}$
- Candidates for points of inflection:
  $f''(x) = 0 \ \Rightarrow \ x = k\pi$
- The sign of the second derivative – where it's positive the function is convex, where negative concave:

	$(0,\ \frac{\pi}{2})$	$(\frac{\pi}{2},\ \pi)$
$f''(x)$	$+$	$-$
$f(x)$	∪	∩

  The function is convex on $\left[k\pi,\ \frac{\pi}{2} + k\pi \right)$ and concave on $\left(\frac{\pi}{2} + k\pi,\ \pi + k\pi \right]$
- Points of inflection: $[k\pi;\ 0]$
- Range – from the monotonicity, extrema and limits:
  $R(f) = \mathbb{R}$

Graph



$y = \operatorname{tg} x$

Notable points:

- inflection $(-2\pi, 0)$
- inflection $(-\pi, 0)$
- inflection $(0, 0)$
- inflection $(\pi, 0)$
- inflection $(2\pi, 0)$

Local extrema in red, points of inflection in purple, a hollow circle – a point where the function isn't defined; asymptotes dashed.

Hyperbolic tangent – horizontal asymptotes y = ±1.

Input: `tanh x` Analyze function

Result

Domain: $D(f) = \mathbb{R}$

Range: $R(f) = (-1,\ 1)$

Parity: odd

x-intercepts: $[0;\ 0]$

Increasing: $\mathbb{R}$

Convex: $(-\infty,\ 0]$

Concave: $[0,\ \infty)$

Points of inflection: $[0;\ 0]$

Asymptotes: $y = -1,\ y = 1$

Steps

- Domain: $D(f) = \mathbb{R}$
- $f(-x) = \tanh(-x) = -\tanh x = -f(x)$ – the function is odd (its graph is symmetric about the origin).
- Intersections with the x-axis: $f(x) = 0 \ \Rightarrow\ x = 0$, i.e. $[0;\ 0]$
- Intersection with the y-axis: $f(0) = 0$, i.e. $[0;\ 0]$
- The sign of the function between its zeros and the points outside the domain:

	$(-\infty,\ 0)$	$(0,\ \infty)$
$f(x)$	$-$	$+$

  The function is positive for $x \in (0,\ \infty)$ and negative for $x \in (-\infty,\ 0)$
- The limits at the ends of the domain

$$\lim_{x\to-\infty} f(x) = -1$$
$$\lim_{x\to\infty} f(x) = 1$$

- The function has no vertical asymptotes.
- Asymptote at -∞: $k = \lim\limits_{x\to-\infty} \dfrac{f(x)}{x} = 0$, $q = \lim\limits_{x\to-\infty} f(x) = -1$ – the line $y = -1$
- Asymptote at +∞: $k = \lim\limits_{x\to\infty} \dfrac{f(x)}{x} = 0$, $q = \lim\limits_{x\to\infty} f(x) = 1$ – the line $y = 1$
- The first derivative: $f'(x) = \dfrac{1}{\cosh^2 x}$
- The equation $f'(x) = 0$ has no solution – there are no stationary points.
- The sign of the first derivative – where it's positive the function increases, where negative it decreases:

	$(-\infty,\ \infty)$
$f'(x)$	$+$
$f(x)$	$\nearrow$

  The function is increasing on $\mathbb{R}$
- The function has no local extrema.
- The second derivative: $f''(x) = -\dfrac{2\tanh x}{\cosh^2 x}$
- Candidates for points of inflection: $f''(x) = 0 \ \Rightarrow\ x = 0$
- The sign of the second derivative – where it's positive the function is convex, where negative concave:

	$(-\infty,\ 0)$	$(0,\ \infty)$
$f''(x)$	$+$	$-$
$f(x)$	$\cup$	$\cap$

  The function is convex on $(-\infty,\ 0]$ and concave on $[0,\ \infty)$
- At $x = 0$: $f'''(0) = -2 < 0$ – the first non-zero derivative (from the third on) is of an odd order: a point of inflection.
- Points of inflection: $[0;\ 0]$
- Range – from the monotonicity, extrema and limits: $R(f) = (-1,\ 1)$

Only the derivative is periodic – the extrema values grow with k.

Input: `x + 2sin x` Analyze function

Result

Domain: $D(f) = \mathbb{R}$

Parity: odd

Period: $p = 2\pi$

x-intercepts: $[0;\ 0]$

Increasing: $\left[-\frac{2\pi}{3} + 2k\pi,\ \frac{2\pi}{3} + 2k\pi\right]$

Decreasing: $\left[\frac{2\pi}{3} + 2k\pi,\ \frac{4\pi}{3} + 2k\pi\right]$

Local maximum: $\left[\frac{2\pi}{3} + 2k\pi;\ \sqrt{3} + \frac{2\pi}{3} + 2k\pi\right]$

Local minimum: $\left[\frac{4\pi}{3} + 2k\pi;\ \frac{4\pi}{3} - \sqrt{3} + 2k\pi\right]$

Convex: $[\pi + 2k\pi,\ 2\pi + 2k\pi]$

Concave: $[2k\pi,\ \pi + 2k\pi]$

Points of inflection:
$[2k\pi;\ 2k\pi],\ [\pi + 2k\pi;\ \pi + 2k\pi]$

Steps

- Domain: $D(f) = \mathbb{R}$
- The function itself isn't periodic, but its derivative is (period 2π) – the monotonicity, extrema and convexity repeat with this period, k ∈ $\mathbb{Z}$.
- $f(-x) = -x - 2\sin x = -f(x)$ – the function is odd (its graph is symmetric about the origin).
- Intersections with the x-axis: $f(x) = 0 \Rightarrow x = 0$, i.e. $[0;\ 0]$
- Intersection with the y-axis: $f(0) = 0$, i.e. $[0;\ 0]$
- The limits at the ends of the domain
  $\lim_{x\to-\infty} f(x) = -\infty$
  $\lim_{x\to\infty} f(x) = \infty$
- The first derivative: $f'(x) = 2\cos x + 1$
- Stationary points
  $f'(x) = 0 \Rightarrow x = \frac{2\pi}{3} + 2k\pi,\ x = \frac{4\pi}{3} + 2k\pi$
- The sign of the first derivative – where it's positive the function increases, where negative it decreases:

	$(0, \frac{2\pi}{3})$	$(\frac{2\pi}{3}, \frac{4\pi}{3})$	$(\frac{4\pi}{3}, 2\pi)$
$f'(x)$	+	−	+
$f(x)$	↗	↘	↗

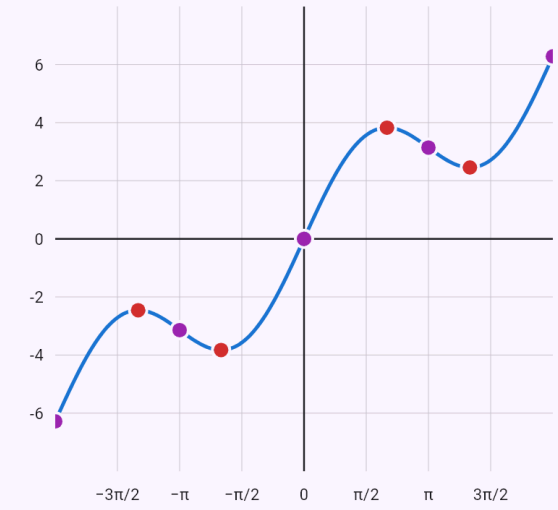
  The function is increasing on $\left[-\frac{2\pi}{3} + 2k\pi,\ \frac{2\pi}{3} + 2k\pi\right]$ and decreasing on $\left[\frac{2\pi}{3} + 2k\pi,\ \frac{4\pi}{3} + 2k\pi\right]$

- Local maximum $\left[\frac{2\pi}{3} + 2k\pi;\ \sqrt{3} + \frac{2\pi}{3} + 2k\pi\right]$, local minimum $\left[\frac{4\pi}{3} + 2k\pi;\ \frac{4\pi}{3} - \sqrt{3} + 2k\pi\right]$
- The second derivative: $f''(x) = -2\sin x$
- Candidates for points of inflection:
  $f''(x) = 0 \Rightarrow x = k\pi$
- The sign of the second derivative – where it's positive the function is convex, where negative concave:

	$(0, \pi)$	$(\pi, 2\pi)$
$f''(x)$	−	+
$f(x)$	∩	∪

  The function is convex on $[\pi + 2k\pi,\ 2\pi + 2k\pi]$ and concave on $[2k\pi,\ \pi + 2k\pi]$
- Points of inflection: $[2k\pi;\ 2k\pi],\ [\pi + 2k\pi;\ \pi + 2k\pi]$

Graph



$y = x + 2\sin x$

Notable points:

- inflection $(-2\pi, -2\pi)$
- maximum $\left(-\frac{4\pi}{3}, \sqrt{3} - \frac{4\pi}{3}\right)$
- inflection $(-\pi, -\pi)$
- minimum $\left(-\frac{2\pi}{3}, -\sqrt{3} - \frac{2\pi}{3}\right)$
- inflection $(0, 0)$
- maximum $\left(\frac{2\pi}{3}, \sqrt{3} + \frac{2\pi}{3}\right)$
- inflection (π, π)
- minimum $\left(\frac{4\pi}{3}, \frac{4\pi}{3} - \sqrt{3}\right)$
- inflection $(2\pi, 2\pi)$

Local extrema in red, points of inflection in purple, a hollow circle – a point where the function isn't defined; asymptotes dashed.

Absolute value – corners, one-sided derivatives.

Input: `|x^2 - 1|` Analyze function

Result

Domain: $D(f) = \mathbb{R}$

Range: $R(f) = [0,\ \infty)$

Parity: even

x-intercepts: $[-1;\ 0],\ [1;\ 0]$

Increasing: $[-1,\ 0],\ [1,\ \infty)$

Decreasing: $(-\infty,\ -1],\ [0,\ 1]$

Local maximum: $[0;\ 1]$

Local minimum: $[-1;\ 0],\ [1;\ 0]$

Convex: $(-\infty,\ -1],\ [1,\ \infty)$

Concave: $[-1,\ 1]$

Steps

- Domain: $D(f) = \mathbb{R}$
- Remove the absolute values by the signs of their arguments on each interval

$$f(x) = \begin{cases} x^2 - 1, & x \in (-\infty,\ -1) \\ 1 - x^2, & x \in [-1,\ 1) \\ x^2 - 1, & x \in [1,\ \infty) \end{cases}$$

- – the function is even (its graph is symmetric about the y-axis).

$$f(-x) = \left|(-x)^2 - 1\right| = \left|x^2 - 1\right| = f(x)$$

- Intersections with the x-axis:
$f(x) = 0 \ \Rightarrow\ x = -1,\ x = 1$, i.e. $[-1;\ 0],\ [1;\ 0]$
- Intersection with the y-axis: $f(0) = 1$, i.e. $[0;\ 1]$
- The sign of the function between its zeros and the points outside the domain:

	$(-\infty,\ -1)$	$(-1,\ 1)$	$(1,\ \infty)$
$f(x)$	$+$	$+$	$+$

The function is positive for
$x \in (-\infty,\ -1) \cup (-1,\ 1) \cup (1,\ \infty)$ and negative for
$x \in \emptyset$

- The limits at the ends of the domain

$$\lim_{x \to -\infty} f(x) = \infty$$

$$\lim_{x \to \infty} f(x) = \infty$$

- The function has no vertical asymptotes.
- No asymptote at -∞: $k = \lim\limits_{x \to -\infty} \dfrac{f(x)}{x} = -\infty$
- No asymptote at +∞: $k = \lim\limits_{x \to \infty} \dfrac{f(x)}{x} = \infty$
- The first derivative

$$f'(x) = \frac{2x^3 - 2x}{\left|x^2 - 1\right|} = \frac{2x(x-1)(x+1)}{\left|x^2 - 1\right|}$$

- Stationary points: $f'(x) = 0 \ \Rightarrow\ x = 0$
- At $x = -1$ there is no derivative – the one-sided derivatives differ ($f'_-(-1) = -2$, $f'_+(-1) = 2$), the graph has a corner there.
- At $x = 1$ there is no derivative – the one-sided derivatives differ ($f'_-(1) = -2$, $f'_+(1) = 2$), the graph has a corner there.
- The sign of the first derivative – where it's positive the function increases, where negative it decreases:

	$(-\infty,\ -1)$	$(-1,\ 0)$	$(0,\ 1)$	$(1,\ \infty)$
$f'(x)$	$-$	$+$	$-$	$+$
$f(x)$	↘	↗	↘	↗

The function is increasing on $[-1,\,0],\ [1,\,\infty)$ and decreasing on $(-\infty,\,-1],\ [0,\,1]$

- At $x = 0$: $f''(0) = -2 < 0$ – a local maximum.
- Local maximum $[0;\ 1]$, local minimum $[-1;\ 0],\ [1;\ 0]$
- The second derivative

$$f''(x) = \frac{2(x^2-1)^3}{\left|x^2-1\right|^3} = \frac{2(x-1)^3(x+1)^3}{\left|x^2-1\right|^3}$$

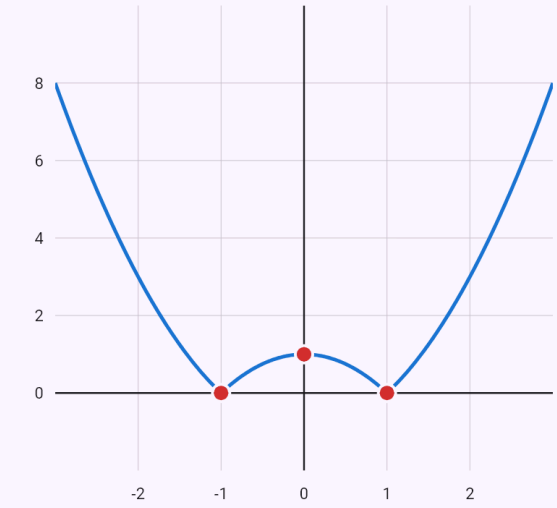
- The equation $f''(x) = 0$ has no solution.
- The sign of the second derivative – where it's positive the function is convex, where negative concave:

	$(-\infty,\,-1)$	$(-1,\,1)$	$(1,\,\infty)$
$f''(x)$	$+$	$-$	$+$
$f(x)$	$\cup$	$\cap$	$\cup$

The function is convex on $(-\infty,\,-1],\ [1,\,\infty)$ and concave on $[-1,\,1]$

- The function has no points of inflection.
- Range – from the monotonicity, extrema and limits: $R(f) = [0,\,\infty)$

Graph



$y = |x^2 - 1|$

Notable points:

- minimum $(-1, 0)$
- maximum $(0, 1)$
- minimum $(1, 0)$

Local extrema in red, points of inflection in purple, a hollow circle – a point where the function isn't defined; asymptotes dashed.

Cube root – a vertical tangent at zero.

Input: `x^(1/3)` Analyze function

Result

Domain: $D(f) = \mathbb{R}$

Range: $R(f) = \mathbb{R}$

Parity: odd

x-intercepts: $[0;\ 0]$

Increasing: $\mathbb{R}$

Convex: $(-\infty,\ 0]$

Concave: $[0,\ \infty)$

Points of inflection: $[0;\ 0]$

Steps

- Domain: $D(f) = \mathbb{R}$
- $f(-x) = \sqrt[3]{(-x)} = \sqrt[3]{-x} = -f(x)$ – the function is odd (its graph is symmetric about the origin).
- Intersections with the x-axis: $f(x) = 0 \ \Rightarrow \ x = 0$, i.e. $[0;\ 0]$
- Intersection with the y-axis: $f(0) = 0$, i.e. $[0;\ 0]$
- The sign of the function between its zeros and the points outside the domain:

	$(-\infty,\ 0)$	$(0,\ \infty)$
$f(x)$	$-$	$+$

  The function is positive for $x \in (0,\ \infty)$ and negative for $x \in (-\infty,\ 0)$
- The limits at the ends of the domain

$$\lim_{x \to -\infty} f(x) = -\infty$$
$$\lim_{x \to \infty} f(x) = \infty$$

- The function has no vertical asymptotes.
- No asymptote at -∞: $k = \lim_{x \to -\infty} \frac{f(x)}{x} = 0$, but $q = \lim_{x \to -\infty} f(x) = -\infty$
- No asymptote at +∞: $k = \lim_{x \to \infty} \frac{f(x)}{x} = 0$, but $q = \lim_{x \to \infty} f(x) = \infty$
- The first derivative: $f'(x) = \frac{1}{3\sqrt[3]{x^2}}$
- The equation $f'(x) = 0$ has no solution – there are no stationary points.
- At $x = 0$ the derivative is infinite ($f'_-(0) = \infty$, $f'_+(0) = \infty$) – the graph has a vertical tangent there.
- The sign of the first derivative – where it's positive the function increases, where negative it decreases:

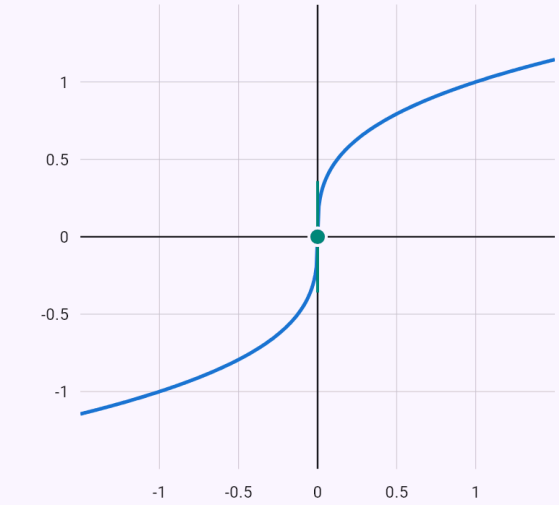
	$(-\infty,\ 0)$	$(0,\ \infty)$
$f'(x)$	$+$	$+$
$f(x)$	↗	↗

  The function is increasing on $\mathbb{R}$
- The function has no local extrema.
- The second derivative: $f''(x) = -\frac{2}{9x\sqrt[3]{x^2}}$
- The equation $f''(x) = 0$ has no solution.
- The sign of the second derivative – where it's positive the function is convex, where negative concave:

	$(-\infty,\ 0)$	$(0,\ \infty)$
$f''(x)$	$+$	$-$
$f(x)$	∪	∩

  The function is convex on $(-\infty,\ 0]$ and concave on $[0,\ \infty)$
- Points of inflection: $[0;\ 0]$
- Range – from the monotonicity, extrema and limits: $R(f) = \mathbb{R}$

Graph



$y = x^{\frac{1}{3}}$

Notable points:

inflection $(0, 0)$ vertical tangent $(0, 0)$

Local extrema in red, points of inflection in purple, a hollow circle – a point where the function isn't defined; asymptotes dashed.

A nested absolute value – by pieces, an even function.

Input: `|-x^2 + 2|x| + 3|` Analyze function

Result

Domain: $D(f) = \mathbb{R}$

Range: $R(f) = [0,\, \infty)$

Parity: even

x-intercepts: $[-3;\ 0],\ [3;\ 0]$

Increasing: $[-3,\, -1],\ [0,\, 1],\ [3,\, \infty)$

Decreasing: $(-\infty,\, -3],\ [-1,\, 0],\ [1,\, 3]$

Local maximum: $[-1;\ 4],\ [1;\ 4]$

Local minimum: $[-3;\ 0],\ [0;\ 3],\ [3;\ 0]$

Convex: $(-\infty,\, -3],\ [3,\, \infty)$

Concave: $[-3,\, 0],\ [0,\, 3]$

Steps

- Domain: $D(f) = \mathbb{R}$
- Remove the absolute values by the signs of their arguments on each interval

$$f(x) = \begin{cases} x^2 + 2x - 3, & x \in (-\infty,\, -3) \\ -x^2 - 2x + 3, & x \in [-3,\, 0) \\ -x^2 + 2x + 3, & x \in [0,\, 3) \\ x^2 - 2x - 3, & x \in [3,\, \infty) \end{cases}$$

- – the function is even (its graph is symmetric about the y-axis).

$$f(-x) = \left|-(-x)^2 + 2|(-x)| + 3\right| = \left|-x^2 + 2|x| + 3\right| =$$

- Intersections with the x-axis:
  $f(x) = 0 \ \Rightarrow\ x = -3,\ x = 3$, i.e. $[-3;\ 0],\ [3;\ 0]$
- Intersection with the y-axis: $f(0) = 3$, i.e. $[0;\ 3]$
- The sign of the function between its zeros and the points outside the domain:

	$(-\infty,\, -3)$	$(-3,\, 3)$	$(3,\, \infty)$
$f(x)$	$+$	$+$	$+$

The function is positive for
$x \in (-\infty,\, -3) \cup (-3,\, 3) \cup (3,\, \infty)$ and negative for
$x \in \emptyset$

- The limits at the ends of the domain
  $\lim\limits_{x \to -\infty} f(x) = \infty$
  $\lim\limits_{x \to \infty} f(x) = \infty$
- The function has no vertical asymptotes.
- No asymptote at -∞: $k = \lim\limits_{x \to -\infty} \dfrac{f(x)}{x} = -\infty$
- No asymptote at +∞: $k = \lim\limits_{x \to \infty} \dfrac{f(x)}{x} = \infty$
- The first derivative

$$f'(x) = \frac{-6x^3 + 2x^3|x| + 6x - 2x|x|}{|x|\left|-x^2 + 2|x| + 3\right|} = \frac{2x(x-1)(x+}{|x|\left|-x^2 + \right.}$$

- Stationary points: $f'(x) = 0 \ \Rightarrow\ x = -1,\ x = 1$
- At $x = -3$ there is no derivative – the one-sided derivatives differ ($f'_-(-3) = -4$, $f'_+(-3) = 4$), the graph has a corner there.
- At $x = 0$ there is no derivative – the one-sided derivatives differ ($f'_-(0) = -2$, $f'_+(0) = 2$), the graph has a corner there.
- At $x = 3$ there is no derivative – the one-sided derivatives differ ($f'_-(3) = -4$, $f'_+(3) = 4$), the graph has a corner there.
- The sign of the first derivative – where it's positive the function increases, where negative it decreases:

	$(-\infty, -3)$	$(-3, -1)$	$(-1, 0)$	$(0, 1)$	$(1, 3)$
$f'(x)$	$-$	$+$	$-$	$+$	$-$
$f(x)$	↘	↗	↘	↗	↘

The function is increasing on $[-3, -1]$, $[0, 1]$, $[3, \infty)$ and decreasing on $(-\infty, -3]$, $[-1, 0]$, $[1, 3]$

- At $x = -1$: $f''(-1) = -2 < 0$ – a local maximum.
- At $x = 1$: $f''(1) = -2 < 0$ – a local maximum.
- Local maximum $[-1;\ 4]$, $[1;\ 4]$, local minimum $[-3;\ 0]$, $[0;\ 3]$, $[3;\ 0]$
- The second derivative

$$f''(x) = \frac{12x^8 + 24x^6 + 36x^6|x| + 2x^6|x|^3 - 84x^4 - 96}{}$$

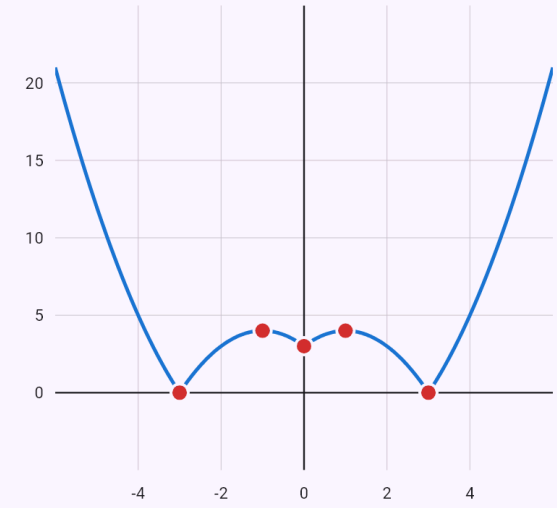
- The equation $f''(x) = 0$ has no solution.
- The sign of the second derivative – where it's positive the function is convex, where negative concave:

	$(-\infty, -3)$	$(-3, 0)$	$(0, 3)$	$(3, \infty)$
$f''(x)$	$+$	$-$	$-$	$+$
$f(x)$	∪	∩	∩	∪

The function is convex on $(-\infty, -3]$, $[3, \infty)$ and concave on $[-3, 0]$, $[0, 3]$

- The function has no points of inflection.
- Range – from the monotonicity, extrema and limits: $R(f) = [0, \infty)$

Graph



$y = |-x^2 + 2|x| + 3|$

Notable points:

- minimum $(-3, 0)$
- maximum $(-1, 4)$
- minimum $(0, 3)$
- maximum $(1, 4)$
- minimum $(3, 0)$

Local extrema in red, points of inflection in purple, a hollow circle – a point where the function isn't defined; asymptotes dashed.

A sum of absolute values – a constant piece and two oblique asymptotes.

Input: `|x - 2| + |x + 3|` Analyze function

Result

Domain: $D(f) = \mathbb{R}$

Range: $R(f) = [5,\ \infty)$

Parity: neither even nor odd

Increasing: $[2,\ \infty)$

Decreasing: $(-\infty,\ -3]$

Constant: $[-3,\ 2]$

Local minimum: $f(x) = 5,\ x \in [-3,\ 2]$

Asymptotes: $y = -2x - 1,\ y = 2x + 1$

Steps

- Domain: $D(f) = \mathbb{R}$
- Remove the absolute values by the signs of their arguments on each interval

$$f(x) = \begin{cases} -2x - 1, & x \in (-\infty,\ -3) \\ 5, & x \in [-3,\ 2) \\ 2x + 1, & x \in [2,\ \infty) \end{cases}$$

- – neither f(x) nor -f(x): the function is neither even nor odd.

$f(-x) = |(-x) - 2| + |(-x) + 3| = |-x - 2| + |3 - x$

- The equation $f(x) = 0$ has no solution – the graph doesn't cross the x-axis.
- Intersection with the y-axis: $f(0) = 5$, i.e. $[0;\ 5]$
- The sign of the function between its zeros and the points outside the domain:

	$(-\infty,\ \infty)$
$f(x)$	$+$

The function is positive for $x \in \mathbb{R}$ and negative for $x \in \emptyset$

- The limits at the ends of the domain

$$\lim_{x \to -\infty} f(x) = \infty$$
$$\lim_{x \to \infty} f(x) = \infty$$

- The function has no vertical asymptotes.
- Asymptote at -∞: $k = \lim_{x \to -\infty} \frac{f(x)}{x} = -2$, $q = \lim_{x \to -\infty} (f(x) + 2x) = -1$ – the line $y = -2x - 1$
- Asymptote at +∞: $k = \lim_{x \to \infty} \frac{f(x)}{x} = 2$, $q = \lim_{x \to \infty} (f(x) - 2x) = 1$ – the line $y = 2x + 1$
- The first derivative

$$f'(x) = \frac{x}{|x - 2|} + \frac{x}{|x + 3|} - \frac{2}{|x - 2|} + \frac{3}{|x + 3|} = \frac{x|x -}$$

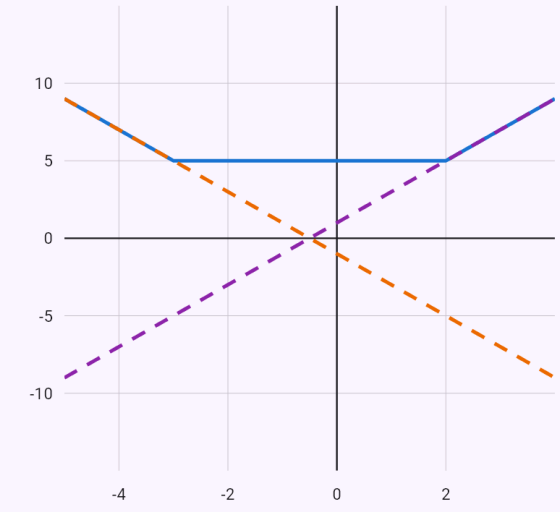
- The equation $f'(x) = 0$ has no solution – there are no stationary points.
- At $x = -3$ there is no derivative – the one-sided derivatives differ ($f'_-(-3) = -2$, $f'_+(-3) = 0$), the graph has a corner there.
- At $x = 2$ there is no derivative – the one-sided derivatives differ ($f'_-(2) = 0$, $f'_+(2) = 2$), the graph has a corner there.
- The sign of the first derivative – where it's positive the function increases, where negative it decreases:

	$(-\infty,\ -3)$	$(-3,\ 2)$	$(2,\ \infty)$
$f'(x)$	$-$	0	$+$
$f(x)$	↘	→	↗

The function is increasing on $[2,\ \infty)$ and decreasing on $(-\infty,\ -3]$. On $[-3,\ 2]$ it is constant (f' = 0).

- Local minimum $f(x) = 5,\ x \in [-3,\ 2]$, no local maximum.
- The second derivative: $f''(x) = 0$
- The second derivative is zero – the graph is (part of) a line, neither convex nor concave.
- Range – from the monotonicity, extrema and limits: $R(f) = [5,\ \infty)$

Graph



$y = |x - 2| + |x + 3|$
$y = -2x - 1$
$y = 2x + 1$

Local extrema in red, points of inflection in purple, a hollow circle – a point where the function isn't defined; asymptotes dashed.

Absolute extrema on an interval – stationary points and end points.

Input: `extrema of x^3 - 3x on [-2, 3]`

Result

$$\max f = 18, \quad \min f = -2$$

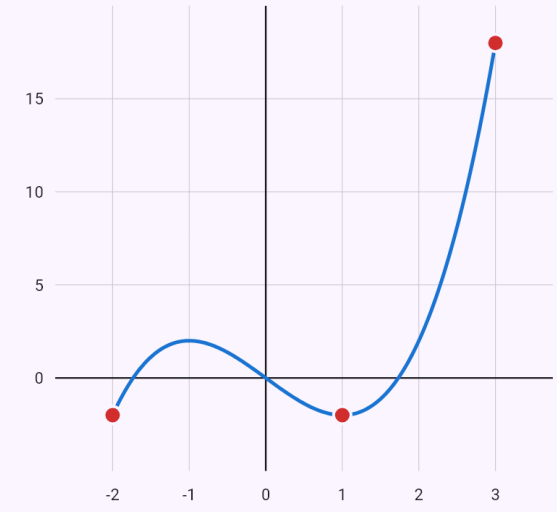
Steps

- The derivative: $f'(x) = 3x^2 - 3$
- Stationary points ($f'(x) = 0$): $x = -1,\ x = 1$; all of them lie in the interval $\langle -2;\ 3 \rangle$
- A continuous function attains its largest and smallest value on a closed interval (the extreme value theorem) – at a stationary point, at a point without a derivative or at an end point. We compare the values

x	-2	-1	1	3
$f(x)$	-2	2	-2	18

- The largest value on $\langle -2;\ 3 \rangle$ is $f(3) = 18$, the smallest $f(-2) = f(1) = -2$

Graph



$y = x^3 - 3x$

Notable points:

- maximum $(3, 18)$
- minimum $(-2, -2)$
- minimum $(1, -2)$

Local extrema in red, points of inflection in purple, a hollow circle – a point where the function isn't defined; asymptotes dashed.

The inverse of a linear fractional function, with a check.

Input: `inverse of f(x) = (2x+1)/(x-3)`

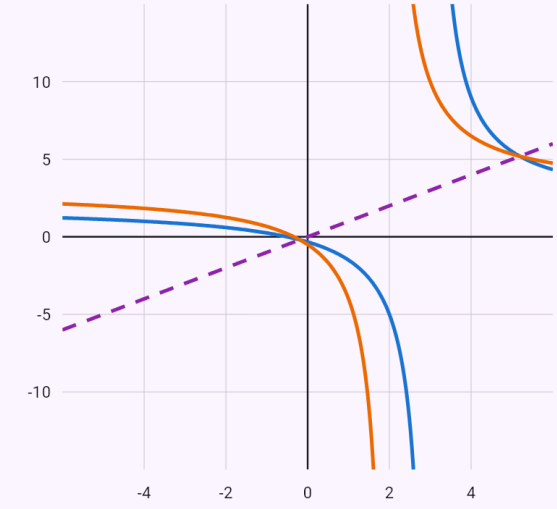
Result

$$f^{-1}(x) = \frac{3x+1}{x-2}$$

Steps

- The function $f(x) = \frac{2x+1}{x-3}$, its domain's conditions: $x - 3 \neq 0$
- We solve $y = f(x)$ for x: $y = \frac{2x+1}{x-3} \Rightarrow x = \frac{3y+1}{y-2}$
- We swap x and y: $f^{-1}(x) = \frac{3x+1}{x-2}$. The inverse's domain is the range of f: $x - 2 \neq 0$
- Check by composing: $f\left(f^{-1}(x)\right) = f\left(\frac{3x+1}{x-2}\right) = x$ ✓

Graph



$y = \frac{2x+1}{x-3}$

f^{-1}

$y = x$

The inverse's graph is symmetric to the graph of f about the line y = x.

A parabola's vertex by completing the square.

Input: `vertex of the parabola y = x^2 - 4x + 7`

Result

$$V = [2;\ 3],\ y = (x-2)^2 + 3$$

Steps

- Completing the square: $y = x^2 - 4x + 7 = (x-2)^2 + 3$
- Vertex $V = [2;\ 3]$, axis $x = 2$; the parabola opens upwards, range $H(f) = \langle 3; \infty)$
- The graph is the graph of $y = x^2$ shifted by the vector $(2;\ 3)$

A sine graph by compressing, shifting and stretching.

Input: `posuny grafu y = 2sin(3(x - 1)) + 1`

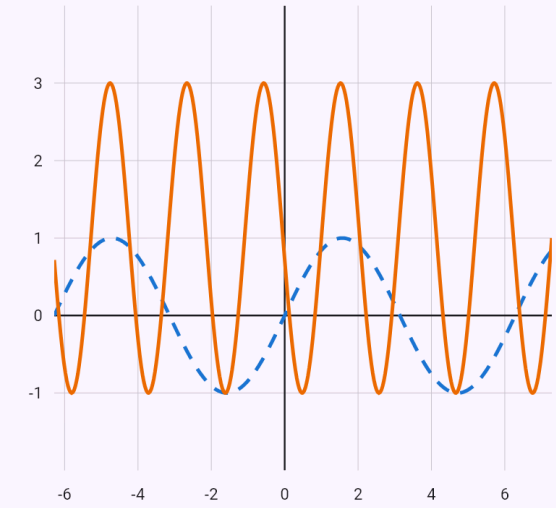
Result

$$y = 2\sin(3(x-1)) + 1$$

Steps

- We start from the graph of the basic function $y = \sin x$
- We compress the graph along the x axis by the factor 3 (towards the y axis): $y = \sin(3x)$
- We shift the graph by 1 to the right along the x axis: $y = \sin(3(x-1))$
- We stretch the graph along the y axis by the factor 2: $y = 2\sin(3(x-1))$
- We shift the graph by 1 up along the y axis: $y = 2\sin(3(x-1)) + 1$

Graph



$y = \sin x$

$y = 2\sin(3(x-1)) + 1$

Dashed: the basic function's graph; solid: the resulting graph.

A root shifted and reflected in the x axis.

Input: `posuny grafu y = -sqrt(x + 2) + 3`

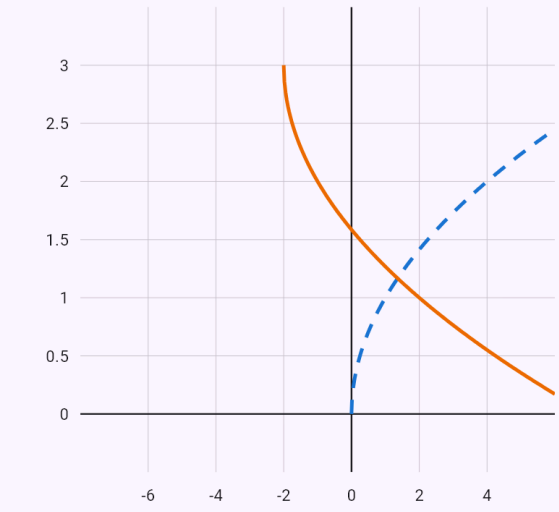
Result

$$y = -\sqrt{x+2} + 3$$

Steps

- We start from the graph of the basic function $y = \sqrt{x}$
- We shift the graph by 2 to the left along the x axis: $y = \sqrt{x+2}$
- We reflect the graph in the x axis: $y = -\sqrt{x+2}$
- We shift the graph by 3 up along the y axis: $y = -\sqrt{x+2} + 3$

Graph



$y = \sqrt{x}$
$y = -\sqrt{x+2} + 3$

Dashed: the basic function's graph; solid: the resulting graph.

A composite function f(g(x)).

Input: `f(x) = x^2 + 1, g(x) = 2x - 3, f(g(x))`

Result

$$f(g(x)) = 4x^2 - 12x + 10$$

Steps

- Into the outer function f we substitute the inner function g for x: $f(g(x)) = (2x-3)^2 + 1 = 4x^2 - 12x + 10$

Known limitations

- A function without even a periodic derivative (x·sin x) is analyzed only numerically on ⟨−2π; 2π⟩, without the period; inverse trigonometric functions (arcsin, arctan) are not supported yet.
- Absolute extrema only on a closed interval and for a continuous function; the inverse only where a formula expresses it – for a function that isn't one-to-one (x²) the inverse on a part of the domain isn't looked for yet.

28. Functions of several variables

Partial derivatives (first and second, mixed) and the gradient of a function of two or three variables – differentiating by one variable, the others are constants.

Local extrema of a function of two variables: the stationary points from f_x = 0, f_y = 0, each decided by the Hessian's determinant D = f_xx·f_yy − f_xy² (a minimum, a maximum, a saddle). Constrained extrema with one constraint ("extrema of xy, x + y = 1") by Lagrange's method and the bordered Hessian.

The directional derivative (the gradient times a unit vector), the total differential with an approximate value and a surface's tangent plane. A double integral is entered by iterated limits or as a region of inequalities; the app splits the region into normal parts, shows the inner and the outer integral and draws the region; the 3D switch shows the integrand's surface over it. The chips below the result change the order of integration and switch to polar coordinates.

Vector analysis: the divergence, the curl (for a plane field a single number ∂Q/∂x − ∂P/∂y), the Laplacian and the potential of a conservative field – the app first checks the curl is zero, then finds φ by integrating the components one by one and checks ∇φ = F. A field is entered as a vector (P; Q; R) or (P; Q), also with the unit vectors i, j, k or named "F = …" on a line of its own. For a plane field the graph shows the field's arrows and the potential's level curves. Chips below the result switch between the divergence, curl and potential of the same field.

Line integrals: of the first kind ∫_C f ds (a curve's length, a wire's mass) and of the second kind ∫_C F·dr (the work of a force, a circulation ∮ around a closed curve). The curve is given by a parametrization r(t) or a preset – a segment, a circle, a graph y = f(x), a polyline; the steps show the tangent vector, the substitution into the field and the integral in t. For a conservative field the app checks the result by the potential φ(B) − φ(A) (path independence), for a closed plane curve (a circle, an ellipse, a convex polygon) by Green's theorem – it also computes the double integral over the region inside. Chips below the result switch which way is the main one. The "Line integral" form (the "Applications" section, the Multivariable calculus field) offers the curve's presets; the graph shows the curve with its orientation over the field's arrows and Green's region, a space curve in 3D, turned by a finger.

Triple integrals: by iterated limits, over a solid of inequalities (∭_V, from a photo too) or with the "Triple integral" form (the "Applications" section, the Multivariable calculus field) with the presets box, ball, cylinder, cone, solid between graphs and inequalities. The app describes the solid as a normal region – the innermost variable between two surfaces over the projection – and computes three integrals one after another (Fubini's theorem). In cylindrical and spherical coordinates it shows the substitution with the Jacobian (r, ρ² sin θ) and the limits in the new variables; when Cartesian coordinates give no exact result, it switches to them by itself, and for a ball, a cylinder or x² + y² in the problem a chip offers them.

The form also computes the volume, the mass with a density ρ, the centroid and the moment of inertia about an axis. A rotatable 3D graph shows the solid.

Letters other than x, y, z (a radius r, a height h, edges a, b, c) are positive parameters: the result is a formula, such as V = 4/3 πr³ for the ball ball(r) – beside the radius r the cylindrical coordinate is written ϱ – or V = abc for a box. The app checks the formula numerically for several values of the parameters; the 3D picture of the solid is drawn for the value 1.

Surface integrals: of the first kind ∬_S f dS (a surface's area, a shell's mass) and of the second kind – the flux ∬_S F·n dS. The surface is given by a preset – a graph z = g(x, y) over a region D, a sphere, a hemisphere, a cylinder, a cone (the whole surface with its discs, or the lateral surface alone) – or by a parametrization r(u, v). The steps show the tangent vectors r_u, r_v, the normal vector r_u × r_v with its orientation (outward for a closed surface, up for a graph, by the parametrization otherwise; outward, inward, up, down to choose), the surface element and the double integral over the parameters; a closed surface is added up from its parts. The flux through a closed surface is checked by Gauss's theorem (the triple integral of the divergence), the flux of a curl by Stokes' theorem – the line integral around the boundary; chips under the result switch the main way. The form "Surface integral" (the "Applications" section, the field Multivariable calculus) offers the surfaces' presets; a rotatable 3D graph shows the surface with its normals, the field's arrows and the boundary.

Wires and shells: the line and surface integral forms (the "What to compute" choice) and the notation quantity=mass / centroid / inertia compute the mass of a wire or a shell with a density f (empty = 1), its centroid and its moment of inertia about an axis (axis=x, y, z). The planimeter lineint(area; C) computes the area inside a closed plane curve as ½∮ x dy − y dx and checks it by Green's theorem.

How to enter it

`partial derivatives x^2 y + y^3`	All first partial derivatives (with "second" the second ones).
`∂/∂x (x^2 y + y^3)`	One partial derivative (also \frac{\partial}{\partial x}).
`gradient x^2 y + y^3`	The gradient.
`extrema of x^2 + y^2 - 2x + 4y`	Local extrema of a function of two variables.
`extrema of xy, x + y = 1`	Constrained extrema – the function and the constraint after a comma.
`directional(x^2 y; [1; 2]; (3; 4))`	Directional derivative: the function, the point, the direction (also in words).

`totaldiff(x^2 y; [1; 2]; h=0.1 0.2)`	Total differential; the point and the increments h are optional (also in words).
`tangentplane(z = x^2 + y^2; [1; 2])`	Tangent plane at a point, for z = f(x, y) or F(x, y, z) = 0 (also in words).
`∫_0^1 ∫_0^x x y dy dx`	A double integral by iterated limits (also \int_0^1 dx \int_0^x ... dy).
`iint(x y; 0 <= x <= 1; 0 <= y <= x)`	A double integral over a region of inequalities (also \iint_D ..., D: ...); you may add order=dxdy or polar.
`div(x y; y z; z x)`	The divergence of a field (also ∇·F, \nabla \cdot, \operatorname{div}, in words “divergence”).
`∇×(y z; x z; x y)`	The curl (also rot, curl, \nabla \times); for a field (P; Q) a single number.
`Δ(x^2 y + y^3)`	The Laplacian of a function of two or three variables (also laplace(...), ∇², \Delta).
`potential (2x + y; x + 2y)`	The potential of a conservative field (also potential(...)); the field may be named: “div F, F = (y; x; 0)”.
`lineint((y; -x); circle([0; 0]; 1))`	A line integral – a field (P; Q) or a function f, then the curve: r(t) = (…); t = a..b, segment(A; B), circle(S; r), graph(y = …; x = a..b), polyline(A; B; …); also cw (clockwise), method=green / method=potential (also in words “line integral … along …”). After the curve also quantity=mass / centroid / inertia (with axis=…); area instead of the function computes the area inside a closed curve.
`lineint(1; circle([0; 0]; 2); quantity=mass)`	A wire's mass with the density f, quantity=centroid its centroid, quantity=inertia its moment of inertia (axis=x, y, z).
`lineint(area; polyline([0; 0]; [2; 0]; [2; 1]; [0; 1]; [0; 0]))`	The planimeter – the area inside a closed plane curve from the integral around its boundary (Green's theorem).

`\oint_C x^2 y \, dx + x \, dy, C: x^2 + y^2 = 1`	From a photo ∮_C or ∫_C with P dx + Q dy, F·dr or f ds; the curve after a comma or on the next line (C: …).
`\int_C x y \, ds, C: x = \cos t, y = \sin t, 0 \le t \le \pi/2`	An integral of the first kind – the curve parametrically with the parameter's range (also t ∈ [a, b]).
`tint(z; x^2 + y^2 + z^2 <= 4; z >= 0; sph)`	A triple integral – the function, then the solid by inequalities or a preset ball(R), ball(R; a; b; c), cylinder(R; h), cylinder(R; h1; h2), cone(R; h); coordinates cart, cyl, sph, the order order=dzdydx, quantity=volume / mass / centroid / inertia (with axis=x). In words “triple integral of z over V: …”, “volume of the solid …”, “mass of the solid … with density …”, “centroid of the solid …”, “moment of inertia of the solid … about the z-axis”.
`\iiint_V z \, dV, V: x^2 + y^2 + z^2 \le 4, z \ge 0`	From a photo ∭ (\iiint_V) with dV or dx dy dz; the solid after a comma (V: …) or under the integral.
`\int_0^1 \int_0^{1-x} \int_0^{1-x-y} x \, dz \, dy \, dx`	By iterated limits – three integrals in a row, the differentials at the end or right after their integral.
`surfint((x; y; z); cylinder(1; 2))`	A surface integral – a function f, a field (P; Q; R) or curl(P; Q; R), then the surface: graph(z = …; inequalities of D), sphere(R), sphere(R; a; b; c), hemisphere(R), cylinder(R; h), cylinder(R; h1; h2), cone(R; h), r(u, v) = (…); u = a..b; v = c..d; lateral (the lateral surface alone), closed (a hemisphere with its base), outward / inward / up / down, method=gauss / method=stokes. In words “surface integral of … over …”, “flux of … through …”. With a function f also quantity=mass / centroid / inertia (with axis=…).
`\oiint_S \vec{F} \cdot d\vec{S}, \vec{F} = (x^2, y^2, z^2), S: x^2 + y^2 + z^2 = 1`	From a photo ∬_S or ∯_S with f dS, F·dS or F·n dS; the surface after a comma or on the next line: S: x² + y² + z² = R² (with z ≥ 0 a hemisphere), x² + y² = R² with bounds of z (a cylinder’s lateral surface), z = g(x, y) with inequalities, r(u, v) with the parameters’ ranges.
`\iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS, \vec{F} = (y, -x, z),`	The flux of a curl (∇ × F, rot F, curl F) – the field named apart; checked by Stokes’ theorem.

`S: z = 4 - x^2 - y^2, z \ge 0`	
`surfint(1; sphere(1); quantity=inertia; axis=z)`	A shell's mass, centroid or moment of inertia with the surface density f.

Examples

Second partial derivatives, the mixed one too.

Input: `second partial derivatives x^2 y + y^3`

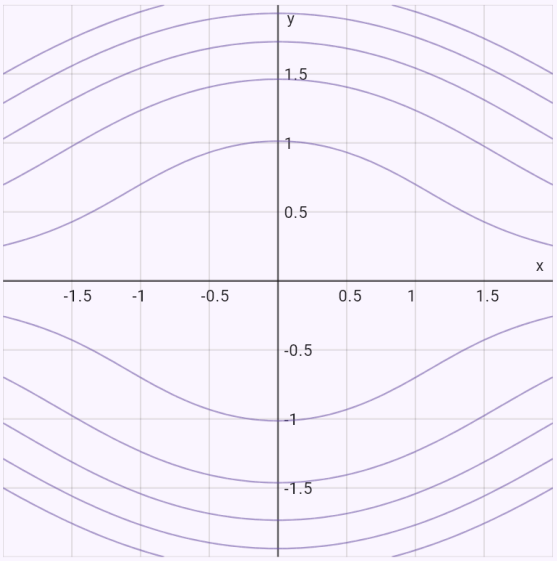
Result

$$f_x = 2xy, \quad f_y = x^2 + 3y^2, \quad f_{xx} = 2$$

Steps

- The partial derivative with respect to x – the other variables are constants: $f_x = \frac{\partial f}{\partial x} = 2xy$
- The partial derivative with respect to y – the other variables are constants: $f_y = \frac{\partial f}{\partial y} = x^2 + 3y^2$
- The partial derivative with respect to xx – the other variables are constants: $f_{xx} = \frac{\partial^2 f}{\partial x^2} = 2y$
- The partial derivative with respect to xy – the other variables are constants: $f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = 2x$
- The partial derivative with respect to yy – the other variables are constants: $f_{yy} = \frac{\partial^2 f}{\partial y^2} = 6y$

Graph



Level curves of f.

Local extrema – a saddle point and a minimum.

Input: `extrema of x^3 - 3xy + y^3`

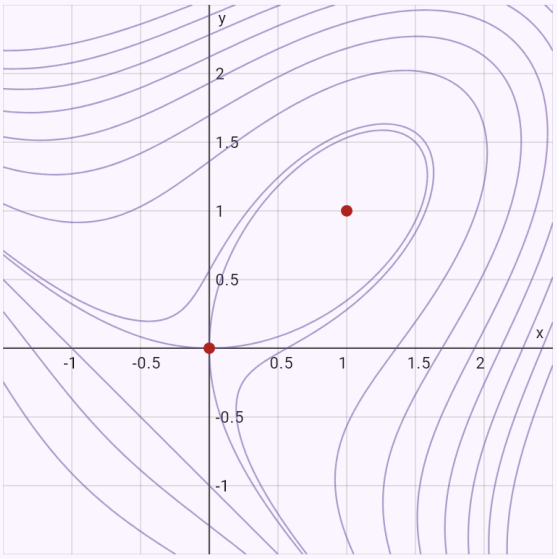
Result

$$\min f(1;\ 1) = -1$$

Steps

- The partial derivative with respect to x – the other variables are constants: $f_x = 3\left(x^2 - y\right)$
- The partial derivative with respect to y – the other variables are constants: $f_y = 3\left(y^2 - x\right)$
- The stationary points $[0;\ 0]$, $[1;\ 1]$ solve the system where both partial derivatives are zero
  $3\left(x^2 - y\right) = 0$
  $3\left(y^2 - x\right) = 0$
- The second partial derivatives $f_{xx} = 6x$, $f_{xy} = -3$, $f_{yy} = 6y$ and the Hessian's determinant
  $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6x & -3 \\ -3 & 6y \end{vmatrix}$
- At $[0;\ 0]$: $D = 0 \cdot 0 - (-3)^2 = -9 < 0$ – a saddle point, no extremum ($f(0;\ 0) = 0$).
- At $[1;\ 1]$: $D = 6 \cdot 6 - (-3)^2 = 27 > 0$ and $f_{xx} = 6 > 0$ – a local minimum $f(1;\ 1) = -1$

Graph



Level curves of f.

A constrained extremum by Lagrange's method.

Input: `extrema of xy, x + y = 1`

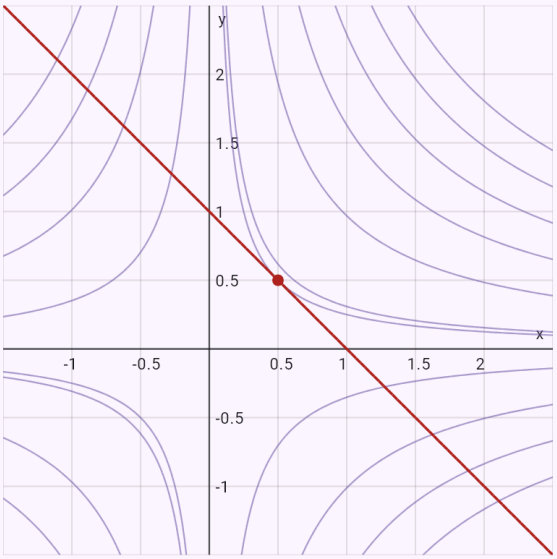
Result

$$\max f\left(\frac{1}{2};\ \frac{1}{2}\right) = \frac{1}{4}$$

Steps

- The Lagrange function with the multiplier λ:
  $L = xy - \lambda(x + y - 1)$. Its partial derivatives are zero and the constraint holds
  $L_x = y - \lambda = 0$
  $L_y = x - \lambda = 0$
  $x + y - 1 = 0$
- Candidates for a constrained extremum (the system's solutions): $\left[\frac{1}{2};\ \frac{1}{2}\right]$
- At $\left[\frac{1}{2};\ \frac{1}{2}\right]$ ($\lambda = \frac{1}{2}$) the bordered Hessian is positive – a constrained local maximum $f\left(\frac{1}{2};\ \frac{1}{2}\right) = \frac{1}{4}$
  $\bar{D} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2$

Graph



Level curves of f.

The directional derivative and the steepest growth.

Input: `directional(x^2 y; [1; 2]; (3; 4))`

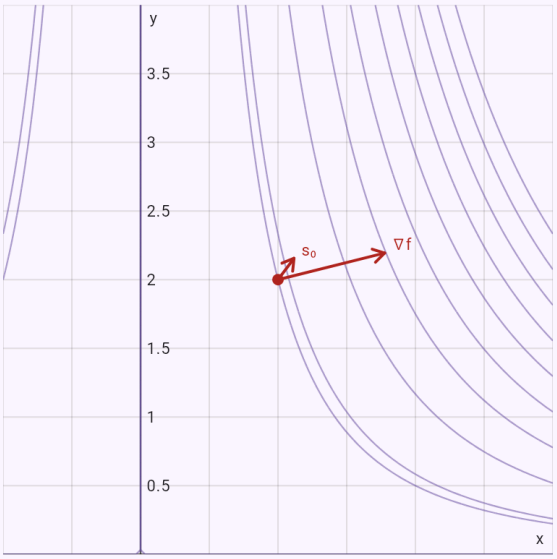
Result

$$\frac{\partial f}{\partial \vec{s}}[1;\ 2] = \frac{16}{5}$$

Steps

- The partial derivative with respect to x – the other variables are constants: $f_x = 2xy$
- The partial derivative with respect to y – the other variables are constants: $f_y = x^2$
- Turn the direction into a unit vector: $|\vec{s}| = 5, \quad \vec{s}_0 = \left(\frac{3}{5};\ \frac{4}{5}\right)$
- The gradient at $[1;\ 2]$: $\nabla f[1;\ 2] = (4;\ 1)$
- The directional derivative is the dot product of the gradient and the unit vector $\frac{\partial f}{\partial \vec{s}}[1;\ 2] = \nabla f \cdot \vec{s}_0 = 4 \cdot \frac{3}{5} + 1 \cdot \frac{4}{5} = \frac{16}{5}$
- The steepest growth is along the gradient, with the rate $|\nabla f[1;\ 2]| = \sqrt{17}$

Graph



Level curves of f. The arrow ∇f points in the direction of steepest growth.

The tangent plane of a paraboloid.

Input: `tangentplane(z = x^2 + y^2; [1; 2])`

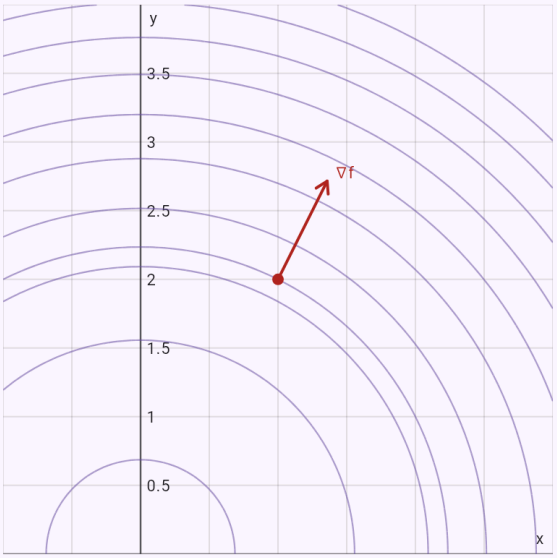
Result

$$z = 2x + 4y - 5$$

Steps

- The partial derivative with respect to x – the other variables are constants: $f_x = 2x$
- The partial derivative with respect to y – the other variables are constants: $f_y = 2y$
- The third coordinate of the point of tangency: $z_0 = f[1;\ 2] = 5$
- Substitute into the tangent plane $z - z_0 = f_x \cdot (x - x_0) + f_y \cdot (y - y_0)$: $z - 5 = 2(x - 1) + 4(y - 2)$
- The plane's normal vector: $\vec{n} = (2;\ 4;\ -1)$

Graph



Level curves of f. The arrow ∇f points in the direction of steepest growth.

A double integral over a triangle.

Input: `∫_0^1 ∫_0^x x y dy dx`

Result

$$\frac{1}{8}$$

Order of integration:

✓ dy dx | dx dy

Coordinates:

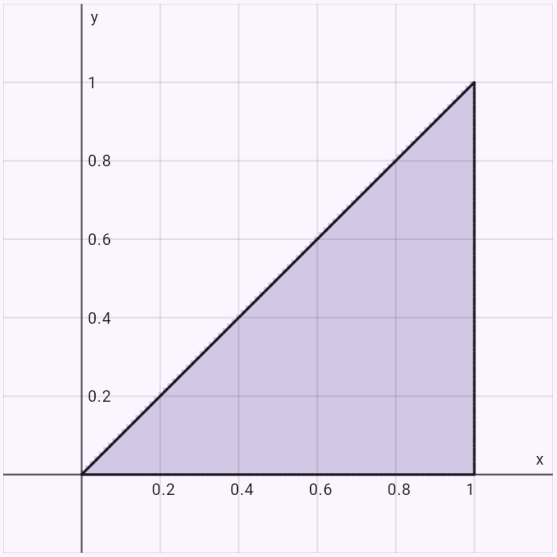
✓ Cartesian | Polar

Steps

- Describe the region as a normal region:
  $0 \le x \le 1,\ 0 \le y \le x$
- By Fubini's theorem, write it as an iterated integral:
  $\int_0^1 \left(\int_0^x xy\, dy \right) dx$
- The inner integral with respect to y (the other variable is a constant): $\int_0^x xy\, dy = \left[\frac{xy^2}{2} \right]_{y=0}^{y=x} = \frac{x^3}{2}$
- The outer integral with respect to x:
  $\int_0^1 \frac{x^3}{2}\, dx = \left[\frac{x^4}{8} \right]_0^1 = \frac{1}{8}$

Graph

✓ 2D | 3D



The region of integration.

The inner integral isn't elementary – the app swaps the order by itself.

Input: `∫_0^1 ∫_x^1 e^(y^2) dy dx`

Result

$$\frac{e-1}{2}$$

Order of integration:

✓ dx dy | dy dx

Coordinates:

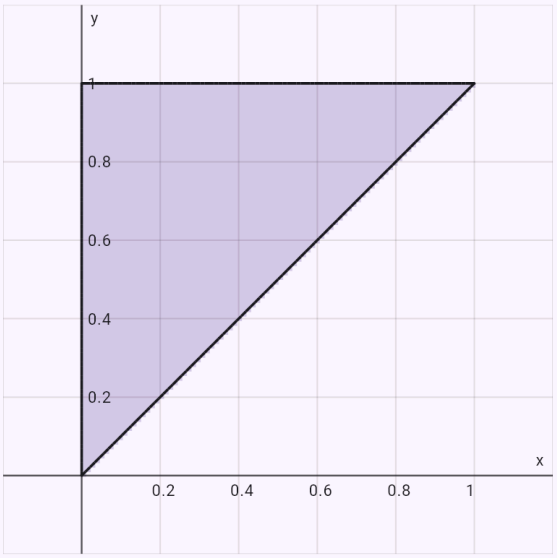
✓ Cartesian | Polar

Steps

- The inner integral with respect to y has no elementary antiderivative, so we swap the order of integration.
- Describe the region as a normal region: $0 \le y \le 1,\ 0 \le x \le y$
- By Fubini's theorem, write it as an iterated integral: $\int_0^1 \left(\int_0^y e^{y^2}\, dx \right) dy$
- The inner integral with respect to x (the other variable is a constant): $\int_0^y e^{y^2}\, dx = \left[x e^{y^2} \right]_{x=0}^{x=y} = y e^{y^2}$
- The outer integral with respect to y: $\int_0^1 y e^{y^2}\, dy = \left[\frac{e^{y^2}}{2} \right]_0^1 = \frac{e-1}{2}$

Graph

✓ 2D | 3D



The region of integration.

A quarter disc in polar coordinates.

Input: `iint(x^2 + y^2; x^2 + y^2 <= 1; x >= 0; y >= 0; polar)`

Result

$$\frac{\pi}{8}$$

Coordinates:

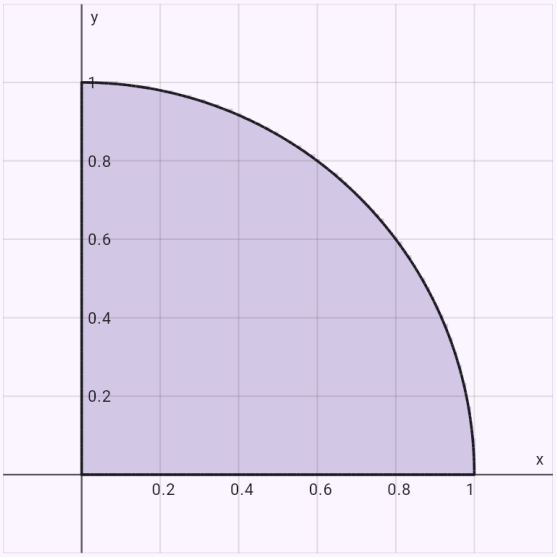
Cartesian | ✓ Polar

Steps

- Polar coordinates: $x = r\cos\varphi,\ y = r\sin\varphi,\ J = r$, r being the Jacobian of the transformation. The integrand: $f \cdot r = r^3$
- The region in polar coordinates: $0 \le \varphi \le \frac{\pi}{2},\ 0 \le r \le 1$
- By Fubini's theorem, write it as an iterated integral: $\int_0^{\frac{\pi}{2}} \left(\int_0^1 r^3\, dr \right) d\varphi$
- The inner integral with respect to r (the other variable is a constant): $\int_0^1 r^3\, dr = \left[\frac{r^4}{4} \right]_{r=0}^{r=1} = \frac{1}{4}$
- The outer integral with respect to φ $\int_0^{\frac{\pi}{2}} \frac{1}{4}\, d\varphi = \left[\frac{\varphi}{4} \right]_0^{\frac{\pi}{2}} = \frac{\pi}{8}$

Graph

✓ 2D | 3D



The region of integration.

The divergence of a field in space as the sum of three partial derivatives.

Input: `div(x y; y z; z x)`

Result

$$\nabla \cdot \vec{F} = x + y + z$$

For this field compute:

✓ divergence | curl | potential

Steps

- The vector field and its components:
  $\vec{F} = (xy;\ yz;\ xz), \quad P = xy,\ Q = yz,\ R = xz$
- The partial derivative with respect to x – the other variables are constants: $\dfrac{\partial P}{\partial x} = y$
- The partial derivative with respect to y – the other variables are constants: $\dfrac{\partial Q}{\partial y} = z$
- The partial derivative with respect to z – the other variables are constants: $\dfrac{\partial R}{\partial z} = x$
- The divergence is the sum of the components' partial derivatives, each by "its" variable
  $\nabla \cdot \vec{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z} = y + z + x = x + y + z$

The curl of a plane vortex – the field's arrows circle the origin.

Input: `rot(-y; x)`

Result

$$(\nabla \times \vec{F})_z = 2$$

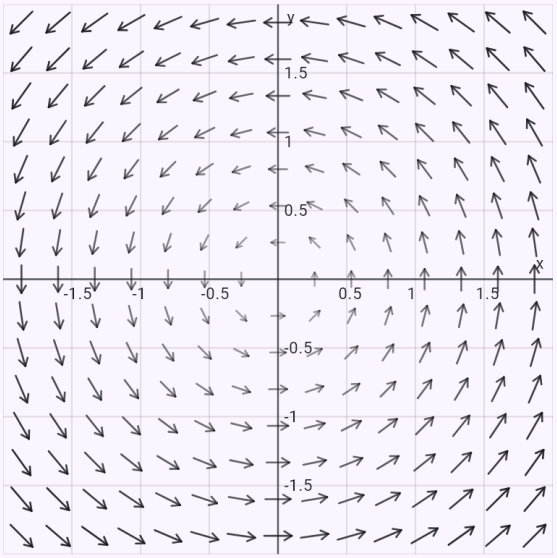
For this field compute:

divergence | ✓ curl | potential

Steps

- The vector field and its components:
  $\vec{F} = (-y;\ x), \quad P = -y,\ Q = x$
- For a plane field the curl is a single number – the z-component, ∂Q/∂x − ∂P/∂y
  $(\nabla \times \vec{F})_z = \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 1 - (-1) = 2$

Graph



The arrows show the field's direction and strength (longer and darker = stronger).

A harmonic function – the Laplacian is zero.

Input: `Δ(x^2 - y^2)`

Result

$$\Delta f = 0$$

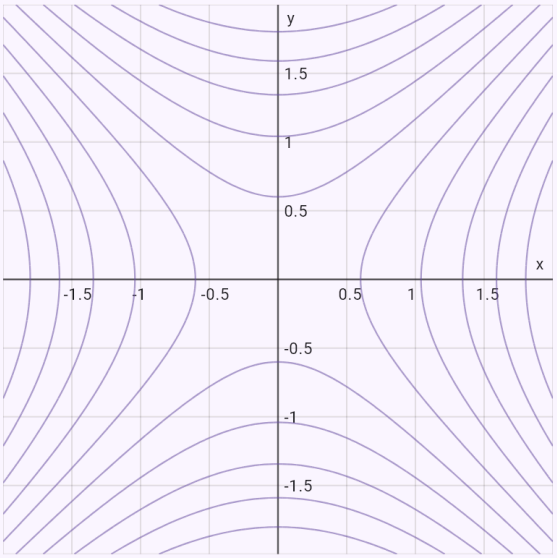
For this function compute:

✓ Laplacian Δf | gradient ∇f

Steps

- The first and second partial derivative with respect to x:
  $$f_x = 2x, \quad f_{xx} = \frac{\partial^2 f}{\partial x^2} = 2$$
- The first and second partial derivative with respect to y:
  $$f_y = -2y, \quad f_{yy} = \frac{\partial^2 f}{\partial y^2} = -2$$
- The Laplacian is the sum of the second partial derivatives: $\Delta f = f_{xx} + f_{yy} = 2 + (-2) = 0$
- Δf = 0 – the function is harmonic.

Graph



Level curves of f.

The potential the school way: P integrated by x, compared with Q, checked; graph with equipotentials.

Input: `potential(2x y; x^2)`

Result

$$\varphi = x^2 y + C$$

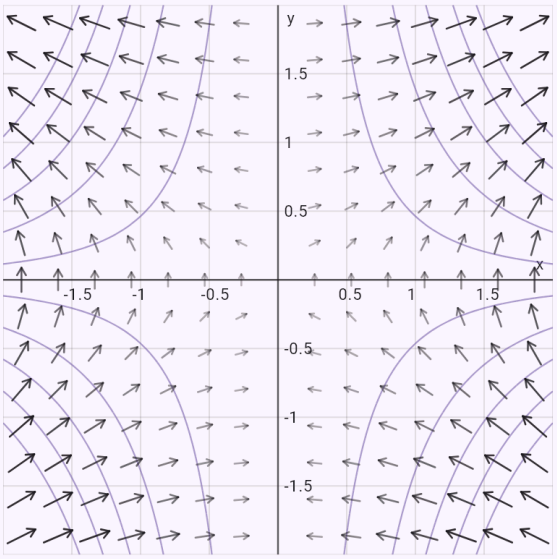
For this field compute:

divergence | curl | ✓ potential

Steps

- The vector field and its components:
  $\vec{F} = (2xy;\ x^2), \quad P = 2xy,\ Q = x^2$
- The curl is zero (in the plane ∂Q/∂x = ∂P/∂y), the field is conservative, so looking for a potential makes sense:
  $\frac{\partial Q}{\partial x} = 2x, \quad \frac{\partial P}{\partial y} = 2x$
- We look for φ with ∇φ = F. We integrate P with respect to x; the constant of integration may depend on the other variables: $\varphi = \int P\,dx = \int 2xy\,dx = x^2 y + C_1(y)$
- We differentiate φ with respect to y and compare with the component Q – the "constant's" derivative remains
  $\frac{\partial \varphi}{\partial y} = x^2 + C_1'(y) = x^2 \ \Rightarrow\ C_1'(y) = 0$
- We integrate the "constant" with respect to y; the new one may depend only on the remaining variables, at the end it is a number C: $C_1(y) = \int 0\,dy = C$
- Check – the potential's gradient is the given field:
  $\nabla\varphi = (2xy;\ x^2) = \vec{F}$

Graph



The arrows show the field's direction and strength (longer and darker = stronger). Level curves of the potential φ (equipotentials) – the field is perpendicular to them everywhere.

The work of a conservative field – by the parametrization, checked by the potential φ = x²y.

Input: `lineint((2x y; x^2); r(t) = (t; t^3); t = 0..1)`

Result

$$\int_C \vec{F} \cdot d\vec{r} = 1$$

Compute:

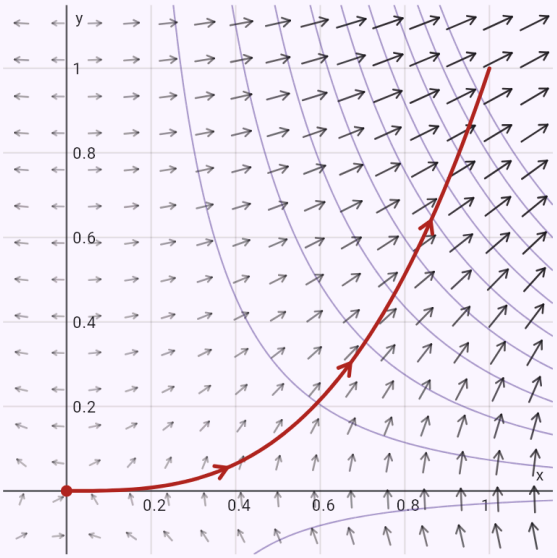
by parametrization | by the potential

Edit in the form

Steps

- A line integral of the second kind of a vector field along the oriented curve C (for example the work of a force along a path): $\int_C \vec{F} \cdot d\vec{r}, \quad \vec{F} = (2xy;\ x^2)$
- The curve's parametrization: $\vec{r}(t) = (t;\ t^3), \quad 0 \le t \le 1$
- The tangent vector – the parametrization's derivative: $\vec{r}'(t) = (1;\ 3t^2)$
- We substitute the parametrization into the field and take the dot product with the tangent vector
  $\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = P\,x' + Q\,y' = 2t^4 + t^2 \cdot 3t^2 = 5t^4$
- What remains is an ordinary definite integral over the parameter's interval: $\int_0^1 5t^4\,dt = 1$
- Check: the field is conservative, it has a potential φ and the integral is independent of the path – the difference φ(B) − φ(A) is enough
  $\varphi = x^2 y, \quad A = [0;\ 0],\ B = [1;\ 1], \quad \varphi(B) - \varphi(A) =$

Graph



The curve C with arrows in the direction of its orientation (the dot = the starting point). The arrows show the field's direction and strength (longer and darker = stronger). Level curves of the potential φ (equipotentials) – the field is perpendicular to them everywhere.

A circulation around the unit circle from a photo, checked by Green's theorem in polar coordinates.

Input: `\oint_C x^2 y \, dx + x \, dy, C: x^2 + y^2 = 1`

Result

$$\oint_C \vec{F} \cdot d\vec{r} = \frac{3\pi}{4}$$

Compute:

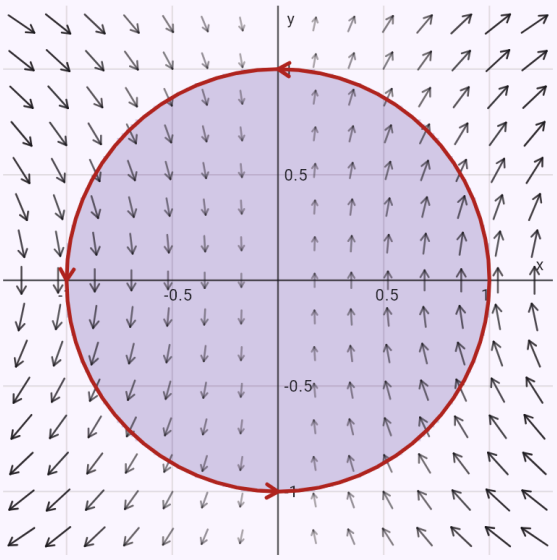
by parametrization | by Green's theorem

Edit in the form

Steps

- The curve is closed – the integral of the second kind around it is the field's circulation:
  $\oint_C \vec{F} \cdot d\vec{r}, \quad \vec{F} = (x^2 y;\ x)$
- The circle with centre S and radius r, traced once (parametrization by the angle t)
  $S = [0;\ 0],\ r = 1: \quad \vec{r}(t) = (\cos t;\ \sin t), \quad 0 \le t \le 2\pi$
- The tangent vector – the parametrization's derivative:
  $\vec{r}\,'(t) = (-\sin t;\ \cos t)$
- We substitute the parametrization into the field and take the dot product with the tangent vector
  $\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = P\,x' + Q\,y' = \cos^2 t \sin t \cdot (-\sin t) + \cdots$
- What remains is an ordinary definite integral over the parameter's interval:
  $\int_0^{2\pi} \cos^4 t \, dt = \frac{3\pi}{4}$
- Check by Green's theorem: the curve is positively oriented (counterclockwise), the integral around it equals the double integral over the region D inside
  $\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA, \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$
- The region D inside the curve – the double integral over it: $D:\ x^2 + y^2 \le 1$
- Polar coordinates: $x = r\cos\varphi,\ y = r\sin\varphi,\ J = r$, r being the Jacobian of the transformation. The integrand: $f \cdot r = -r^3 \cos^2\varphi + r$
- The region in polar coordinates: $0 \le \varphi \le 2\pi,\ 0 \le r \le 1$
- By Fubini's theorem, write it as an iterated integral
  $\int_0^{2\pi} \left(\int_0^1 \left(-r^3 \cos^2\varphi + r\right) dr \right) d\varphi$
- The inner integral with respect to r (the other variable is a constant)
  $\int_0^1 \left(-r^3 \cos^2\varphi + r\right) dr = \left[-\frac{r^4 \cos^2\varphi}{4} + \frac{r^2}{2}\right]_{r=0}^{r=1} = \frac{2 -}{}$
- The outer integral with respect to φ
  $\int_0^{2\pi} \left(\frac{2 - \cos^2\varphi}{4}\right) d\varphi = \left[\frac{3\varphi}{8} - \frac{\cos\varphi \cdot \sin\varphi}{8}\right]_0^{2\pi} = \frac{3\pi}{4}$
- Both sides of Green's theorem agree: $\oint_C \vec{F} \cdot d\vec{r} = \frac{3\pi}{4}$

Graph



The curve C with arrows in the direction of its orientation (the dot = the starting point). The arrows show the field's direction and strength (longer and darker = stronger). Shaded is the region D inside the curve from Green's theorem.

The length of a parabola – the integral of 1 of the first kind.

Input: `lineint(1; graph(y = x^2; x = 0..1))`

Result

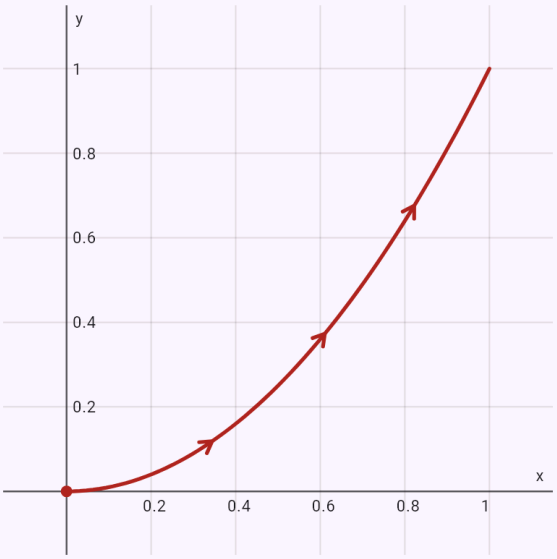
$$\int_C ds = \frac{\sqrt{5}}{2} + \frac{\operatorname{arsinh} 2}{4}$$

Edit in the form

Steps

- A line integral of the first kind of f along the curve C – we add up f times the length ds of small pieces of the curve:
  $\int_C ds$
- A function's graph – x itself is the parameter (parametrization r(x) = (x; f(x))):
  $y = x^2: \quad \vec{r}(x) = \left(x;\ x^2\right), \quad 0 \le x \le 1$
- The tangent vector – the parametrization's derivative:
  $\vec{r}'(x) = (1;\ 2x)$
- Its length; the arc length element is ds = |r'(t)| dt:
  $|\vec{r}'(x)| = \sqrt{1^2 + 2x^2} = \sqrt{4x^2 + 1}$
- We substitute the parametrization into f and multiply by the tangent vector's length:
  $f(\vec{r}(x)) \cdot |\vec{r}'(x)| = \sqrt{4x^2 + 1}$
- What remains is an ordinary definite integral over the parameter's interval
  $\int_0^1 \left(\sqrt{4x^2 + 1}\right) dx = \frac{2\sqrt{5} + \operatorname{arsinh} 2}{4}$
- The integral of 1 ds is the curve's length (with a density f instead of 1 it would be the mass of a wire).

Graph



The curve C with arrows in the direction of its orientation (the dot = the starting point).

A helix in space – a rotatable 3D graph with the field's arrows.

Input: `lineint((-y; x; z); r(t) = (cos t; sin t; t); t = 0..2π)`

Result

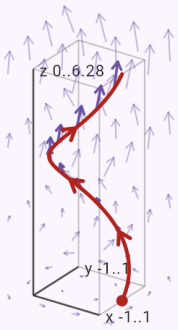
$$\int_C \vec{F} \cdot d\vec{r} = 2\pi + 2\pi^2$$

Edit in the form

Steps

- A line integral of the second kind of a vector field along the oriented curve C (for example the work of a force along a path): $\int_C \vec{F} \cdot d\vec{r}, \quad \vec{F} = (-y;\ x;\ z)$
- The curve's parametrization: $\vec{r}(t) = (\cos t;\ \sin t;\ t), \quad 0 \le t \le 2\pi$
- The tangent vector – the parametrization's derivative: $\vec{r}'(t) = (-\sin t;\ \cos t;\ 1)$
- We substitute the parametrization into the field and take the dot product with the tangent vector $\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = P\,x' + Q\,y' + R\,z' = (-\sin t) \cdot (-\sin$
- What remains is an ordinary definite integral over the parameter's interval: $\int_0^{2\pi} (t+1)\,dt = 2\pi(\pi+1)$

Graph



The curve C with arrows in the direction of its orientation (the dot = the starting point). The arrows show the field, stronger on the curve. Turn it by dragging a finger.

A semicircle's centroid – the mass and two static moments.

Input: `lineint(1; r(t) = (cos t; sin t); t = 0..π; quantity=centroid)`

Result

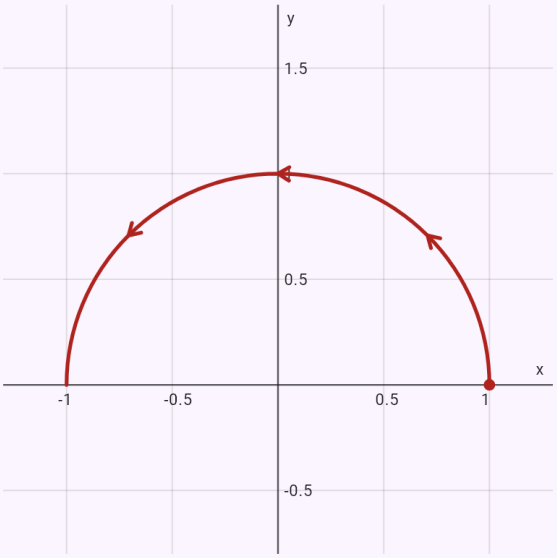
$$T = \left[0;\ \frac{2}{\pi}\right]$$

Edit in the form

Steps

- A line integral of the first kind of f along the curve C – we add up f times the length ds of small pieces of the curve: $\int_C ds$
- The wire's centroid: the static moments (the integral of a coordinate times the density) over the mass, for y (and z) likewise
  $T = \left[\frac{S_x}{m};\ \frac{S_y}{m}\right], \quad m = \int_C ds, \quad S_x = \int_C x ds$
- The curve's parametrization:
  $\vec{r}(t) = (\cos t;\ \sin t), \quad 0 \le t \le \pi$
- The tangent vector – the parametrization's derivative:
  $\vec{r}'(t) = (-\sin t;\ \cos t)$
- Its length; the arc length element is ds = |r'(t)| dt:
  $|\vec{r}'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$
- We substitute the parametrization into f and multiply by the tangent vector's length: $f(\vec{r}(t)) \cdot |\vec{r}'(t)| = 1$
- What remains is an ordinary definite integral over the parameter's interval: $\int_0^\pi 1\, dt = \pi$
- The static moment for the coordinate x – the same steps with the integrand $x \cdot \rho$: $S_x = \int_C x\, ds = 0$
- The static moment for the coordinate y – the same steps with the integrand $y \cdot \rho$: $S_y = \int_C y\, ds = 2$
- The centroid's coordinates are the moments over the mass: $T = \left[0;\ \frac{2}{\pi}\right]$

Graph



The curve C with arrows in the direction of its orientation (the dot = the starting point).

An ellipse's area by the planimeter: ½∮ x dy − y dx = πab.

Input: `lineint(area; r(t) = (3 cos t; 2 sin t); t = 0..2π)`

Result

$$A = 6\pi$$

Compute:

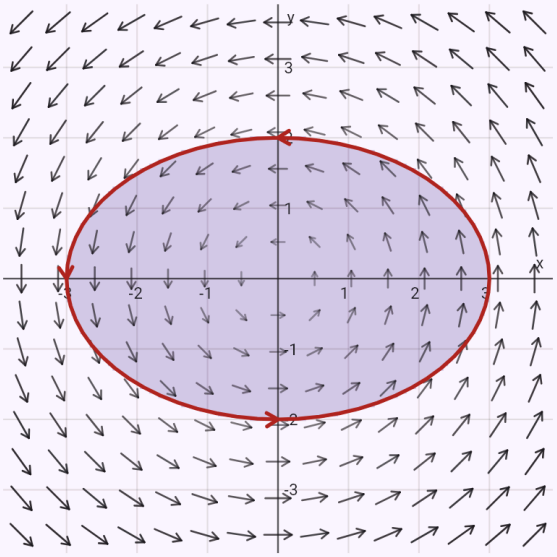
✓ by parametrization | by Green's theorem

Edit in the form

Steps

- The curve is closed – the integral of the second kind around it is the field's circulation:
  $\oint_C \vec{F}\cdot d\vec{r}, \quad \vec{F} = \left(-\frac{y}{2};\ \frac{x}{2}\right)$
- The area inside a closed curve as a line integral around its boundary (the planimeter) – by Green's theorem with Q_x − P_y = 1: $A = \frac{1}{2}\oint_C x\,dy - y\,dx$
- The curve's parametrization:
  $\vec{r}(t) = (3\cos t;\ 2\sin t), \quad 0 \le t \le 2\pi$
- The tangent vector – the parametrization's derivative:
  $\vec{r}'(t) = (-3\sin t;\ 2\cos t)$
- We substitute the parametrization into the field and take the dot product with the tangent vector
  $\vec{F}(\vec{r}(t))\cdot\vec{r}'(t) = P\,x' + Q\,y' = (-\sin t)\cdot(-3\sin t) + \ldots$
- What remains is an ordinary definite integral over the parameter's interval: $\int_0^{2\pi} 3\,dt = 6\pi$
- Check by Green's theorem: the curve is positively oriented (counterclockwise), the integral around it equals the double integral over the region D inside
  $\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA, \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$
- The region D inside the curve – the double integral over it: $D:\ \frac{x^2}{9} + \frac{y^2}{4} \le 1$
- Describe the region as a normal region:
  $-3 \le x \le 3,\ -\frac{2\sqrt{9-x^2}}{3} \le y \le \frac{2\sqrt{9-x^2}}{3}$
- By Fubini's theorem, write it as an iterated integral
  $\int_{-3}^{3}\left(\int_{-\frac{2\sqrt{9-x^2}}{3}}^{\frac{2\sqrt{9-x^2}}{3}} 1\,dy\right)dx$
- The inner integral with respect to y (the other variable is a constant)
  $\int_{-\frac{2\sqrt{9-x^2}}{3}}^{\frac{2\sqrt{9-x^2}}{3}} 1\,dy = [y]_{y=-\frac{2\sqrt{9-x^2}}{3}}^{y=\frac{2\sqrt{9-x^2}}{3}} = \frac{4\sqrt{9-x^2}}{3}$
- The outer integral with respect to x
  $\int_{-3}^{3}\left(\frac{4\sqrt{9-x^2}}{3}\right)dx = \left[\frac{2x\sqrt{9-x^2}}{3} + 6\arcsin\left(\frac{x}{3}\right)\right]_{-3}^{3}$
- Both sides of Green's theorem agree: $\oint_C \vec{F}\cdot d\vec{r} = 6\pi$

Graph



The curve C with arrows in the direction of its orientation (the dot = the starting point). The arrows show the field's direction and strength (longer and darker = stronger). Shaded is the region D inside the curve from Green's theorem.

The upper half-ball in spherical coordinates – the Jacobian ρ² sin θ, the limits of ρ, θ, φ.

Input: `tint(z; x^2 + y^2 + z^2 <= 4; z >= 0; sph)`

Result

$$4\pi \approx 12.566$$

Compute in coordinates:

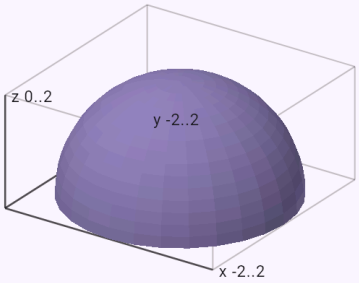
Cartesian | cylindrical | ✓ spherical

Edit in the form

Steps

- The triple integral of the function over the solid V:
$$\iiint_V z\,dV$$
- Spherical coordinates – J = ρ² sin θ is the Jacobian of the transformation (dV = ρ² sin θ dρ dθ dφ); the substitution and the integrand
$$x = \rho\sin\theta\cos\varphi,\ y = \rho\sin\theta\sin\varphi,\ z = \rho\cos\theta$$
$$f \cdot \rho^2\sin\theta = \rho^3\cos\theta\cdot\sin\theta$$
- The solid in spherical coordinates:
$$0 \le \varphi \le 2\pi,\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le \rho \le 2$$
- By Fubini's theorem, write it as a threefold iterated integral
$$\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{2}\rho^3\cos\theta\cdot\sin\theta\,d\rho\,d\theta\,d\varphi$$
- The innermost integral with respect to ρ (the other variables are constants)
$$\int_0^{2}\rho^3\cos\theta\cdot\sin\theta\,d\rho = \left[\frac{\rho^4\cos\theta\cdot\sin\theta}{4}\right]_{\rho=0}^{\rho=2} = 4\cos\theta\cdot\sin\theta$$
- The middle integral with respect to θ
$$\int_0^{\frac{\pi}{2}}4\cos\theta\cdot\sin\theta\,d\theta = \left[2\sin^2\theta\right]_{\theta=0}^{\theta=\frac{\pi}{2}} = 2$$
- The outer integral with respect to φ:
$$\int_0^{2\pi}2\,d\varphi = [2\varphi]_0^{2\pi} = 4\pi$$

Graph



The solid V integrated over. Turn it by dragging a finger.

A tetrahedron by iterated limits – three integrals one after another.

Input: `\int_0^1 \int_0^{1-x} \int_0^{1-x-y} x \, dz \, dy \, dx`

Result

$$\frac{1}{24}$$

Compute in coordinates:

✓ Cartesian | cylindrical | spherical

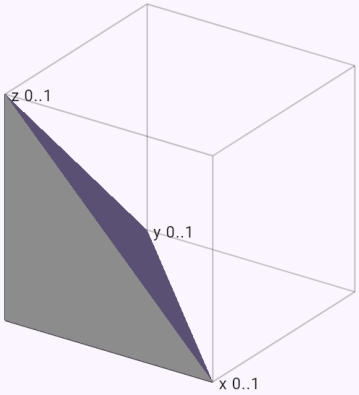
Edit in the form

Steps

- The triple integral of the function over the solid V:
  $$\iiint_V x \, dV$$
- Describe the solid as a normal region – z lies between two surfaces over the solid's projection:
  $0 \le x \le 1,\ 0 \le y \le 1 - x,\ 0 \le z \le -x - y + 1$
- By Fubini's theorem, write it as a threefold iterated integral:
  $$\int_0^1 \int_0^{1-x} \int_0^{-x-y+1} x \, dz \, dy \, dx$$
- The innermost integral with respect to z (the other variables are constants):
  $$\int_0^{-x-y+1} x \, dz = [xz]_{z=0}^{z=-x-y+1} = x(-x - y + 1)$$
- The middle integral with respect to y
  $$\int_0^{1-x} (x(-x - y + 1)) \, dy = \left[-\frac{x(x + y - 1)^2}{2} \right]_{y=0}^{y=1-x} =$$
- The outer integral with respect to x

$$\int_0^1 \left(\frac{x(x-1)^2}{2} \right) dx = \left[\frac{x^4}{8} - \frac{x^3}{3} + \frac{x^2}{4} \right]_0^1 = \frac{1}{24}$$

Graph



The solid V integrated over. Turn it by dragging a finger.

The volume of a cone in cylindrical coordinates.

Input: `tint(1; cone(1; 2); cyl; quantity=volume)`

Result

$$V = \frac{2\pi}{3} \approx 2.0944$$

Compute in coordinates:

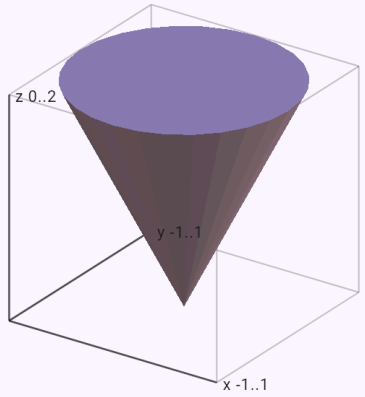
Cartesian | ✓ cylindrical

Edit in the form

Steps

- The solid's volume is the triple integral of one:
  $V = \iiint_V dV$
- Cylindrical coordinates − J = r is the Jacobian of the transformation (dV = r dr dφ dz); the substitution and the integrand
  $x = r\cos\varphi,\ y = r\sin\varphi,\ z = z$
  $f \cdot r = r$
- The solid in cylindrical coordinates:
  $0 \le \varphi \le 2\pi,\ 0 \le r \le 1,\ 2r \le z \le 2$
- By Fubini's theorem, write it as a threefold iterated integral: $\int_0^{2\pi}\int_0^1\int_{2r}^2 r\,dz\,dr\,d\varphi$
- The innermost integral with respect to z (the other variables are constants):
  $\int_{2r}^2 r\,dz = [zr]_{z=2r}^{z=2} = r(2-2r)$
- The middle integral with respect to r
  $\int_0^1 (r(2-2r))\,dr = \left[-\frac{2r^3}{3} + r^2\right]_{r=0}^{r=1} = \frac{1}{3}$
- The outer integral with respect to φ
  $\int_0^{2\pi} \frac{1}{3}\,d\varphi = \left[\frac{\varphi}{3}\right]_0^{2\pi} = \frac{2\pi}{3}$

Graph



The solid V integrated over. Turn it by dragging a finger.

The centroid of a half-ball – the mass and three first moments.

Input: `tint(1; x^2 + y^2 + z^2 <= 1; z >= 0; quantity=centroid)`

Result

$$T\left[0;\ 0;\ \frac{3}{8}\right]$$

The solid or the function contains x² + y² + z² – spherical coordinates are often simpler:

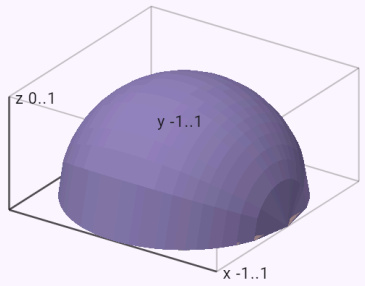
✓ Cartesian | cylindrical | spherical

Edit in the form

Steps

- The solid's centroid: each coordinate is a first moment (the integral of the coordinate times the density) divided by the mass m, alike for y and z:
$$x_T = \frac{1}{m}\iiint_V x\,dV,\ m = \iiint_V dV$$
- Describe the solid as a normal region – z lies between two surfaces over the solid's projection
$$-1 \le x \le 1,\ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2},\ 0 \le z \le \sqrt{-}$$
- By Fubini's theorem, write it as a threefold iterated integral
$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{-x^2-y^2+1}} dz\,dy\,dx$$
- The innermost integral with respect to z (the other variables are constants)
$$\int_{0}^{\sqrt{-x^2-y^2+1}} dz = [z]_{z=0}^{z=\sqrt{-x^2-y^2+1}} = \sqrt{-x^2-y^2+1}$$
- The middle integral with respect to y
$$\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(\sqrt{-x^2-y^2+1}\right) dy = \left[\frac{(1-x^2)\arcsin\left(\frac{\sqrt{}}{2}\right.\right.$$
- The outer integral with respect to x
$$\int_{-1}^{1}\left(\frac{\pi(1-x^2)}{2}\right)dx = \left[-\frac{\pi x^3}{6}+\frac{\pi x}{2}\right]_{-1}^{1} = \frac{2\pi}{3}$$
- The first moment for the coordinate x – the same steps with the integrand $x \cdot \rho$
$$\iiint_V x\,dV = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{-x^2-y^2+1}} x\,dz\,dy\,dx = 0$$
- The first moment for the coordinate y – the same steps with the integrand $y \cdot \rho$
$$\iiint_V y\,dV = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{-x^2-y^2+1}} y\,dz\,dy\,dx = 0$$
- The first moment for the coordinate z – the same steps with the integrand $z \cdot \rho$
$$\iiint_V z\,dV = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{-x^2-y^2+1}} z\,dz\,dy\,dx = \frac{\pi}{4}$$
- The centroid's coordinates are the moments divided by the mass
$$x_T = \frac{0}{\frac{2\pi}{3}} = 0$$
$$y_T = \frac{0}{\frac{2\pi}{3}} = 0$$
$$z_T = \frac{\frac{\pi}{4}}{\frac{2\pi}{3}} = \frac{3}{8}$$

Graph



The solid V integrated over. Turn it by dragging a finger.

The moment of inertia of a cylinder about its axis.

Input: `tint(1; cylinder(1; 2); cyl; quantity=inertia; axis=z)`

Result

$$I_z = \pi \approx 3.1416$$

Compute in coordinates:

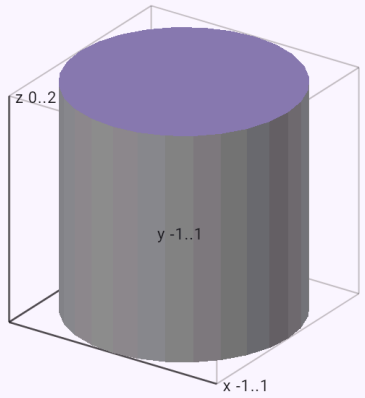
Cartesian | ✓ cylindrical

Edit in the form

Steps

- The moment of inertia about the z-axis – the density times the squared distance d from the axis:
  $$I_z = \iiint_V d^2\, dV = \iiint_V \left(x^2 + y^2\right) dV$$
- Cylindrical coordinates – J = r is the Jacobian of the transformation (dV = r dr dφ dz); the substitution and the integrand
  $$x = r\cos\varphi,\ y = r\sin\varphi,\ z = z$$
  $$f \cdot r = r^3$$
- The solid in cylindrical coordinates:
  $$0 \le \varphi \le 2\pi,\ 0 \le r \le 1,\ 0 \le z \le 2$$
- By Fubini's theorem, write it as a threefold iterated integral: $\int_0^{2\pi}\int_0^1\int_0^2 r^3\, dz\, dr\, d\varphi$
- The innermost integral with respect to z (the other variables are constants): $\int_0^2 r^3\, dz = \left[zr^3\right]_{z=0}^{z=2} = 2r^3$
- The middle integral with respect to r:
  $$\int_0^1 2r^3\, dr = \left[\frac{r^4}{2}\right]_{r=0}^{r=1} = \frac{1}{2}$$
- The outer integral with respect to φ
  $$\int_0^{2\pi} \frac{1}{2}\, d\varphi = \left[\frac{\varphi}{2}\right]_0^{2\pi} = \pi$$

Graph



The solid V integrated over. Turn it by dragging a finger.

The outward flux through a sphere – parametrized by the angles θ, φ and checked by Gauss's theorem.

Input: `surfint((x; y; z); sphere(2))`

Result

$$\oiint_S \vec{F}\cdot\vec{n}\,dS = 32\pi$$

Compute:

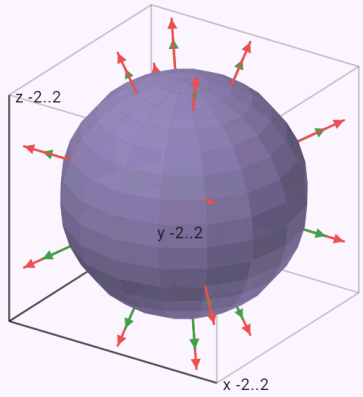
by the parametrization | by Gauss's theorem

Edit in the form

Steps

- The flux of the vector field through the surface S – a surface integral of the second kind:
  $\oiint_S \vec{F}\cdot\vec{n}\,dS, \quad \vec{F} = (x;\ y;\ z)$
- A parametrization of the sphere by the angles θ (from the positive z-axis) and φ (about the z-axis)
  $\vec{r}(\theta,\varphi) = (2\cos\varphi\cdot\sin\theta;\ 2\sin\varphi\cdot\sin\theta;\ 2\cos\theta), \quad 0 \le$
- The partial derivatives by the parameters – the surface's tangent vectors
  $\vec{r}_\theta = (2\cos\varphi\cdot\cos\theta;\ 2\cos\theta\cdot\sin\varphi;\ -2\sin\theta), \quad \vec{r}_\varphi = ($
- The normal vector (the tangent vectors' cross product) oriented out of the solid
  $\vec{N} = \vec{r}_\theta\times\vec{r}_\varphi = \left(4\sin^2\theta\cos\varphi;\ 4\sin^2\theta\sin\varphi;\ 4\cos\theta\cdot\mathrm{si}\right.$
- Substitute the parametrization into the field and take the dot product with the normal vector (F · n dS = F · N du dv): $\vec{F}(\vec{r})\cdot\vec{N} = 8\sin\theta$
- The double integral over the parameters' rectangle
  $\int_0^{\pi}\int_0^{2\pi} 8\sin\theta\,d\varphi\,d\theta = \int_0^{\pi} 16\pi\sin\theta\,d\theta = 32\pi$
- Check by Gauss's theorem: the outward flux through the closed surface is the triple integral of the divergence over the solid inside
- The triple integral of the function over the solid V:
  $\iiint_V 3\,dV$
- Describe the solid as a normal region – z lies between two surfaces over the solid's projection
  $-2 \le x \le 2,\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2},\ -\sqrt{-x^2-y}$
- By Fubini's theorem, write it as a threefold iterated integral
  $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{-\sqrt{-x^2-y^2+4}}^{\sqrt{-x^2-y^2+4}} 3\,dz\,dy\,dx$
- The innermost integral with respect to z (the other variables are constants)
  $\int_{-\sqrt{-x^2-y^2+4}}^{\sqrt{-x^2-y^2+4}} 3\,dz = [3z]_{z=-\sqrt{-x^2-y^2+4}}^{z=\sqrt{-x^2-y^2+4}} = 6\sqrt{-x^2-y^2}$
- The middle integral with respect to y
  $\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(6\sqrt{-x^2-y^2+4}\right)dy = \left[\left(12-3x^2\right)\arcsin\right.$
- The outer integral with respect to x
  $\int_{-2}^{2}\left(\pi\left(12-3x^2\right)\right)dx = \left[-\pi x^3+12\pi x\right]_{-2}^{2} = 32\pi$
- Gauss's theorem gives the same value:
  $\oiint_S \vec{F}\cdot\vec{n}\,dS = 32\pi$

Graph



The surface S with its normals (green) and the field's arrows (red). Turn it by dragging a finger.

The area of a paraboloid – the graph's surface element, a double integral in polar coordinates.

Input: `surfint(1; graph(z = x^2 + y^2; x^2 + y^2 <= 1))`

Result

$$\iint_S dS = \frac{\left(-1+5\sqrt{5}\right)\pi}{6}$$

Edit in the form

Steps

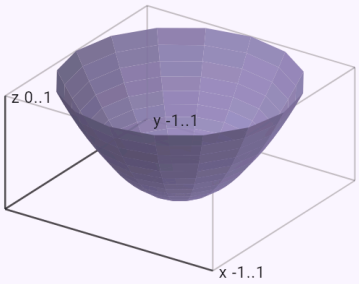
- The area of the surface S is the surface integral of one: $\iint_S dS$
- The surface is the graph of z = g(x, y) over the region D in the xy-plane: $z = g(x,y) = x^2 + y^2, \quad D: \ x^2 + y^2 \le 1$
- The partial derivatives of g: $g_x = 2x, \quad g_y = 2y$
- The graph's surface element: $dS = \sqrt{1 + g_x^2 + g_y^2}\, dA = \sqrt{4x^2 + 4y^2 + 1}\, dA$
- Substitute the parametrization into f and multiply by the surface element: $f(x,y,g(x,y))\sqrt{1 + g_x^2 + g_y^2} = \sqrt{4x^2 + 4y^2 + 1}$
- The surface integral over a graph is a double integral over its projection D: $\iint_S \cdots = \iint_D \left(\sqrt{4x^2 + 4y^2 + 1}\right) dA$
- Polar coordinates: $x = r\cos\varphi, \ y = r\sin\varphi, \ J = r$, r being the Jacobian of the transformation. The integrand: $f \cdot r = r\sqrt{4r^2 + 1}$
- The region in polar coordinates: $0 \le \varphi \le 2\pi, \ 0 \le r \le 1$
- By Fubini's theorem, write it as an iterated integral: $\int_0^{2\pi} \left(\int_0^1 \left(r\sqrt{4r^2 + 1} \right) dr \right) d\varphi$
- The inner integral with respect to r (the other variable is a constant)

$$\int_0^1 \left(r\sqrt{4r^2 + 1} \right) dr = \left[\frac{\sqrt{\left(4r^2 + 1\right)^3}}{12} \right]_{r=0}^{r=1} = \frac{-1 + 5\sqrt{}}{12}$$

- The outer integral with respect to φ

$$\int_0^{2\pi} \left(\frac{-1 + 5\sqrt{5}}{12} \right) d\varphi = \left[\frac{\left(-1 + 5\sqrt{5}\right)\varphi}{12} \right]_0^{2\pi} = \frac{(-1 +}{}$$

Graph



The surface S integrated over. Turn it by dragging a finger.

The flux through a closed cylinder by Gauss's theorem, checked part by part – the lateral surface and

two discs.

Input: `surfint((x; y; z); cylinder(1; 2); method=gauss)`

Result

$$\oiint_S \vec{F} \cdot \vec{n}\, dS = 6\pi$$

Compute:

by the parametrization | ✓ by Gauss's theorem

Edit in the form

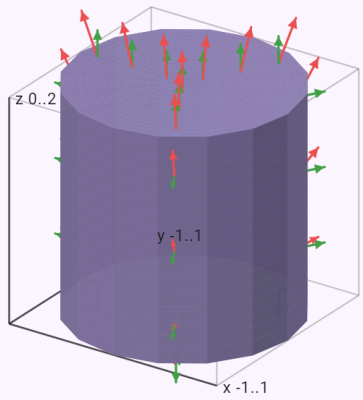
Steps

- The flux of the vector field through the surface S – a surface integral of the second kind:
  $\oiint_S \vec{F} \cdot \vec{n}\, dS, \quad \vec{F} = (x;\ y;\ z)$
- By Gauss's theorem, the outward flux through a closed surface equals the triple integral of the divergence over the solid V inside
- The triple integral of the function over the solid V:
  $\iiint_V 3\, dV$
- Describe the solid as a normal region – z lies between two surfaces over the solid's projection:
  $-1 \le x \le 1,\ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2},\ 0 \le z \le 2$
- By Fubini's theorem, write it as a threefold iterated integral: $\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{0}^{2} 3\, dz\, dy\, dx$
- The innermost integral with respect to z (the other variables are constants): $\int_0^2 3\, dz = [3z]_{z=0}^{z=2} = 6$
- The middle integral with respect to y

$$\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 6\, dy = [6y]_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} = 12\sqrt{1-x^2}$$

- The outer integral with respect to x
  $\int_{-1}^{1} \left(12\sqrt{1-x^2}\right) dx = \left[6x\sqrt{1-x^2} + 6\arcsin x\right]_{-1}^{1} =$
- The check by the surface's parametrization gives the same value: $\oiint_S \vec{F} \cdot \vec{n}\, dS = 6\pi$

Graph



The surface S with its normals (green) and the field's arrows (red). Turn it by dragging a finger.

The flux of a curl through a hemisphere – checked by Stokes' theorem around the equator.

Input: `surfint(curl((-y; x; 0)); hemisphere(1))`

Result

$$\iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}\, dS = 2\pi$$

Compute:

by the parametrization | by Stokes' theorem

Edit in the form

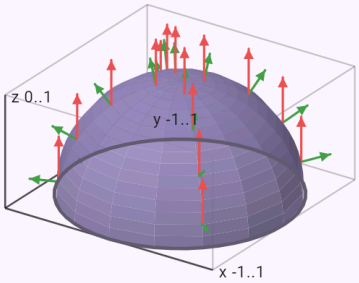
Steps

- The flux of the field's curl through S – a surface integral of the second kind, also computable by Stokes' theorem
$$\iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}\, dS, \quad \vec{F} = (-y;\ x;\ 0)$$
- The field's curl
$$\nabla \times \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z};\ \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x};\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = (0;$$
- A parametrization of the upper hemisphere – θ only up to π/2
$$\vec{r}(\theta, \varphi) = (\cos\varphi \cdot \sin\theta;\ \sin\varphi \cdot \sin\theta;\ \cos\theta), \quad 0 \le \theta \le$$
- The partial derivatives by the parameters – the surface's tangent vectors
$$\vec{r}_\theta = (\cos\varphi \cdot \cos\theta;\ \cos\theta \cdot \sin\varphi;\ -\sin\theta), \quad \vec{r}_\varphi = (-\text{sir}$$
- The normal vector (the tangent vectors' cross product) oriented out of the solid
$$\vec{N} = \vec{r}_\theta \times \vec{r}_\varphi = \left(\sin^2\theta\cos\varphi;\ \sin^2\theta\sin\varphi;\ \cos\theta \cdot \sin\theta\right)$$
- Substitute the parametrization into the field and take the dot product with the normal vector (F · n dS = F · N du dv): $\vec{F}(\vec{r}) \cdot \vec{N} = \sin(2\theta)$
- The double integral over the parameters' rectangle
$$\int_0^{\frac{\pi}{2}} \int_0^{2\pi} \sin(2\theta)\, d\varphi\, d\theta = \int_0^{\frac{\pi}{2}} 2\pi \sin(2\theta)\, d\theta = 2\pi$$
- Check by Stokes' theorem: the flux of the curl equals the circulation of the field around the boundary (right-hand rule)

$$\iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}\, dS = \oint_{\partial S} \vec{F} \cdot d\vec{r}$$

- The edges of the parameters' rectangle that map to a point or are traversed twice in opposite directions (a seam) cancel – the surface's real boundary remains.
- A part of the boundary
$$\vec{r}(\varphi) = (\cos\varphi;\ \sin\varphi;\ 0), \quad \varphi:\ 0 \to 2\pi$$
- The line integral along it:
$$\vec{F}(\vec{r}) \cdot \vec{r}' = 1, \quad \int_0^{2\pi} 1\, d\varphi = 2\pi$$
- Stokes' theorem gives the same value:
$$\iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}\, dS = 2\pi$$

Graph



The surface S with its normals (green) and the field's arrows (red). The boundary with an arrow of its direction. Turn it by dragging a finger.

A surface integral of the first kind over a hemisphere given by its equation.

Input: `\iint_S z \, dS, S: x^2 + y^2 + z^2 = 4, z \ge 0`

Result

$$\iint_S z \, dS = 8\pi$$

Edit in the form

Steps

- The surface integral of the first kind of f over the surface S: $\iint_S z \, dS$
- A parametrization of the upper hemisphere – θ only up to π/2

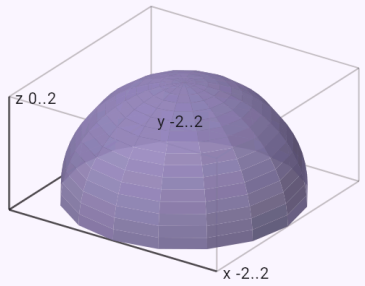
  $\vec{r}(\theta, \varphi) = (2\cos\varphi \cdot \sin\theta;\ 2\sin\varphi \cdot \sin\theta;\ 2\cos\theta), \quad 0 \le$
- The partial derivatives by the parameters – the surface's tangent vectors

  $\vec{r}_\theta = (2\cos\varphi \cdot \cos\theta;\ 2\cos\theta \cdot \sin\varphi;\ -2\sin\theta), \quad \vec{r}_\varphi = ($
- The cross product of the tangent vectors is a normal vector, its length gives the surface element dS = |r_u × r_v| du dv

  $\vec{r}_\theta \times \vec{r}_\varphi = \left(4\sin^2\theta\cos\varphi;\ 4\sin^2\theta\sin\varphi;\ 4\cos\theta \cdot \sin\theta\right),$
- Substitute the parametrization into f and multiply by the surface element: $f(\vec{r})\,|\vec{r}_\theta \times \vec{r}_\varphi| = 8\cos\theta \cdot \sin\theta$
- The double integral over the parameters' rectangle

$$\int_0^{\frac{\pi}{2}} \int_0^{2\pi} 8\cos\theta \cdot \sin\theta \, d\varphi \, d\theta = \int_0^{\frac{\pi}{2}} 8\pi \sin(2\theta) \, d\theta = 8\pi$$

Graph



The surface S integrated over. Turn it by dragging a finger.

A spherical shell's moment of inertia about the z-axis.

Input: `surfint(1; sphere(1); quantity=inertia; axis=z)`

Result

$$I_z = \frac{8\pi}{3}$$

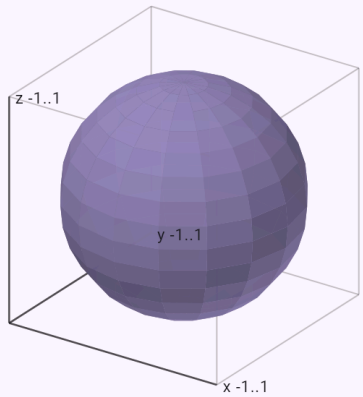
Edit in the form

Steps

- The surface integral of the first kind of f over the surface S: $\iint_S (x^2+y^2)\,dS$
- The shell's moment of inertia about the axis z – the density times the squared distance d from the axis: $I_z = \iint_S d^2\,dS, \quad d^2 = x^2+y^2$
- A parametrization of the sphere by the angles θ (from the positive z-axis) and φ (about the z-axis) $\vec{r}(\theta,\varphi) = (\cos\varphi\cdot\sin\theta;\ \sin\varphi\cdot\sin\theta;\ \cos\theta), \quad 0 \le \theta \le$
- The partial derivatives by the parameters – the surface's tangent vectors $\vec{r}_\theta = (\cos\varphi\cdot\cos\theta;\ \cos\theta\cdot\sin\varphi;\ -\sin\theta), \quad \vec{r}_\varphi = (-\sin$
- The cross product of the tangent vectors is a normal vector, its length gives the surface element dS = |r_u × r_v| du dv $\vec{r}_\theta \times \vec{r}_\varphi = \left(\sin^2\theta\cos\varphi;\ \sin^2\theta\sin\varphi;\ \cos\theta\cdot\sin\theta\right), \quad |\vec{r}_\theta$
- Substitute the parametrization into f and multiply by the surface element: $f(\vec{r})\,|\vec{r}_\theta \times \vec{r}_\varphi| = \sin^3\theta$
- The double integral over the parameters' rectangle

$$\int_0^{\pi}\int_0^{2\pi} \sin^3\theta\,d\varphi\,d\theta = \int_0^{\pi} 2\pi\sin^3\theta\,d\theta = \frac{8\pi}{3}$$

Graph



The surface S integrated over. Turn it by dragging a finger.

A map of the integral theorems

All the integral theorems say the same: the integral of a derivative over a region equals an integral over its boundary. The boundary of a segment is two points, of a surface a curve, of a solid a closed surface; the generalized Stokes theorem ∫_Ω dω = ∫_∂Ω ω says it all at once. The app uses the theorems as a second way of computing and as a check (the chips under a result); the glossary shows the same map under "Integral theorems".

How to enter it

`∫ f'(x) dx (a..b) = f(b) − f(a)`	The Newton–Leibniz formula – inside: the derivative on [a, b], on the boundary: the values at the endpoints.
`∫_C ∇ϕ · dr = ϕ(B) − ϕ(A)`	The gradient theorem (a potential, path independence) – inside: the gradient along a curve, on the boundary:

	the potential at its ends.
`∬_D (Q_x − P_y) dA = ∮_∂D P dx + Q dy`	Green’s theorem – inside: ∂Q/∂x − ∂P/∂y over a plane region, on the boundary: the circulation around the closed curve.
`∬_S (∇ × F) · n dS = ∮_∂S F · dr`	Stokes’ theorem – inside: the flux of the curl through a surface, on the boundary: the circulation around its edge.
`∭_V div F dV = ∯_∂V F · n dS`	Gauss’s theorem – inside: the divergence over a solid, on the boundary: the outward flux through the closed surface.

Physical applications of the integral theorems

The “Applications” section (the Physics and Multivariable calculus fields, electricity also in Electrical engineering, flow and heat in Engineering) has models with units: Gauss's law (a sphere, a line, a plane), Ampère's and the Biot–Savart law (a wire, a solenoid, a toroid, a current loop), electromagnetic induction (a magnetic flux, a changing field, a generator), a capacitor (parallel-plate, spherical, cylindrical), the energy of a field, work in a force field (gravity, a spring, the Coulomb force), fluid flow (a flow rate, a parabolic profile, the continuity equation, circulation and curl), heat conduction (a plane and a cylindrical wall) and the gravitational field by Gauss's law.

The steps start with the integral law, then explain the symmetry (on the chosen surface or curve the field is constant and points along the normal or the tangent), show the evaluated integral, the enclosed charge, current or mass and the formula to substitute into. The electromagnetic laws come with their differential form – the glossary term “Maxwell's equations” has both forms side by side. The graph shows the field against the distance (E(r), B(r), g(r)), a velocity profile, the temperature in a wall or a generator's voltage.

Examples

The field inside a uniformly charged ball – the enclosed charge Q r³/R³, the field grows linearly.

Input: `model(gauss; find=E[V_m]; Q=1[nC]; R=10[cm]; r=5[cm]; mode=sphere; charge=ball)`

Result

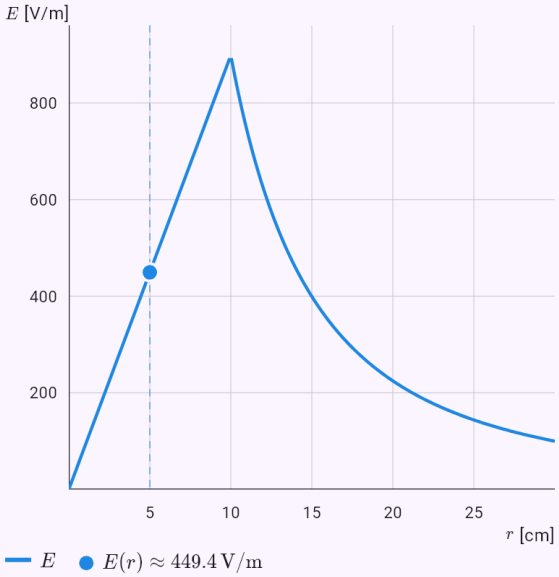
$$E \approx 449.4\,\mathrm{V/m}$$

Steps

- Find E; given: $Q = 1\,\mathrm{nC}, \quad R = 10\,\mathrm{cm}, \quad r = 5\,\mathrm{cm}$
- Converting the units
  $Q = 1\,\mathrm{nC} = 10^{-9}\,\mathrm{C}, \quad R = 10\,\mathrm{cm} = 0.1\,\mathrm{m}, \quad r = 5\,\mathrm{cm}$
- A charge Q distributed symmetrically about the centre – on a sphere's surface (a conducting sphere, a shell) or uniformly through its volume. The Gaussian surface is a concentric sphere of radius r; outside, the charge acts as a point charge. The model's equation: $\oint_S \vec{E} \cdot d\vec{S} = \frac{Q_{\text{in}}}{\varepsilon_0}$
- Symmetry: the field is radial and has the same magnitude on every concentric sphere. We take a sphere of radius r as the Gaussian surface – on it the field is parallel to the normal and constant.
- The integral evaluated: $\oint_S \vec{E} \cdot d\vec{S} = \oint_S E\,dS = E \cdot 4\pi r^2$
- The enclosed charge
  $$Q_{\text{in}} = Q\frac{r^3}{R^3} = 10^{-9} \cdot \frac{(0.05)^3}{(0.1)^3} \approx 1.25 \cdot 10^{-10}\,\mathrm{C}$$
- Expressing the sought quantity
  $$E \cdot 4\pi r^2 = \frac{Q_{\text{in}}}{\varepsilon_0} \Rightarrow E = \frac{Q_{\text{in}}}{4\pi\varepsilon_0 r^2}$$
- Substituting
  $$E = \frac{1.25 \cdot 10^{-10}}{4\pi \cdot 8.854 \cdot 10^{-12} \cdot (0.05)^2} \approx 449.4\,\mathrm{V/m}$$

- The same law in differential form (Maxwell's equations):
  $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

A straight wire's magnetic field by Ampère's law around a circle about the axis.

Input: `model(ampere; find=B[uT]; I=10[A]; R=1[mm]; r=5[cm]; mode=wire)`

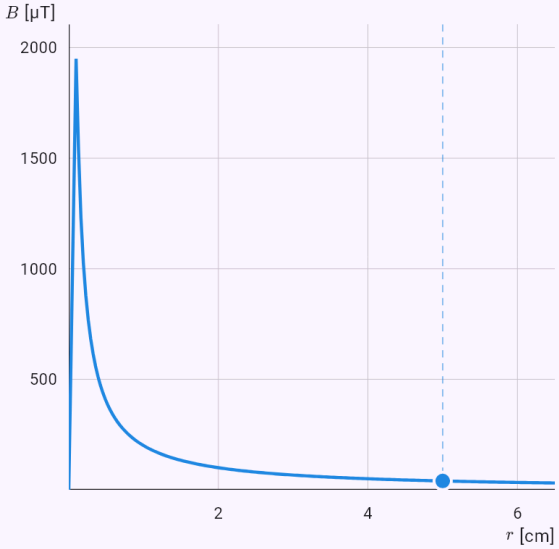
Result

$$B \approx 40\,\mu\mathrm{T}$$

Steps

- Find B; given: $I = 10\,\mathrm{A}, \quad R = 1\,\mathrm{mm}, \quad r = 5\,\mathrm{cm}$
- Converting the units:
  $R = 1\,\mathrm{mm} = 0.001\,\mathrm{m}, \quad r = 5\,\mathrm{cm} = 0.05\,\mathrm{m}$
- A long straight wire of radius R carrying a current I spread uniformly over its cross-section. The curve is a circle around the wire's axis; inside the wire it encloses only part of the current. The model's equation:
  $\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{in}}$
- Symmetry: the field lines are circles around the wire's axis; B is tangent to them and constant on a circle of radius r.
- The integral evaluated: $\oint_C \vec{B} \cdot d\vec{l} = \oint_C B\,dl = B \cdot 2\pi r$
- The enclosed current: $I_{\text{in}} = I$
- Expressing the sought quantity:
  $B \cdot 2\pi r = \mu_0 I_{\text{in}} \Rightarrow B = \dfrac{\mu_0 I_{\text{in}}}{2\pi r}$
- Substituting: $B = \dfrac{1.257 \cdot 10^{-6} \cdot 10}{2\pi \cdot 0.05} \approx 4 \cdot 10^{-5}\,\mathrm{T}$
- The same law in differential form (Maxwell's equations):
  $\nabla \times \vec{B} = \mu_0 \vec{J}$
- Converting the units: $B \approx 4 \cdot 10^{-5}\,\mathrm{T} = 40\,\mu\mathrm{T}$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

The field on a current loop's axis by the Biot–Savart law.

Input: `model(ampere; find=B[uT]; I=5[A]; Rl=10[cm]; z=10[cm]; mode=loop)`

Result

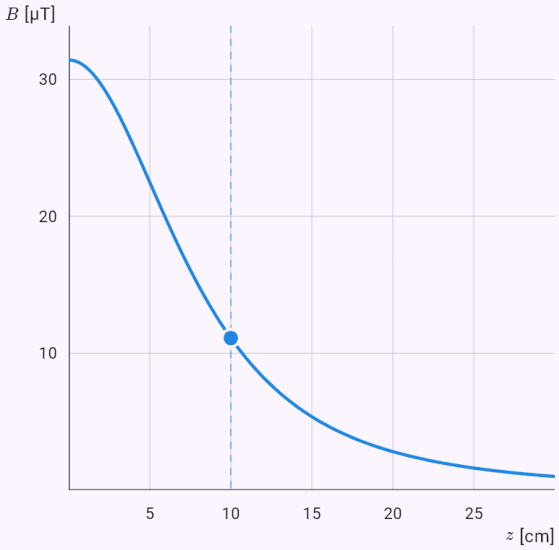
$$B \approx 11.11\,\mu\mathrm{T}$$

Steps

- Find B; given: $I = 5\ \mathrm{A},\quad R = 10\ \mathrm{cm},\quad z = 10\ \mathrm{cm}$
- Converting the units:
  $R = 10\ \mathrm{cm} = 0.1\ \mathrm{m},\quad z = 10\ \mathrm{cm} = 0.1\ \mathrm{m}$
- A circular loop of radius R carrying a current I. The field on its axis at a distance z from the centre follows from the Biot–Savart law – the contributions' perpendicular components cancel, the axial ones add up. The model's equation: $d\vec{B} = \dfrac{\mu_0}{4\pi}\dfrac{I\,d\vec{l}\times\hat{e}}{\rho^2}$
- Every piece dl of the loop is equally far from a point on the axis, ρ = √(R² + z²), and perpendicular to the line joining them. The components perpendicular to the axis cancel around the loop; only the axial ones, dB · R/ρ, add up.
- The integral evaluated
  $$B = \oint_C \frac{\mu_0}{4\pi}\frac{I\,dl}{R^2+z^2}\cdot\frac{R}{\sqrt{R^2+z^2}} = \frac{\mu_0 I}{4\pi}\frac{R}{\left(R^2+z^2\right)^{3/2}}\cdot$$
- Expressing the sought quantity: $B = \dfrac{\mu_0 I R^2}{2\left(R^2+z^2\right)^{3/2}}$
- Substituting
  $$B = \frac{1.257\cdot10^{-6}\cdot5\cdot(0.1)^2}{2\left((0.1)^2+(0.1)^2\right)^{3/2}} \approx 1.111\cdot10^{-5}\ \mathrm{T}$$
- At the loop's centre (z = 0):
  $B(0) = \dfrac{\mu_0 I}{2R} \approx 3.142\cdot10^{-5}\ \mathrm{T}$

- Converting the units: $B \approx 1.111\cdot10^{-5}\ \mathrm{T} = 11.11\,\mu\mathrm{T}$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

A generator's emf amplitude ε = NBAω and its sinusoidal course in the graph.

Input: `model(induction; find=emf[V]; N=200[one]; B=0.1[T]; A=500[cm2]; omega=314.16[rad_s];`

`mode=rotating)`

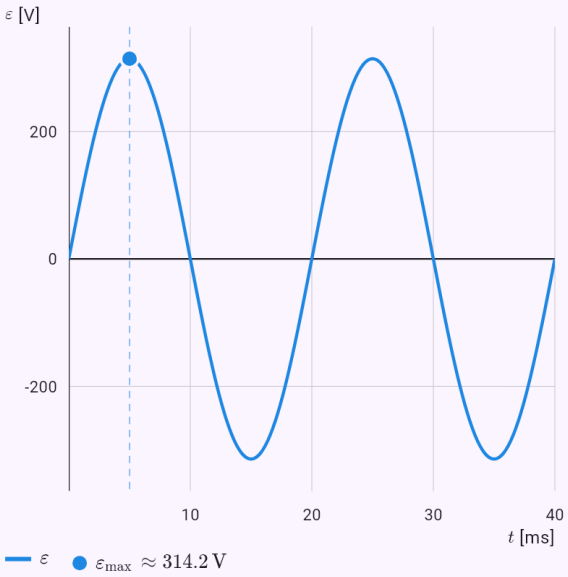
Result

$$\varepsilon_{\max} \approx 314.2\ \text{V}$$

Steps

- Find $\varepsilon_{\max}$; given
  $N = 200, \quad B = 0.1\ \text{T}, \quad A = 500\ \text{cm}^2, \quad \omega = 314.2\ \text{ra}$
- Converting the units: $A = 500\ \text{cm}^2 = 0.05\ \text{m}^2$
- A loop of N turns rotates at an angular speed ω in a uniform field B – the principle of an AC generator. The emf is sinusoidal with the amplitude NBAω. The model's equation: $\oint_C \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\iint_S \vec{B}\cdot d\vec{S}$
- The magnetic flux through one turn: $\Phi(t) = \iint_S \vec{B}\cdot d\vec{S} = BA\cos\omega t$
- The integral evaluated: $\varepsilon(t) = -N\frac{d\Phi}{dt} = NBA\omega\sin\omega t$
- Expressing the sought quantity: $\varepsilon_{\max} = NBA\omega$
- Substituting: $\varepsilon_{\max} = 200\cdot 0.1\cdot 0.05\cdot 314.2 \approx 314.2\ \text{V}$
- The frequency: $f = \frac{\omega}{2\pi} \approx 50\ \text{Hz}$
- The same law in differential form (Maxwell's equations): $\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

Graph



The model's quantities in time, computed from the solution; the answer is marked.

The capacitance of a metre of coaxial cable – the field by Gauss's law, the voltage as the field's integral.

Input: `model(capacitor; find=C[pF]; epsr=2.3[one]; r1=1[mm]; r2=3.5[mm]; L=1[m]; mode=cylindrical)`

Result

$$C \approx 102.1\ \text{pF}$$

Steps

- Find C; given
  $\varepsilon_r = 2.3, \quad R_1 = 1\ \text{mm}, \quad R_2 = 3.5\ \text{mm}, \quad L = 1\ \text{m}$
- Converting the units:
  $R_1 = 1\ \text{mm} = 0.001\ \text{m}, \quad R_2 = 3.5\ \text{mm} = 0.0035\ \text{m}$
- Two coaxial cylindrical electrodes of radii $R_1 < R_2$ and length L – a coaxial cable, for example. The model's equation
  $\oint_S \vec{E}\cdot d\vec{S} = \frac{Q}{\varepsilon_0\varepsilon_r}, \quad U = \int_1^2 \vec{E}\cdot d\vec{l}, \quad C = \frac{Q}{U}$
- The Gaussian surface is a coaxial cylinder of radius r and length L between the electrodes; it encloses the inner electrode's charge Q, and no field passes through its bases.
- The field between the electrodes by Gauss's law
  $E\cdot 2\pi r L = \frac{Q}{\varepsilon_0\varepsilon_r} \Rightarrow E = \frac{Q}{2\pi\varepsilon_0\varepsilon_r L r}$
- The voltage as the field's line integral
  $U = \int_{R_1}^{R_2} \frac{Q}{2\pi\varepsilon_0\varepsilon_r L r}\,dr = \frac{Q}{2\pi\varepsilon_0\varepsilon_r L}\ln\frac{R_2}{R_1}$
- The capacitance
  $C = \frac{Q}{U} = \frac{2\pi\varepsilon_0\varepsilon_r L}{\ln\frac{R_2}{R_1}} = \frac{2\pi\cdot 8.854\cdot 10^{-12}\cdot 2.3\cdot 1}{\ln\frac{0.0035}{0.001}} \approx 1.0$
- Converting the units: $C \approx 1.021\cdot 10^{-10}\ \text{F} = 102.1\ \text{pF}$

The energy of a 1 T magnetic field in a litre of volume.

Input: `model(fieldenergy; find=W[J]; B=1[T]; V=1[l]; mode=magnetic)`

Result

$$W \approx 397.9\,\mathrm{J}$$

Steps

- Find W; given: $B = 1\,\mathrm{T}, \quad V = 1\,\mathrm{l}$
- Converting the units: $V = 1\,\mathrm{l} = 0.001\,\mathrm{m}^3$
- The energy density of a magnetic field in vacuum, w = B²/(2μ₀) – the field inside a coil, for example. The model's equation: $W = \iiint_V w\,dV, \quad w = \frac{B^2}{2\mu_0}$
- Substituting $w = \frac{(1)^2}{2 \cdot 1.257 \cdot 10^{-6}} \approx 397\,900\,\mathrm{J/m}^3$
- In a uniform field the energy density is the same everywhere; the volume integral is the density times the volume.
- Expressing the sought quantity: $W = \iiint_V w\,dV = wV$
- Substituting: $W = 397\,900 \cdot 0.001 \approx 397.9\,\mathrm{J}$
- This is how a coil carrying a current stores its energy: $W = \frac{1}{2}LI^2$

The work of gravity when 1 kg is lifted from the Earth's surface to a height of one radius.

Input: `model(fieldwork; find=W[MJ]; M=5.972e24[kg]; m=1[kg]; r1=6371[km]; r2=12742[km]; mode=gravity)`

Result

$$W \approx -31.28\,\mathrm{MJ}$$

Steps

- Find W; given $M = 5.972 \cdot 10^{24}\,\mathrm{kg}, \quad m = 1\,\mathrm{kg}, \quad r_1 = 6371\,\mathrm{km}, \quad r_2$
- Converting the units $r_1 = 6371\,\mathrm{km} = 6\,371\,000\,\mathrm{m}, \quad r_2 = 12\,740\,\mathrm{km} = 1.27$
- The force is conservative (it has a potential): the work is independent of the path, it depends only on the start and the end. So we take the simplest path – along the line joining them.
- The gravity of a body of mass M acts on a body m moved from a distance r_1 to r_2 from the centre. A negative work of the field means the body had to be lifted. The model's equation: $W = \int_C \vec{F} \cdot d\vec{r}, \quad \vec{F} = -\frac{GMm}{r^2}\,\hat{e}_r$
- The integral evaluated $W = -\int_{r_1}^{r_2} \frac{GMm}{r^2}\,dr = \left[\frac{GMm}{r}\right]_{r_1}^{r_2} = GMm\left(\frac{1}{r_2} - \frac{1}{r}\right.$
- Substituting $W = 6.674 \cdot 10^{-11} \cdot 5.972 \cdot 10^{24} \cdot 1 \cdot \left(\frac{1}{1.274 \cdot 10^7} - \frac{1}{6\,3}\right.$
- The potential energy $E_p = -\frac{GMm}{r}: \quad E_{p1} \approx -6.256 \cdot 10^7\,\mathrm{J}, \quad E_{p2} \approx -3.1$
- A conservative force's work equals the loss of potential energy: $\Delta E_p = -W \approx 3.128 \cdot 10^7\,\mathrm{J}$
- Converting the units: $W \approx -3.128 \cdot 10^7\,\mathrm{J} = -31.28\,\mathrm{MJ}$

The flow rate of laminar flow – the integral of the parabolic profile in polar coordinates.

Input: `model(flow; find=Q[l_s]; vmax=2[m_s]; R=5[cm]; mode=parabolic)`

Result

$$Q \approx 7.854\,\mathrm{l/s}$$

Steps

- Find Q; given: $v_{\max} = 2\,\mathrm{m/s}, \quad R = 5\,\mathrm{cm}$
- Converting the units: $R = 5\,\mathrm{cm} = 0.05\,\mathrm{m}$
- Laminar (Poiseuille) flow: the fluid rests at the wall and is fastest on the axis; the velocity profile is the parabola v = v_max(1 − r²/R²). The model's equation

$$Q = \iint_S \vec{v} \cdot d\vec{S}, \quad v(r) = v_{\max}\left(1 - \frac{r^2}{R^2}\right)$$

- The speed depends only on the distance r from the axis, so the cross-section is integrated in polar coordinates (dS = r dr dφ).
- The integral evaluated

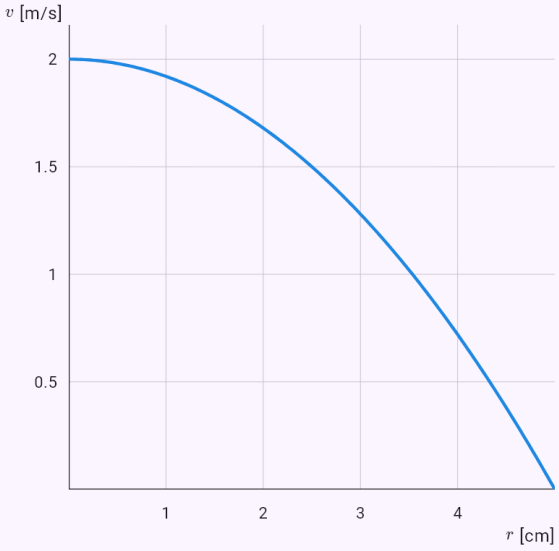
$$Q = \int_0^{2\pi}\int_0^{R} v_{\max}\left(1 - \frac{r^2}{R^2}\right) r\,dr\,d\varphi = 2\pi v_{\max}\left[\frac{r^2}{2} - \cdot\right.$$

- Substituting

$$Q = \frac{\pi \cdot (0.05)^2 \cdot 2}{2} \approx 0.007854\,\mathrm{m^3/s}$$

- The mean speed over the cross-section: $\bar{v} = \frac{v_{\max}}{2} \approx 1\,\mathrm{m/s}$
- Converting the units: $Q \approx 0.007854\,\mathrm{m^3/s} = 7.854\,\mathrm{l/s}$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

The heat loss of an insulated pipe – the temperature falls logarithmically in the wall.

Input: `model(heatflux; find=P[W]; k=0.04[W_mK]; L=1[m]; r1=2[cm]; r2=4[cm]; T1=80[degC];`

`T2=20[degC]; mode=cylinder)`

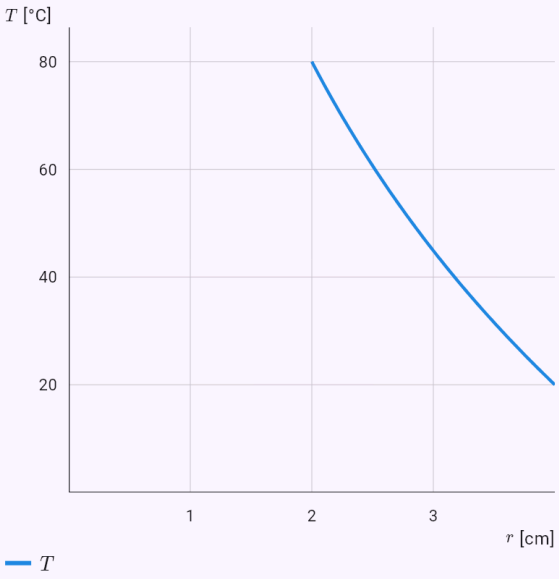
Result

$$P \approx 21.76\,\mathrm{W}$$

Steps

- Find P; given
  $k = 0.04\,\mathrm{W/(m\cdot K)}, \quad L = 1\,\mathrm{m}, \quad r_1 = 2\,\mathrm{cm}, \quad r_2 = 4$
- Converting the units:
  $r_1 = 2\,\mathrm{cm} = 0.02\,\mathrm{m}, \quad r_2 = 4\,\mathrm{cm} = 0.04\,\mathrm{m}$
- A cylindrical wall – a pipe's insulation of inner radius r_1, outer radius r_2 and length L. Heat passes through cylinders of growing area, so the temperature falls logarithmically. The model's equation:
  $\vec{q} = -k\nabla T, \quad P = \iint_S \vec{q}\cdot d\vec{S}$
- A steady state without heat sources – the same heat flow passes through every layer: $\nabla\cdot\vec{q} = 0$
- The integral evaluated
  $P = \iint_S \vec{q}\cdot d\vec{S} = -k\frac{dT}{dr}\cdot 2\pi r L \Rightarrow \int_{T_1}^{T_2} dT = -\frac{P}{2\pi k L}$
- Expressing the sought quantity
  $T_2 - T_1 = -\frac{P}{2\pi k L}\ln\frac{r_2}{r_1} \Rightarrow P = \frac{2\pi k L(T_1 - T_2)}{\ln\frac{r_2}{r_1}}$
- Substituting
  $P = \frac{2\pi\cdot 0.04\cdot 1\cdot(80-20)}{\ln\frac{0.04}{0.02}} \approx 21.76\,\mathrm{W}$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

The gravitational acceleration halfway to the Earth's centre.

Input: `model(gravityfield; find=g[m_s2]; M=5.972e24[kg]; R=6371[km]; r=3185.5[km];`

`mode=ball)`

Result

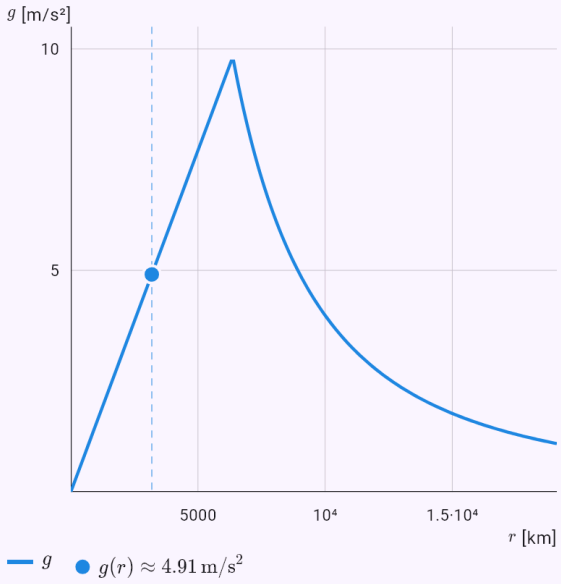
$$g \approx 4.91\ \mathrm{m/s^2}$$

Steps

- Find g; given
  $M = 5.972 \cdot 10^{24}\ \mathrm{kg}, \quad R = 6371\ \mathrm{km}, \quad r = 3186\ \mathrm{km}$
- Converting the units
  $R = 6371\ \mathrm{km} = 6\,371\,000\ \mathrm{m}, \quad r = 3186\ \mathrm{km} = 3\,186\,0($
- A homogeneous ball (a planet of constant density). Inside, the gravitational acceleration grows linearly with the distance from the centre. The model's equation:
  $\oint_S \vec{g} \cdot d\vec{S} = -4\pi G M_{\mathrm{in}}$
- Symmetry: the field is radial and has the same magnitude on every concentric sphere. We take a sphere of radius r as the Gaussian surface – on it the field is parallel to the normal and constant.
- The integral evaluated: $\oint_S \vec{g} \cdot d\vec{S} = -g \cdot 4\pi r^2$
- The enclosed mass
  $M_{\mathrm{in}} = M\frac{r^3}{R^3} = 5.972 \cdot 10^{24} \cdot \frac{(3\,186\,000)^3}{(6\,371\,000)^3} \approx 7.465 \cdot 10^{}$
- Expressing the sought quantity:
  $g \cdot 4\pi r^2 = 4\pi G M_{\mathrm{in}} \Rightarrow g = \frac{G M_{\mathrm{in}}}{r^2}$
- Substituting
  $g = \frac{6.674 \cdot 10^{-11} \cdot 7.465 \cdot 10^{23}}{(3\,186\,000)^2} \approx 4.91\ \mathrm{m/s^2}$

- The same law in differential form (by Gauss's theorem):
  $\nabla \cdot \vec{g} = -4\pi G\rho$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

Known limitations

- Extrema only for functions of two variables, constrained ones with one constraint. A double integral only over a bounded region whose boundary is at most quadratic in one variable; the surface z = f(x, y) is drawn in 3D only for the double integral so far. Vector fields only in the variables x, y (z), other letters are constants; arrows are drawn only for a plane field. The potential is found by integrating the components – if an integral isn't elementary, it isn't found. Line integrals: Green's theorem is checked only for a circle, an ellipse and a convex polygon; a closed curve made of a graph and a segment can be entered only as a polyline or piecewise parametrically. A triple integral only over a bounded solid that is normal with respect to one variable (one lower and one upper surface over the whole projection) and whose inequalities are at most quadratic in it; spherical coordinates only for a solid whose angles θ and φ form a rectangle.
- Surface integrals: Gauss's theorem is checked only for the presets (a sphere, a closed hemisphere, a cylinder, a cone), Stokes' only for the flux of a curl written as curl or ∇ × F; a graph z = g(x, y) only over

a region the double integral handles; enter a sphere with a bound other than $z \geq 0$ as a graph or by a parametrization.

- The physical applications of the integral theorems compute only symmetric cases (a sphere, an infinite line and plane, a long coil, a uniform field) where the integral becomes a product; enter a general distribution of charge or current as a line or surface integral. They neglect the displacement current in the Ampère–Maxwell law and model materials only by a capacitor's relative permittivity.

29. Taylor polynomial

The app builds the Taylor (at 0 the Maclaurin) polynomial of a given degree: a table of the derivatives at the point, substitution into the formula and the remainder in Lagrange form. It can also find the degree itself – the smallest one for which the error at a given point or on an interval stays below a given bound. The graph shows the function and the polynomial; a slider changes the degree so you can see the approximation improve.

High degrees (up to about 1000) are computed exactly and shortened in the solution and the graph; the degree slider is then logarithmic. Functions whose derivatives grow quickly (tan x) go by dividing series, the error by Cauchy's estimate in the complex plane or an alternating series' estimate. If even that fails, the app explains why and gives advice.

How to enter it

`taylor(sin x, 0, 5)`	The function, the point and the degree.
`maclaurin(e^x, 4)`	The Maclaurin polynomial – the point is 0.
`taylor(e^x, 0, error=0.001, x=0.5)`	The degree by the error at the point x (on an interval enter r=… instead of x).
`ln(1 + x)`	A function with the “Taylor polynomial” chip – fill in the point, the degree or the accuracy there.

Examples

A polynomial of degree 4 with the remainder.

Input: `taylor(ln(1 + x), 0, 4)`

Result

$$T_4(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

Steps

- The Maclaurin polynomial (the Taylor polynomial at 0):
  $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k, \quad n = 4$
- The derivatives and their values at a

$$\begin{aligned}
f(x) &= \ln(x+1) & f(0) &= 0 \\
f'(x) &= \frac{1}{x+1} & f'(0) &= 1 \\
f''(x) &= -\frac{1}{(x+1)^2} & f''(0) &= -1 \\
f'''(x) &= \frac{2}{(x+1)^3} & f'''(0) &= 2 \\
f^{(4)}(x) &= -\frac{6}{(x+1)^4} & f^{(4)}(0) &= -6
\end{aligned}$$

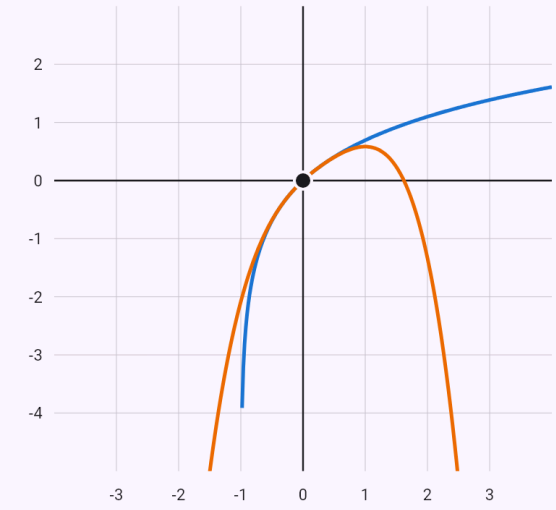
- Filling in the values of the derivatives

$$\begin{aligned}
T_4(x) &= \frac{1}{1!}x - \frac{1}{2!}x^2 + \frac{2}{3!}x^3 - \frac{6}{4!}x^4 \\
&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}
\end{aligned}$$

- The remainder (the error) in Lagrange's form, ξ between a and x: $R_4(x) = \frac{f^{(5)}(\xi)}{5!}x^5 = \frac{24}{(\xi+1)^5 \cdot 5!}x^5$

Graph

Degree: 4



$y = \ln(1+x)$

$T_4(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$

The point around which the polynomial approximates the function:

- $(0, 0)$

Use the slider above the graph to pick the degree – with a higher degree the polynomial approximates the function better, and further from the point.

The degree by the required accuracy – an approximate value of e^0.5.

Input: `taylor(e^x, 0, error=0.001, x=0.5)`

Result

$$T_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

$$f(0.5) \approx 1.6484$$

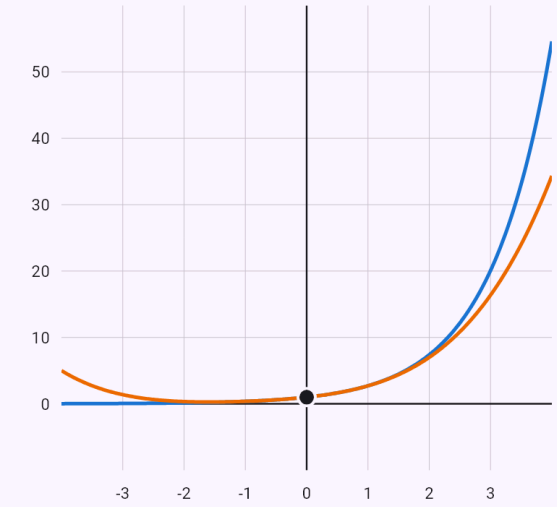
Steps

- The Maclaurin polynomial (the Taylor polynomial at 0):
$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k, \quad n = 4$$
- The derivatives and their values at a
$$\begin{aligned} f(x) &= e^x & f(0) &= 1 \\ f'(x) &= e^x & f'(0) &= 1 \\ f''(x) &= e^x & f''(0) &= 1 \\ f'''(x) &= e^x & f'''(0) &= 1 \\ f^{(4)}(x) &= e^x & f^{(4)}(0) &= 1 \end{aligned}$$
- Filling in the values of the derivatives
$$\begin{aligned} T_4(x) &= 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \end{aligned}$$
- The remainder (the error) in Lagrange's form, ξ between a and x: $R_4(x) = \frac{f^{(5)}(\xi)}{5!}x^5 = \frac{e^\xi}{5!}x^5$
- The lowest degree n whose remainder estimate at x keeps within the error
$$M_k = \max_{t \in [0,\, 0.5]} \left| f^{(k)}(t) \right|, \quad h = |x - a| = 0.5$$
$$n = 1: \quad |R_1| \le \frac{1.6487}{2!} \cdot 0.5^2 \approx 0.2063 > 10^{-3}$$
$$n = 2: \quad |R_2| \le \frac{1.6487}{3!} \cdot 0.5^3 \approx 0.0344 > 10^{-3}$$
$$n = 3: \quad |R_3| \le \frac{1.6487}{4!} \cdot 0.5^4 \approx 4.3 \cdot 10^{-3} > 10^{-3}$$
$$n = 4: \quad |R_4| \le \frac{1.6487}{5!} \cdot 0.5^5 \approx 4.3 \cdot 10^{-4} \le 10^{-3}$$

- The approximate value:
$$f(0.5) \approx T_4(0.5) = \frac{211}{128} \approx 1.6484$$

Graph

Degree: 4



$y = e^x$

$T_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$

The point around which the polynomial approximates the function:

- $(0, 1)$

Use the slider above the graph to pick the degree – with a higher degree the polynomial approximates the function better, and further from the point.

Degree 999 – a high degree is shortened in the solution.

Input: `taylor(ln(1 + x), 0, chyba=0.001, x=1)`

Result

$$T_{999}(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots - \frac{x^{998}}{998} + \frac{x^{999}}{999}$$

$$f(1) \approx 0.6936$$

Steps

- The Maclaurin polynomial (the Taylor polynomial at 0):

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k, \quad n = 999$$

- The derivatives and their values at a

$$f(x) = \ln(x+1) \qquad f(0) = 0$$
$$f'(x) = \frac{1}{x+1} \qquad f'(0) = 1$$
$$f''(x) = -\frac{1}{(x+1)^2} \qquad f''(0) = -1$$
$$f'''(x) = \frac{2}{(x+1)^3} \qquad f'''(0) = 2$$
$$\vdots$$
$$f^{(998)}(x) = -\frac{997!}{(x+1)^{998}} \qquad f^{(998)}(0) = -997!$$
$$f^{(999)}(x) = \frac{998!}{(x+1)^{999}} \qquad f^{(999)}(0) = 998!$$

- The remainder (the error) in Lagrange's form, ξ between a and x

$$R_{999}(x) = \frac{f^{(1000)}(\xi)}{1000!} x^{1000} = \frac{-999!}{(\xi+1)^{1000} \cdot 1000!} x^{1000}$$

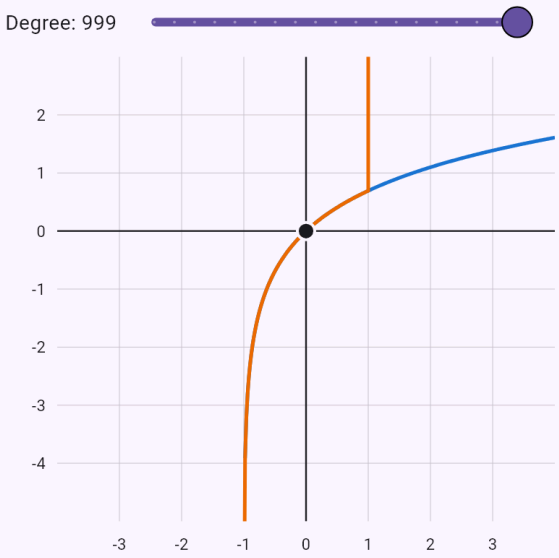
- The lowest degree n whose remainder estimate at x keeps within the error

$$M_k = \max_{t \in [0,1]} \left| f^{(k)}(t) \right|, \quad h = |x - a| = 1$$
$$n = 0: \quad |R_0| \le \frac{1}{1!} \cdot 1^1 \approx 1.001 > 10^{-3}$$
$$n = 1: \quad |R_1| \le \frac{1}{2!} \cdot 1^2 \approx 0.5005 > 10^{-3}$$
$$\cdots$$
$$n = 998: \quad |R_{998}| \le \frac{4.03 \cdot 10^{2561}}{999!} \cdot 1^{999} \approx 1.001 \cdot 10^{-3}$$
$$n = 999: \quad |R_{999}| \le \frac{4.02 \cdot 10^{2564}}{1000!} \cdot 1^{1000} \approx 10^{-3} \le 10^{-}$$

- The approximate value: $f(1) \approx T_{999}(1) = 0.6936$

Graph

Degree: 999



$y = \ln(1+x)$

$T_{999}(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots - \frac{x^{998}}{998} +$

The point around which the polynomial approximates the function:

- $(0, 0)$

Use the slider above the graph to pick the degree – with a higher degree the polynomial approximates the function better, and further from the point.

Tangent – coefficients from series, the error by Cauchy's estimate.

Input: `taylor(tan x, 0, chyba=0.000001, x=1)`

Result

$$T_{43}(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots$$

$$f(1) \approx 1.5574077$$

Steps

- The Maclaurin polynomial (the Taylor polynomial at 0):

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k, \quad n = 43$$

- The derivatives and their values at a

$$\begin{aligned} f(x) &= \tan x & f(0) &= 0 \\ f'(x) &= \frac{1}{\cos^2 x} & f'(0) &= 1 \\ f''(x) &= \frac{2\tan x}{\cos^2 x} & f''(0) &= 0 \\ f'''(x) &= \frac{2}{\cos^4 x} + \frac{4\tan^2 x}{\cos^2 x} & f'''(0) &= 2 \\ &\vdots \end{aligned}$$

- The coefficients come from the arithmetic of series;

formulas of the high derivatives would be huge

$$\operatorname{tg} x = \frac{\sin x}{\cos x}, \quad c_n = \frac{s_n - \sum_{k=1}^{n} q_k\, c_{n-k}}{q_0}$$

- The derivatives are too large, so we use Cauchy's estimate on a circle in the complex plane

$$|R_n| \le \frac{M\, q^{n+1}}{1-q}, \quad q = \frac{h}{\rho}, \quad M = \max_{|z-a|=\rho} |f(z)|$$

$$R \approx 1.569, \quad \rho = 1.49, \quad M \approx 12.369, \quad h = 1$$

$$n = 43: \quad |R_{43}| \le 8.91 \cdot 10^{-7} \le 10^{-6}$$

- The approximate value: $f(1) \approx T_{43}(1) = 1.5574077$

Known limitations

- Degrees above 1000 aren't computed; a point on or beyond the edge of convergence can't be handled by Cauchy's estimate.

30. Indefinite integrals

The app integrates with the methods taught at school and says at every step what it does. It checks the result – differentiates it term by term by the rules and edits the sum to the integrand (a common denominator, identities like sin²x + cos²x = 1) – and draws the integrand and the antiderivative.

Roots of powers of the variable (∛(x⁴), √(x³)) are rewritten as powers with rational exponents; an absolute value (|cos x|, √(1 − sin²x)) is written with the sign function sgn.

If an integral has no elementary antiderivative (such as e^(−x²)), the app says so and gives it in terms of a special function (erf, Si, …).

How to enter it

`int(x e^x, x)`	An integral with the variable – limits or an application (area, volume, …) can be added in the fields below the input.
`int(x^2 sin x)`	Without the variable when there is only one.
`∫ x^2 dx`	With the integral sign, also from a photo.
`x e^x`	A function with the “Integrate” chip.

Basic integrals and substitution

The integral of a sum term by term, constants taken out, table integrals (also with a linear inner function) and substitution when the integrand contains a function together with the derivative of its inside.

Examples

Term by term from the table.

Input: `int(3x^2 - 2/x + e^x)`

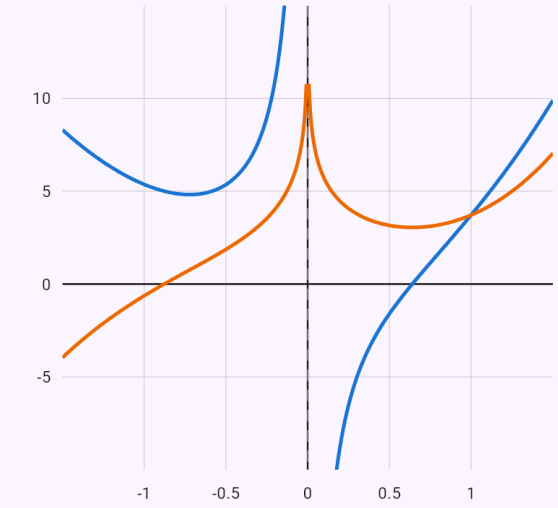
Result

$$\int \left(3x^2 + e^x - \frac{2}{x}\right) dx = x^3 + e^x - 2$$

Steps

- The integral of a sum is the sum of the integrals
  $$\int \left(3x^2 + e^x - \frac{2}{x}\right) dx = \int 3x^2\, dx + \int e^x\, dx + \int -\frac{2}{x}$$
- The constant taken out of the integral:
  $$\int 3x^2\, dx = 3\int x^2\, dx$$
- From the table of basic integrals: $\int x^2\, dx = \frac{x^3}{3}$
- From the table of basic integrals: $\int e^x\, dx = e^x$
- The constant taken out of the integral:
  $$\int -\frac{2}{x}\, dx = -2\int \frac{1}{x}\, dx$$
- From the table of basic integrals: $\int \frac{1}{x}\, dx = \ln|x|$
- Check – the result (the antiderivative) differentiated gives the integrand:
  $$\left(x^3 + e^x - 2\ln|x|\right)' = 3x^2 + e^x - \frac{2}{x}$$

Graph



$f(x) = 3x^2 + e^x - \frac{2}{x}$

$F(x) = x^3 + e^x - 2\ln|x|$

The integrand f and the antiderivative F for C = 0 (another C shifts it up or down). Where f is positive, F increases; where negative, F decreases.

The substitution t = x².

Input: `int(2x cos(x^2))`

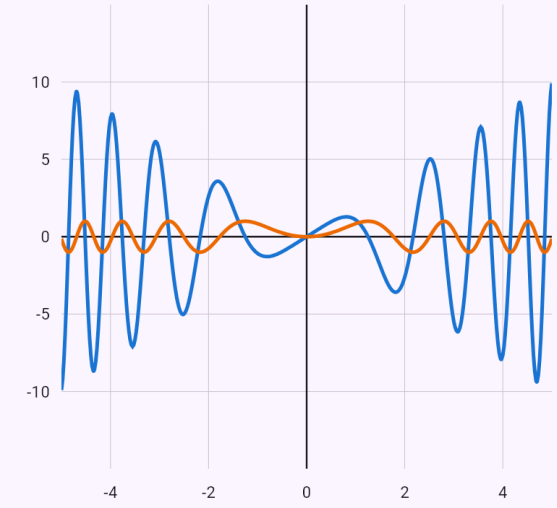
Result

$$\int 2x \cos\left(x^2\right) dx = \sin\left(x^2\right) + C$$

Steps

- The constant taken out of the integral:
  $\int 2x \cos\left(x^2\right) dx = 2 \int x \cos\left(x^2\right) dx$
- Substituting $t = x^2$, $dt = 2x\,dx$:
  $\int x \cos\left(x^2\right) dx = \int \frac{\cos t}{2}\, dt$
- The constant taken out of the integral:
  $\int \frac{\cos t}{2}\, dt = \frac{1}{2} \int \cos t\, dt$
- From the table of basic integrals: $\int \cos t\, dt = \sin t$
- Back to the original variable: $\frac{\sin t}{2} = \frac{\sin\left(x^2\right)}{2}$
- Check – the result (the antiderivative) differentiated gives the integrand
  $\left(\sin\left(x^2\right)\right)' = \cos\left(x^2\right) \cdot 2x = 2x \cos\left(x^2\right)$

Graph



$f(x) = 2x \cos\left(x^2\right)$
$F(x) = \sin\left(x^2\right)$

The integrand f and the antiderivative F for C = 0 (another C shifts it up or down). Where f is positive, F increases; where negative, F decreases.

An integral with constants a, b.

Input: `int(1/(a - bx)^3)`

Result

$$\int \frac{1}{(a - bx)^3}\, dx = \frac{1}{2b(bx - a)^2} + C$$

Conditions: $b \neq 0$

Steps

- From the table – the inner function is linear, so the result is divided by its coefficient of the variable:
  $\int \frac{1}{(a - bx)^3}\, dx = \frac{1}{2b(bx - a)^2}$
- Check – the result (the antiderivative) differentiated gives the integrand
  $\left(\frac{1}{2b(bx - a)^2}\right)' = \frac{1}{2b} \cdot \left(-\frac{2}{(a - bx)^3}\right) \cdot (-b) = \frac{1}{(a -$

Integration by parts

The app integrates a product of a polynomial with an exponential, a trigonometric function or a logarithm by parts, repeatedly if needed; integrals like $e^x \sin x$ by cyclic integration by parts.

Examples

Integration by parts twice in a row.

Input: `int(x^2 e^x)`

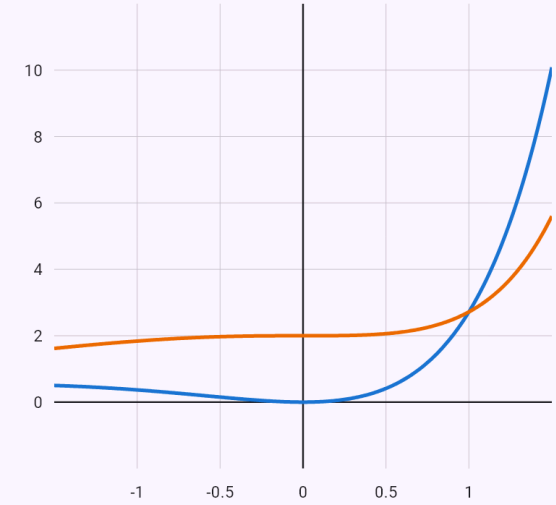
Result

$$\int x^2 e^x \, dx = x^2 e^x - 2xe^x + 2e^x + C$$

Steps

- By parts (∫u·v' = u·v − ∫u'·v), u = x^2, v' = e^x, so u' = $2x$, v = e^x: $\int x^2 e^x \, dx = x^2 e^x - \int 2xe^x \, dx$
- The constant taken out of the integral: $\int 2xe^x \, dx = 2 \int xe^x \, dx$
- By parts (∫u·v' = u·v − ∫u'·v), u = x, v' = e^x, so u' = 1, v = e^x: $\int xe^x \, dx = xe^x - \int e^x \, dx$
- From the table of basic integrals: $\int e^x \, dx = e^x$
- Check – the result (the antiderivative) differentiated gives the integrand: $\left(x^2 e^x - 2xe^x + 2e^x\right)' = x^2 e^x$
- The term differentiated by the rules for a composite function, a product and a quotient: $\left(x^2 e^x\right)' = e^x \cdot x^2 + e^x \cdot 2x = x^2 e^x + 2xe^x$
- The term differentiated by the rules for a composite function, a product and a quotient: $(-2xe^x)' = -2 \cdot (e^x \cdot x + e^x) = -2xe^x - 2e^x$
- Added and simplified $\left(x^2 e^x - 2xe^x + 2e^x\right)' = x^2 e^x + 2xe^x - 2xe^x - 2e^x + 2$

Graph



The integrand f and the antiderivative F for C = 0 (another C shifts it up or down). Where f is positive, F increases; where negative, F decreases.

Rational functions

The app divides a rational function if needed, factors the denominator and splits the function into partial fractions, which it integrates into logarithms and arctangents.

A denominator that can't be factored over the rationals ($x^4 + 1$, $x^5 + 1$) is factored over the reals – each pair of complex conjugate roots gives a quadratic factor with no real roots.

Examples

Partial fractions.

Input: `int((x + 1)/(x^2 - 3x + 2))`

Result

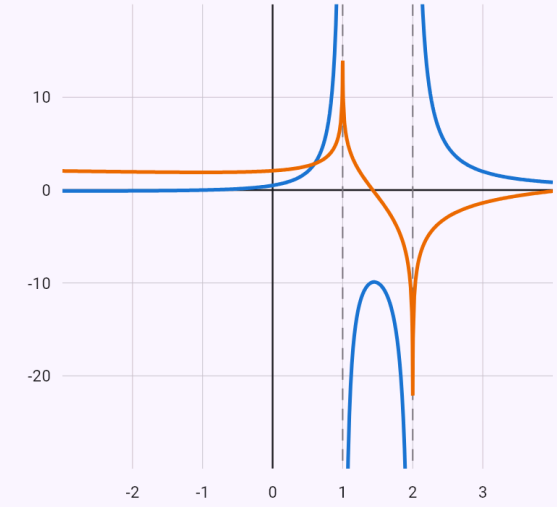
$$\int \frac{x+1}{x^2-3x+2}\,dx = 3\ln|x-2| - 2\ln$$

Steps

- The rational function split into a polynomial and partial fractions:
  $$\int \frac{x+1}{x^2-3x+2}\,dx = \int \left(\frac{3}{x-2} - \frac{2}{x-1}\right) dx$$
- The integral of a sum is the sum of the integrals
  $$\int \left(\frac{3}{x-2} - \frac{2}{x-1}\right) dx = \int \frac{3}{x-2}\,dx + \int -\frac{2}{x-1}\,d$$
- The constant taken out of the integral:
  $$\int \frac{3}{x-2}\,dx = 3\int \frac{1}{x-2}\,dx$$
- From the table of basic integrals:
  $$\int \frac{1}{x-2}\,dx = \ln|x-2|$$
- The constant taken out of the integral:
  $$\int -\frac{2}{x-1}\,dx = -2\int \frac{1}{x-1}\,dx$$
- From the table of basic integrals:
  $$\int \frac{1}{x-1}\,dx = \ln|x-1|$$
- Check – the result (the antiderivative) differentiated gives the integrand:
  $$(3\ln|x-2| - 2\ln|x-1|)' = \frac{x+1}{x^2-3x+2}$$
- The term differentiated by the rules for a composite function, a product and a quotient:
  $$(3\ln|x-2|)' = 3\cdot\frac{1}{x-2} = \frac{3}{x-2}$$
- The term differentiated by the rules for a composite function, a product and a quotient:
  $$(-2\ln|x-1|)' = -2\cdot\frac{1}{x-1} = -\frac{2}{x-1}$$

- Added and simplified
  $$(3\ln|x-2| - 2\ln|x-1|)' = \frac{3}{x-2} - \frac{2}{x-1} = \frac{3(x-}{(x-}$$

Graph



$f(x) = \frac{x+1}{x^2-3x+2}$

$F(x) = 3\ln|x-2| - 2\ln|x-1|$

The integrand f and the antiderivative F for C = 0 (another C shifts it up or down). Where f is positive, F increases; where negative, F decreases.

Factoring the denominator x⁴ + 1 over the reals.

Input: `int(1/(x^4 + 1))`

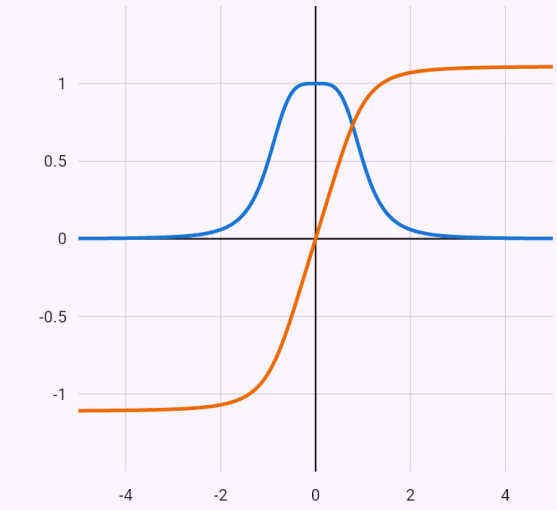
Result

$$\int \frac{1}{x^4+1}\,dx = \frac{\sqrt{2}}{4}\arctan\left(\sqrt{2}\left(x - \right.\right.$$

Steps

- Factor the denominator over ℝ: its complex roots come in conjugate pairs r, r̄, and each pair gives a quadratic factor (x − r)(x − r̄) = x² − 2·Re r·x + |r|² with no real roots:
  $x^4+1 = \left(x^2+\sqrt{2}x+1\right)\left(x^2-\sqrt{2}x+1\right)$
- The rational function split into a polynomial and partial fractions
  $$\int \frac{1}{x^4+1}\,dx = \int \left(\frac{\sqrt{2}x+2}{4\left(x^2+\sqrt{2}x+1\right)} + \frac{2-\sqrt{2}x}{4\left(x^2-\sqrt{2}x\right.\right.$$
- The integral of a sum is the sum of the integrals
  $$\int \left(\frac{\sqrt{2}x+2}{4\left(x^2+\sqrt{2}x+1\right)} + \frac{2-\sqrt{2}x}{4\left(x^2-\sqrt{2}x+1\right)}\right)dx = \int$$
- The denominator has no real roots: a multiple of its derivative split off the numerator gives ln, the rest − after completing the square − gives arctan
  $$\int \frac{\sqrt{2}x+2}{4\left(x^2+\sqrt{2}x+1\right)}\,dx = \frac{\sqrt{2}}{8}\int \frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1}\,dx +$$
- The denominator has no real roots: a multiple of its derivative split off the numerator gives ln, the rest − after completing the square − gives arctan
  $$\int \frac{2-\sqrt{2}x}{4\left(x^2-\sqrt{2}x+1\right)}\,dx = -\frac{\sqrt{2}}{8}\int \frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1}\,dx +$$

Graph



$f(x) = \frac{1}{x^4+1}$

$F(x) = \frac{\sqrt{2}}{4}\arctan\left(\sqrt{2}\left(x-\frac{\sqrt{2}}{2}\right)\right) + \frac{\sqrt{2}\,\text{ar}}{}$

The integrand f and the antiderivative F for C = 0 (another C shifts it up or down). Where f is positive, F increases; where negative, F decreases.

Checking partial fractions over a common denominator.

Input: `int(1/(x^3 + 1))`

Result

$$\int \frac{1}{x^3+1}\,dx = \frac{\sqrt{3}}{3}\arctan\left(\frac{\sqrt{3}(2x-\right.$$

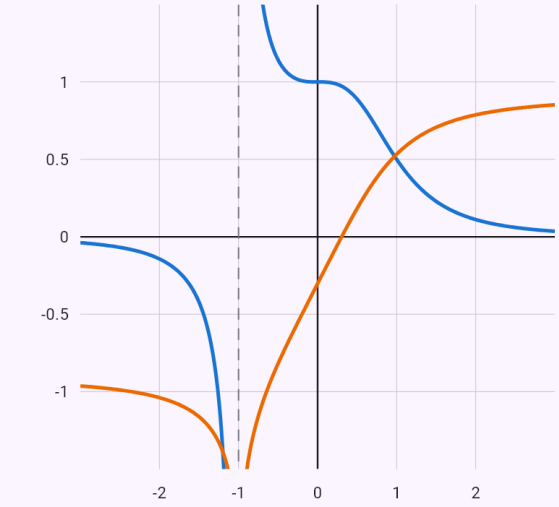
Steps

- The rational function split into a polynomial and partial fractions
$$\int \frac{1}{x^3+1}\,dx = \int \left(\frac{1}{3(x+1)} + \frac{2-x}{3\left(x^2-x+1\right)}\right) dx$$
- The integral of a sum is the sum of the integrals
$$\int \left(\frac{1}{3(x+1)} + \frac{2-x}{3\left(x^2-x+1\right)}\right) dx = \int \frac{2-x}{3\left(x^2-x+1\right)}$$
- The constant taken out of the integral:
$$\int \frac{2-x}{3\left(x^2-x+1\right)}\,dx = \frac{1}{3}\int \frac{2-x}{x^2-x+1}\,dx$$
- The denominator has no real roots: a multiple of its derivative split off the numerator gives ln, the rest – after completing the square – gives arctan
$$\int \frac{2-x}{x^2-x+1}\,dx = -\frac{1}{2}\int \frac{2x-1}{x^2-x+1}\,dx + \frac{3}{2}\int \frac{}{(x-}$$
- The constant taken out of the integral:
$$\int \frac{1}{3(x+1)}\,dx = \frac{1}{3}\int \frac{1}{x+1}\,dx$$
- From the table of basic integrals:
$$\int \frac{1}{x+1}\,dx = \ln|x+1|$$
- Check – the result (the antiderivative) differentiated gives the integrand
$$\left(\frac{\sqrt{3}}{3}\arctan\left(\frac{\sqrt{3}(2x-1)}{3}\right) - \frac{\ln\left(x^2-x+1\right)}{6} + \frac{\ln|x}{}\right.$$
- The term differentiated by the rules for a composite function, a product and a quotient
$$\left(\frac{\sqrt{3}}{3}\arctan\left(\frac{\sqrt{3}(2x-1)}{3}\right)\right)' = \frac{\sqrt{3}}{3}\cdot\frac{1}{\frac{(2x-1)^2}{3}+1}\cdot\frac{2}{}$$
- The term differentiated by the rules for a composite function, a product and a quotient

$$\left(-\frac{\ln\left(x^2-x+1\right)}{6}\right)' = -\frac{1}{6}\cdot\frac{1}{x^2-x+1}\cdot(2x-1) =$$

- The term differentiated by the rules for a composite function, a product and a quotient
$$\left(\frac{\ln|x+1|}{3}\right)' = \frac{1}{3}\cdot\frac{1}{x+1} = \frac{1}{3(x+1)}$$
- Added and simplified
$$\left(\frac{\sqrt{3}}{3}\arctan\left(\frac{\sqrt{3}(2x-1)}{3}\right) - \frac{\ln\left(x^2-x+1\right)}{6} + \frac{\ln|x}{}\right.$$

Graph



$f(x) = \frac{1}{x^3+1}$

$F(x) = \frac{\sqrt{3}}{3}\arctan\left(\frac{\sqrt{3}(2x-1)}{3}\right) - \frac{\ln\left(x^2-\right.}{6}$

The integrand f and the antiderivative F for C = 0 (another C shifts it up or down). Where f is positive, F increases; where negative, F decreases.

Trigonometric functions

Identities first (sin²x = (1 − cos 2x)/2, products into sums), for an odd power the substitution t = sin x or t = cos x, for a rational function of sine and cosine a substitution by parity, and the Weierstrass substitution t = tan(x/2) only as the last resort.

Examples

An odd power of sine – the substitution t = cos x.

Input: `int(sin(x)^3 cos(x)^2)`

Result

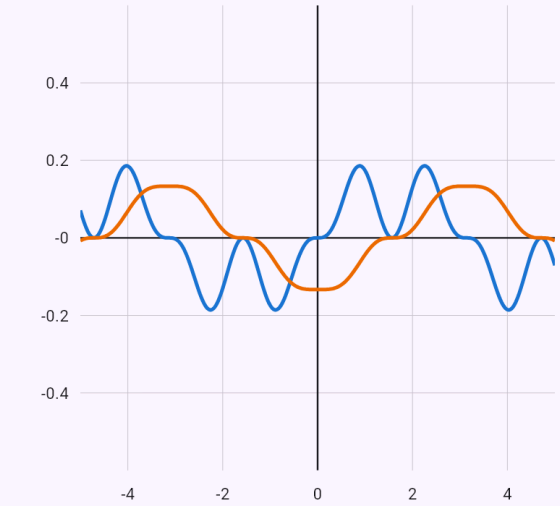
$$\int \cos^2 x \sin^3 x \, dx = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3}$$

Steps

- An odd power – the substitution $t = \cos x$,
  $\sin^2 x = 1 - \cos^2 x$
  $dt = -\sin x \, dx$
  $\int \cos^2 x \sin^3 x \, dx = \int \left(t^4 - t^2\right) dt$
- The integral of a sum is the sum of the integrals:
  $\int \left(t^4 - t^2\right) dt = \int t^4 \, dt + \int -t^2 \, dt$
- From the table of basic integrals: $\int t^4 \, dt = \frac{t^5}{5}$
- The constant taken out of the integral:
  $\int -t^2 \, dt = -\int t^2 \, dt$
- From the table of basic integrals: $\int t^2 \, dt = \frac{t^3}{3}$
- Back to the original variable:
  $t = \cos x$: $\frac{\cos^5 x}{5} - \frac{\cos^3 x}{3}$
- Check – the result (the antiderivative) differentiated gives the integrand: $\left(\frac{\cos^5 x}{5} - \frac{\cos^3 x}{3}\right)' = \cos^2 x \sin^3 x$
- The term differentiated by the rules for a composite function, a product and a quotient
  $\left(\frac{\cos^5 x}{5}\right)' = \frac{1}{5} \cdot 5 \cos^4 x \cdot (-\sin x) = -\cos^4 x \sin x$
- The term differentiated by the rules for a composite function, a product and a quotient
  $\left(-\frac{\cos^3 x}{3}\right)' = -\frac{1}{3} \cdot 3 \cos^2 x \cdot (-\sin x) = \cos^2 x \sin x$

- Added and simplified with the identity
  $\sin^2 x + \cos^2 x = 1$
  $\left(\frac{\cos^5 x}{5} - \frac{\cos^3 x}{3}\right)' = -\cos^4 x \sin x + \cos^2 x \sin x = ($

Graph



$f(x) = \cos^2 x \sin^3 x$

$F(x) = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3}$

The integrand f and the antiderivative F for C = 0 (another C shifts it up or down). Where f is positive, F increases; where negative, F decreases.

The check: the derivative term by term, edited by the identity 1 + tan²x = 1/cos²x.

Input: `int(1/cos(x)^4)`

Result

$$\int \frac{1}{\cos^4 x}\,dx = \tan x + \frac{\tan^3 x}{3} + C$$

Steps

- The function doesn't change when sine and cosine change sign together – the substitution $t = \tan x$

  $\sin x = \frac{t}{\sqrt{1+t^2}},\ \cos x = \frac{1}{\sqrt{1+t^2}},\ dx = \frac{dt}{(1+t^2)}$

  $$\int \frac{1}{\cos^4 x}\,dx = \int$$

- The integral of a sum is the sum of the integrals: $\int (t^2+1)\,dt = \int t^2\,dt + \int 1\,dt$
- From the table of basic integrals: $\int t^2\,dt = \frac{t^3}{3}$
- From the table of basic integrals: $\int 1\,dt = t$
- Back to the original variable:

  $t = \tan x:\quad \tan x + \frac{\tan^3 x}{3}$

- Check – the result (the antiderivative) differentiated gives the integrand: $\left(\tan x + \frac{\tan^3 x}{3}\right)' = \frac{1}{\cos^4 x}$
- The term differentiated by the rules for a composite function, a product and a quotient

  $$\left(\frac{\tan^3 x}{3}\right)' = \frac{1}{3}\cdot 3\tan^2 x\cdot\frac{1}{\cos^2 x} = \frac{\tan^2 x}{\cos^2 x}$$

- Added and simplified with the identity $1 + \tan^2 x = \frac{1}{\cos^2 x}$

  $$\left(\tan x + \frac{\tan^3 x}{3}\right)' = \frac{1}{\cos^2 x} + \frac{\tan^2 x}{\cos^2 x} = \frac{\tan^2 x + 1}{\cos^2 x} =$$

Hyperbolic functions

Table integrals (∫sinh x dx = cosh x + C, ∫1/cosh²x dx = tanh x + C, …), ∫tanh x dx by the logarithm of the derivative, by parts (x sinh x, arsinh x), squares by the formulas sinh²x = (cosh 2x − 1)/2, tanh²x = 1 − 1/cosh²x; the rest by the definition through exponentials.

Examples

The formula sinh²x = (cosh 2x − 1)/2.

Input: `int(sinh(x)^2)`

Result

$$\int \sinh^2 x \, dx = -\frac{x}{2} + \frac{\sinh(2x)}{4} + C$$

Steps

- Using the formula $\sinh^2 \alpha = \frac{\cosh 2\alpha - 1}{2}$:
  $\int \sinh^2 x \, dx = \int \frac{\cosh(2x) - 1}{2} \, dx$
- The integral of a sum is the sum of the integrals
  $\int \left(\frac{\cosh(2x)}{2} - \frac{1}{2} \right) dx = \int -\frac{1}{2} \, dx + \int \frac{\cosh(2x)}{2} \, dx$
- From the table of basic integrals: $\int -\frac{1}{2} \, dx = -\frac{x}{2}$
- The constant taken out of the integral:
  $\int \frac{\cosh(2x)}{2} \, dx = \frac{1}{2} \int \cosh(2x) \, dx$
- From the table – the inner function is linear, so the result is divided by its coefficient of the variable:
  $\int \cosh(2x) \, dx = \frac{\sinh(2x)}{2}$

- Check – the result (the antiderivative) differentiated gives the integrand: $\left(-\frac{x}{2} + \frac{\sinh(2x)}{4} \right)' = \sinh^2 x$
- The term differentiated by the rules for a composite function, a product and a quotient
  $\left(\frac{\sinh(2x)}{4} \right)' = \frac{1}{4} \cdot \cosh(2x) \cdot 2 = \frac{\cosh(2x)}{2}$
- Added and simplified with the identity $\cosh 2x = 1 + 2\sinh^2 x$
  $\left(-\frac{x}{2} + \frac{\sinh(2x)}{4} \right)' = -\frac{1}{2} + \frac{\cosh(2x)}{2} = \sinh^2 x$

Roots

Integrals with a root of a linear fractional function by substitution, with a root of a quadratic by an Euler substitution, and binomial integrals by Chebyshev's conditions – when none of them holds, the app says the integral is not elementary.

Examples

An Euler substitution.

Input: `int(sqrt(x^2 + 1))`

Result

$$\int \sqrt{x^2+1}\,dx = \frac{x\sqrt{x^2+1}}{2} + \frac{\ln\left|x+}{}$$

Steps

- Euler's substitution (type 1) $\sqrt{x^2+1} = t - x$ – the integral becomes one of a rational function of t

$$x = \frac{t^2-1}{2t}, \quad dx = \frac{t^2+1}{2t^2}\,dt, \quad \sqrt{x^2+1}$$

$$\int \sqrt{x^2+1}\,dx = \int \frac{(t^2+1)^2}{4t^3}\,dt$$

- The constant taken out of the integral

$$\int \frac{(t^2+1)^2}{4t^3}\,dt = \frac{1}{4}\int \frac{(t^2+1)^2}{t^3}\,dt$$

- The rational function split into a polynomial and partial fractions

$$\int \frac{(t^2+1)^2}{t^3}\,dt = \int \left(t + \frac{2}{t} + \frac{1}{t^3}\right) dt$$

- The integral of a sum is the sum of the integrals

$$\int \left(t + \frac{2}{t} + \frac{1}{t^3}\right) dt = \int t\,dt + \int \frac{2}{t}\,dt + \int \frac{1}{t^3}\,dt$$

- From the table of basic integrals: $\int t\,dt = \frac{t^2}{2}$
- The constant taken out of the integral:

$$\int \frac{2}{t}\,dt = 2\int \frac{1}{t}\,dt$$

- From the table of basic integrals: $\int \frac{1}{t}\,dt = \ln|t|$
- From the table of basic integrals: $\int \frac{1}{t^3}\,dt = -\frac{1}{2t^2}$
- Back to the original variable

$$t = x + \sqrt{x^2+1}: \quad \frac{\ln\left|x+\sqrt{x^2+1}\right|}{2} + \frac{}{(-x^2-1)(6}$$

- Check – the result (the antiderivative) differentiated gives the integrand

$$\left(\frac{x\sqrt{x^2+1}}{2} + \frac{\ln\left|x+\sqrt{x^2+1}\right|}{2}\right)' = \sqrt{x^2+1}$$

- The term differentiated by the rules for a composite function, a product and a quotient

$$\left(\frac{x\sqrt{x^2+1}}{2}\right)' = \frac{1}{2}\cdot\left(\sqrt{x^2+1} + x\cdot\frac{x}{\sqrt{x^2+1}}\right) = \frac{2}{2\sqrt{}}$$

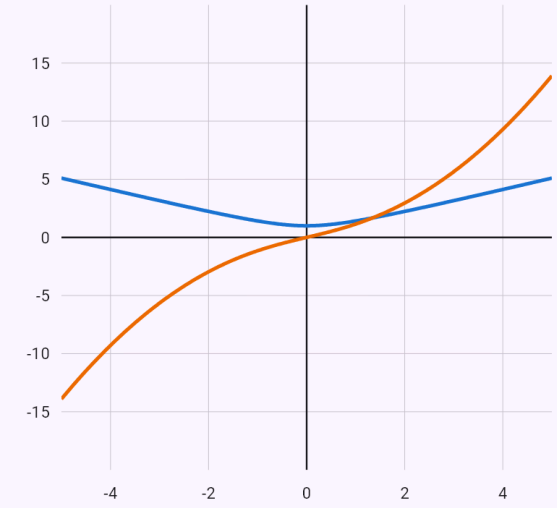
- The term differentiated by the rules for a composite function, a product and a quotient

$$\left(\frac{\ln\left|x+\sqrt{x^2+1}\right|}{2}\right)' = \frac{1}{2}\cdot\frac{1}{x+\sqrt{x^2+1}}\cdot\left(\frac{x}{\sqrt{x^2+1}}\right.$$

- Added and simplified

$$\left(\frac{x\sqrt{x^2+1}}{2} + \frac{\ln\left|x+\sqrt{x^2+1}\right|}{2}\right)' = \frac{2x^2+1}{2\sqrt{x^2+1}} + \frac{}{2\sqrt{}}$$

Graph



$f(x) = \sqrt{x^2+1}$

$F(x) = \frac{x\sqrt{x^2+1}}{2} + \frac{\ln\left|x+\sqrt{x^2+1}\right|}{2}$

The integrand f and the antiderivative F for C = 0 (another C shifts it up or down). Where f is positive, F increases; where negative, F decreases.

Known limitations

- For some integrals we find the result without the working – the app says so in the steps.
- A computation without steps that takes too long is stopped – the app says so.
- For long results where the edit to the integrand wouldn’t be readable (several terms with roots of numbers, e.g. $\int 1/(x^4 + 1)\, dx$) the app leaves the check out.

31. Definite integrals and their applications

The app computes a definite integral by the Newton–Leibniz formula (the fundamental theorem of calculus) and draws the area under the graph – the part above the x-axis in a different colour from the part below. It also handles improper integrals and applications: area, arc length, volume and surface area of a solid of revolution, also for curves given parametrically and in polar coordinates.

Definite integral

The app finds the antiderivative as for an indefinite integral and substitutes the limits. With a substitution it converts the limits right away. On an interval symmetric about zero it uses the symmetry: the odd part's integral is zero without any antiderivative, and √(r² − x²) is recognized as the area of a half disc. Without an elementary antiderivative it gives the exact form with a higher function (such as 2 Si(π)) and computes its number from a power series with an error estimate; where it finds no exact form, it computes the value numerically and says by which method.

Letters in the problem (∫ √(r² − x²) dx from −r to r) are positive parameters: the result is a formula (πr²/2), checked numerically for various values of the parameters, and the graph has a slider for each letter.

How to enter it

`int(x^2, x, 0, 3)`	An integral with the variable and the limits.
`\int_{0}^{\pi} \sin x\,dx`	From a photo or in LaTeX.
`x^2`	A function with the “Integrate” chip – fill in the limits in the fields.

Examples

A substitution with the limits converted.

Input: `int(3x^2 cos(x^3), x, 0, 1)`

Result

$$\int_0^1 3x^2 \cos\left(x^3\right) dx = \sin 1 \approx 0.8415$$

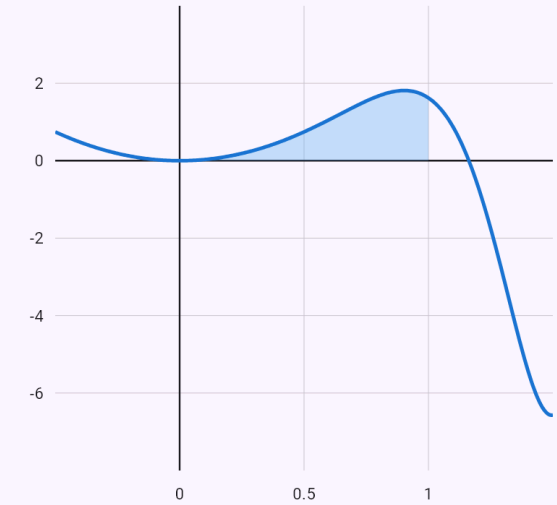
Solve by a numerical method:

Σ Simpson | Σ Trapezoidal | Σ Romberg

Steps

- The constant taken out of the integral:
  $\int 3x^2 \cos\left(x^3\right) dx = 3 \int x^2 \cos\left(x^3\right) dx$
- Substituting $t = x^3$, $dt = 3x^2\, dx$:
  $\int x^2 \cos\left(x^3\right) dx = \int \frac{\cos t}{3}\, dt$
- The constant taken out of the integral:
  $\int \frac{\cos t}{3}\, dt = \frac{1}{3} \int \cos t\, dt$
- From the table of basic integrals: $\int \cos t\, dt = \sin t$
- With the substitution, the limits are recomputed too – no need to go back to the original variable:
  $x = 0 \Rightarrow t = 0, \quad x = 1 \Rightarrow t = 1$
- The Newton–Leibniz formula, F(b) − F(a)
  $\int_0^1 3x^2 \cos\left(x^3\right) dx = [\sin t]_0^1 = \sin 1 - 0 = \sin 1$

Graph



$f(x) = 3x^2 \cos\left(x^3\right)$

above the x-axis (+) below the x-axis (−)

The definite integral is the area between the graph and the x-axis – the part above the axis counts positively, the part below negatively.

The odd part on a symmetric interval is zero, the rest is the area of a half disc – the result is π.

Input: `int((x^3*cos(x/2) + 1/2)*sqrt(4 - x^2), x, -2, 2)`

Result

$$\int_{-2}^{2} \sqrt{4-x^2}\left(x^3 \cos\left(\frac{x}{2}\right)+\frac{1}{2}\right) dx =$$

Solve by a numerical method:

Σ Simpson | Σ Trapezoidal | Σ Romberg

Steps

- The interval is symmetric about zero, so we split the integral into its odd part and the rest

$$\int_{-2}^{2} \sqrt{4-x^2}\left(x^3 \cos\left(\frac{x}{2}\right)+\frac{1}{2}\right) dx = \int_{-2}^{2} x^3\sqrt{4-x^2}\,\mathrm{c}$$

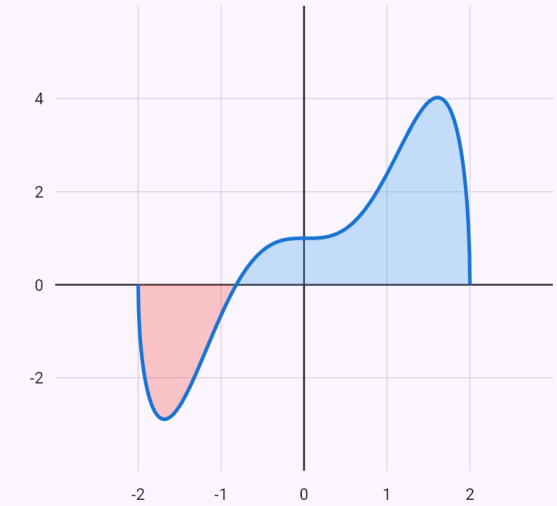
- The function $x^3\sqrt{4-x^2}\cos\left(\frac{x}{2}\right)$ is odd (f(−x) = −f(x)): on ⟨−2; 2⟩ the areas above and below the axis cancel, so the integral is zero without any antiderivative:

$$\int_{-2}^{2} x^3\sqrt{4-x^2}\cos\left(\frac{x}{2}\right) dx = 0$$

- The graph of √(r² − x²) is the upper half of a circle of radius r = 2, so the integral is the area of a half disc, πr²/2

$$\int_{-2}^{2} \frac{\sqrt{4-x^2}}{2}\, dx = \frac{1}{2}\cdot\frac{\pi\cdot 2^2}{2} = \pi$$

Graph



$f(x) = \sqrt{4-x^2}\left(x^3 \cos\left(\frac{x}{2}\right)+\frac{1}{2}\right)$

above the x-axis (+) below the x-axis (−)

The definite integral is the area between the graph and the x-axis – the part above the axis counts positively, the part below negatively.

The antiderivative isn't elementary: exactly 2 Si(π), the value from a power series.

Input: `int(sin(x)/x, x, -pi, pi)`

Result

$$\int_{-\pi}^{\pi} \frac{\sin x}{x}\,dx = 2\,\mathrm{Si}\,(\pi) \approx 3.7039$$

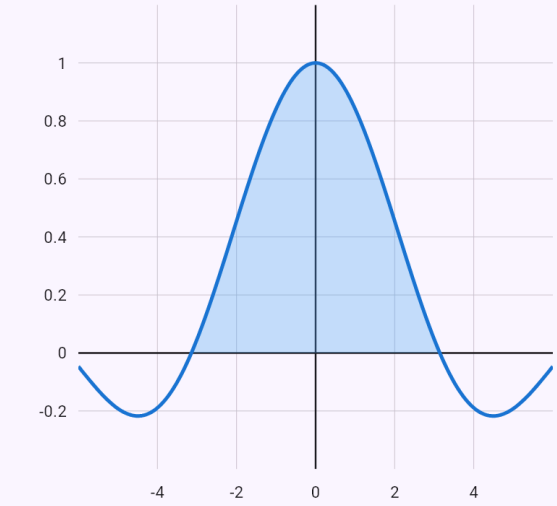
A numerical method computes the number for the exact form too:

Σ Simpson | Σ Trapezoidal | Σ Romberg

Steps

- The function $\frac{\sin x}{x}$ is even (f(−x) = f(x)), its graph is symmetric about the y-axis, so the integral over ⟨−π; π⟩ is twice the one over ⟨0; π⟩: $\int_{-\pi}^{\pi} \frac{\sin x}{x}\,dx = 2\int_{0}^{\pi} \frac{\sin x}{x}\,dx$
- The antiderivative isn't elementary, it is a higher function – the exact result is written with it
  $\int_{-\pi}^{\pi} \frac{\sin x}{x}\,dx = 2\,\mathrm{Si}\,(\pi) \approx 3.7039$
- The number for the exact form comes from the power series of the higher function: $\mathrm{Si}\,(t) = \sum_{k=0}^{\infty} \frac{(-1)^k\,t^{2k+1}}{(2k+1)\cdot(2k+1)!}$. For t = π we add 6 terms. The series alternates, so the error is less than the first term left out, $3.6 \cdot 10^{-5}$ (alternating series estimate): $\mathrm{Si}\,(\pi) \approx 1.8519$

Graph



$f(x) = \frac{\sin x}{x}$

above the x-axis (+) below the x-axis (−)

The definite integral is the area between the graph and the x-axis – the part above the axis counts positively, the part below negatively.

An integral with a parameter r – the area of a half-disc, πr²/2; the slider in the graph changes r.

Input: `int(sqrt(r^2-x^2), x, -r, r)`

Result

$$\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx = \frac{\pi r^2}{2}$$

Steps

- Letters other than the variable are positive parameters: $r > 0$
- A root of a difference of squares – the trigonometric substitution $x = r \sin t$, $dx = r \cos t\, dt$, the root $\sqrt{r^2 - x^2} = r \cos t$: $\int \sqrt{r^2 - x^2}\, dx = \int r^2 \cos^2 t\, dt$
- The constant taken out of the integral: $\int r^2 \cos^2 t\, dt = r^2 \int \cos^2 t\, dt$
- Using the formula $\cos^2 t = \frac{1 + \cos 2t}{2}$: $\int \cos^2 t\, dt = \int \left(\frac{\cos(2t)}{2} + \frac{1}{2}\right) dt$
- The integral of a sum is the sum of the integrals $\int \left(\frac{\cos(2t)}{2} + \frac{1}{2}\right) dt = \int \frac{1}{2}\, dt + \int \frac{\cos(2t)}{2}\, dt$
- From the table of basic integrals: $\int \frac{1}{2}\, dt = \frac{t}{2}$
- The constant taken out of the integral: $\int \frac{\cos(2t)}{2}\, dt = \frac{1}{2} \int \cos(2t)\, dt$
- From the table – the inner function is linear, so the result is divided by its coefficient of the variable: $\int \cos(2t)\, dt = \frac{\sin(2t)}{2}$
- With the substitution, the limits are recomputed too – no need to go back to the original variable: $x = -r \Rightarrow t = -\frac{\pi}{2}$, $x = r \Rightarrow t = \frac{\pi}{2}$

- The Newton–Leibniz formula, F(b) – F(a)

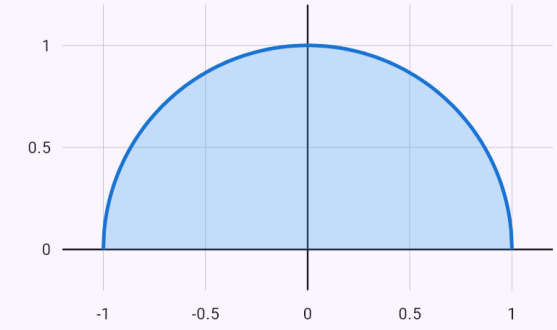
$$\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx = \left[r^2 \left(\frac{t}{2} + \frac{\sin(2t)}{4}\right)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi r^2}{4} - \left($$

Values of the parameters

$r = 1$

$$\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx \approx 1.571$$

Graph



The definite integral is the area between the graph and the x-axis – the part above the axis counts positively, the part below negatively.

Improper integrals

The app handles an infinite limit or an unbounded function with a limit of the antiderivative and says whether the integral converges or diverges. If that limit cannot be computed, it decides by a comparison test.

Integrals $\int_0^{\infty} x^{s-1} \ln^k x/(x^n + a^n)^m\, dx$ ($0 < s < mn$, $k = 0$ to 3) are computed exactly even without an antiderivative: as the k-th derivative with respect to s of the integral with a parameter I(s). The app finds I(s) by the substitution $t = x^n/(x^n + a^n)$ via the beta function and the reflection formula (chip “Beta function”), or over a keyhole contour around the positive half-axis via residues (chip “Residues”). Where

an antiderivative exists, the Newton–Leibniz formula stays the default (chip “Antiderivative”). The value is always checked numerically.

Another improper integral without an antiderivative is compared with 1/x^α at each end (an infinite limit or a point where the function grows beyond all bounds) by the limit comparison test. Only when it converges at every end does the app compute the value numerically (≈); a divergent integral is reported as divergent.

Examples

An infinite upper limit.

Input: `int(1/x^2, x, 1, oo)`

Result

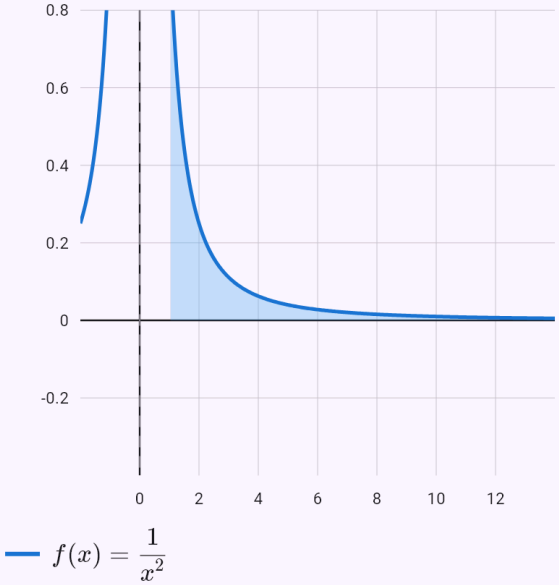
$$\int_1^{\infty} \frac{1}{x^2}\,dx = 1$$

Steps

- From the table of basic integrals: $\int \frac{1}{x^2}\,dx = -\frac{1}{x}$
- An improper integral – at an infinite limit and where the function is unbounded we take the limit of the antiderivative. The limits are finite, the integral converges

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{x\to\infty}\left(-\frac{1}{x}\right) - (-1) = 0 - (-1) = 1$$

Graph



The definite integral is the area between the graph and the x-axis – the part above the axis counts positively, the part below negatively.

A function unbounded at the lower limit.

Input: `int(1/sqrt(x), x, 0, 1)`

Result

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2$$

Solve by a numerical method:

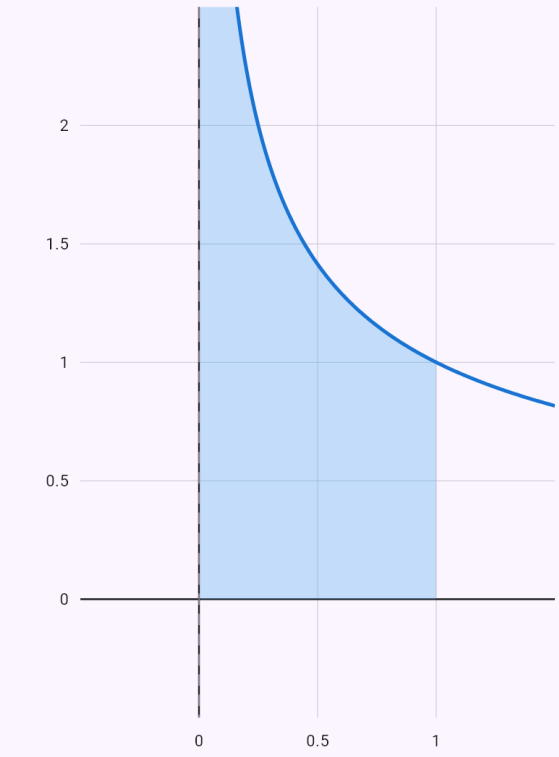
Σ Simpson | Σ Trapezoidal | Σ Romberg

Steps

- From the table of basic integrals: $\int \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}$
- An improper integral – at an infinite limit and where the function is unbounded we take the limit of the antiderivative. The limits are finite, the integral converges

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2 - \lim_{x\to 0^+} 2\sqrt{x} = 2 - 0 = 2$$

Graph



$f(x) = \frac{1}{\sqrt{x}}$

above the x-axis (+) below the x-axis (−)

The definite integral is the area between the graph and the x-axis – the part above the axis counts positively, the part below negatively.

∫₀^∞ x³ ln²x/(1 + x⁶) dx = I''(4) via the beta function – 5√3π³/972.

Input: `int(x^3*ln(x)^2/(1+x^6), x, 0, oo)`

Result

$$\int_0^\infty \frac{x^3(\ln x)^2}{x^6+1}\,dx = \frac{5\sqrt{3}\pi^3}{972} \approx 0.2763$$

Steps

- The logarithm comes from differentiating with respect to the parameter s: $\frac{\partial^2}{\partial s^2}\, x^{s-1} = x^{s-1}\ln^2 x$. So the integral is a derivative of the integral with a parameter $I(s) = \int_0^\infty \frac{x^{s-1}}{x^6+1}\,dx$, which converges for $0 < s < 6$: $\int_0^\infty \frac{x^3(\ln x)^2}{x^6+1}\,dx = I''(4)$. We compute I(s) for a general s, differentiate and substitute.

- The substitution takes the integral to ⟨0; 1⟩, giving Euler's beta function B(p, q) = ∫₀¹ t□⁻¹(1 − t)^(q − 1) dt

$$t = \frac{x^6}{x^6+1} \;\Rightarrow\; x = \left(\frac{t}{1-t}\right)^{\frac{1}{6}}, \quad dx = \frac{1}{6\sqrt[6]{t^5}\sqrt[6]{(1-t)^7}}$$

$$I(s) = \frac{1}{6}\int_0^1 t^{\frac{s}{6}-1}(1-t)^{-\frac{s}{6}}\,dt = \frac{1}{6}\mathrm{B}\left(\frac{s}{6}, 1-\frac{s}{6}\right)$$

- We write the beta function with the gamma function, B(p, q) = Γ(p)Γ(q)/Γ(p + q), and use the reflection formula Γ(p)Γ(1 − p) = π/sin πp

$$\mathrm{B}\left(\frac{s}{6}, 1-\frac{s}{6}\right) = \frac{\Gamma\left(\frac{s}{6}\right)\Gamma\left(1-\frac{s}{6}\right)}{\Gamma(1)} = \frac{\pi}{\sin\frac{\pi s}{6}}$$

$$I(s) = \frac{\pi}{6\sin\frac{\pi s}{6}}$$

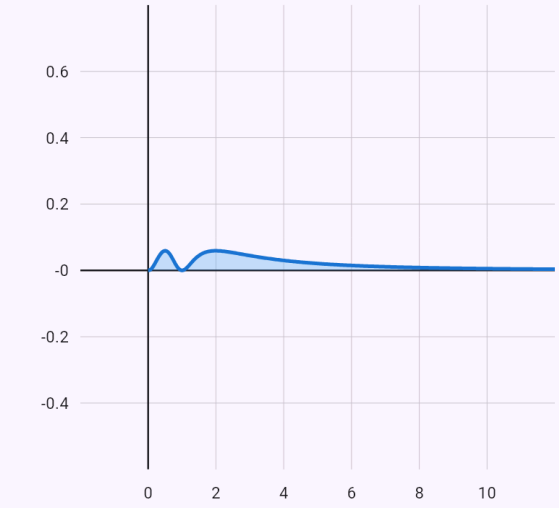
- The derivative with respect to the parameter s (of order 2)

$$I''(s) = \frac{\pi^3\left(\cot^2\left(\frac{\pi s}{6}\right) + \frac{1}{\sin^2\left(\frac{\pi s}{6}\right)}\right)}{216\sin\left(\frac{\pi s}{6}\right)}$$

- We substitute s_0

$$\int_0^\infty \frac{x^3(\ln x)^2}{x^6+1}\,dx = I''(4) = \frac{5\sqrt{3}\pi^3}{972} \approx 0.2763$$

Graph



$f(x) = \frac{x^3(\ln x)^2}{x^6+1}$

above the x-axis (+) below the x-axis (−)

The definite integral is the area between the graph and the x-axis – the part above the axis counts positively, the part below negatively.

The same integral over a keyhole contour via residues.

Input: `int(x^3*ln(x)^2/(1+x^6), x, 0, oo)` Residues

Result

$$\int_0^\infty \frac{x^3(\ln x)^2}{x^6+1}\,dx = \frac{5\sqrt{3}\pi^3}{972} \approx 0.2763$$

Steps

- The logarithm comes from differentiating with respect to the parameter s: $\frac{\partial^2}{\partial s^2} x^{s-1} = x^{s-1}\ln^2 x$. So the integral is a derivative of the integral with a parameter $I(s) = \int_0^\infty \frac{x^{s-1}}{x^6+1}\,dx$, which converges for $0 < s < 6$: $\int_0^\infty \frac{x^3(\ln x)^2}{x^6+1}\,dx = I''(4)$. We compute I(s) for a general s, differentiate and substitute.
- We take the power z^(s − 1) = e^((s − 1) ln z) with the argument of z in (0; 2π) and integrate over the keyhole contour C around the positive real half-axis. On the upper edge z^(s − 1) = x^(s − 1), on the lower one x^(s − 1)·e^(2πis); the large and the small circle contribute nothing in the limit because $0 < s < 6$. Inside lie the poles $z_j = e^{\frac{(2j+1)\pi i}{6}}$, $j = 0, 1, \ldots, 5$ of order 1, and by the residue theorem (for s not an integer)

$$\oint_C \frac{z^{s-1}}{z^6+1}\,dz = \left(1-e^{2\pi i s}\right) I(s) = 2\pi i \sum_{j=0}^{5} \operatorname{Res}_{z=z_j} \frac{z^{s-1}}{z^6+}$$

- The poles are simple; the residue is the numerator over the derivative of the denominator at the pole, with $z_j^n = -a^n$

$$\operatorname{Res}_{z=z_j} = \frac{z_j^{s-1}}{6z_j^5} = \frac{z_j^s}{6z_j^6} = -\frac{z_j^s}{6}$$

- We substitute the powers of the poles, sum the geometric series and simplify. The formula was derived for s not an integer (otherwise 1 − e^(2πis) = 0). Both sides are continuous in s, though, so it holds for s = 4 as well − by the limit s → 4

$$z_j^s = e^{\frac{(2j+1)\pi i s}{6}}$$

$$\sum_{j=0}^{5} e^{\frac{(2j+1)\pi i s}{6}} = \frac{e^{\frac{\pi i s}{6}}\left(1-e^{2\pi i s}\right)}{1-e^{\frac{\pi i s}{3}}}$$

$$\frac{2\pi i\, e^{\frac{\pi i s}{6}}}{1-e^{\frac{\pi i s}{3}}} = -\frac{\pi}{\sin\frac{\pi s}{6}}$$

$$I(s) = \frac{\pi}{6\sin\frac{\pi s}{6}}$$

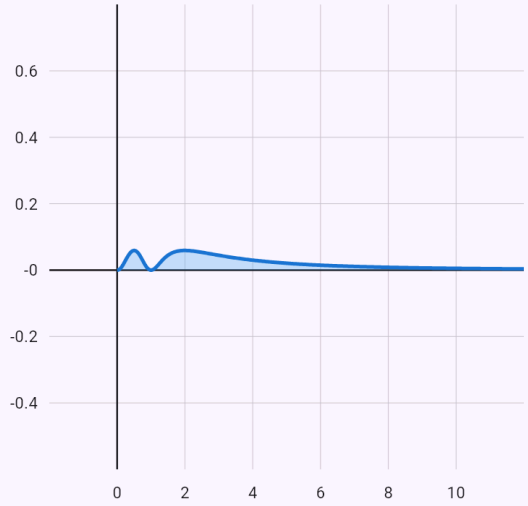
- The derivative with respect to the parameter s (of order 2)

$$I''(s) = \frac{\pi^3\left(\cot^2\left(\frac{\pi s}{6}\right) + \frac{1}{\sin^2\left(\frac{\pi s}{6}\right)}\right)}{216\sin\left(\frac{\pi s}{6}\right)}$$

- We substitute s_0

$$\int_0^\infty \frac{x^3(\ln x)^2}{x^6+1}\,dx = I''(4) = \frac{5\sqrt{3}\pi^3}{972} \approx 0.2763$$

Graph



$f(x) = \frac{x^3(\ln x)^2}{x^6+1}$

above the x-axis (+) below the x-axis (−)

The definite integral is the area between the graph and the x-axis – the part above the axis counts positively, the part below negatively.

No antiderivative: convergence by comparison with 1/x³, then the value numerically.

Input: `int(1/(x^3+ln(x)), x, 2, oo)`

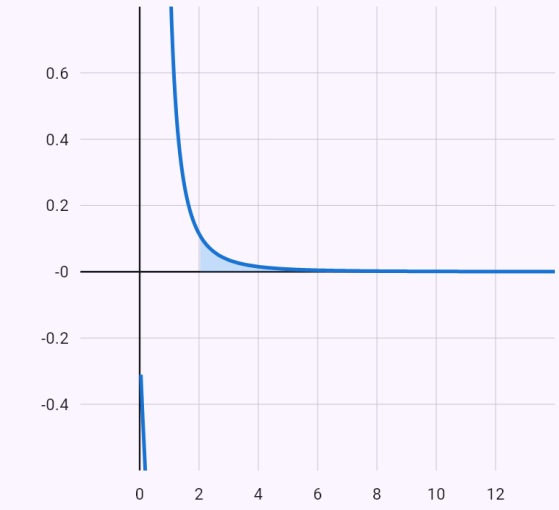
Result

$$\int_2^{\infty} \frac{1}{x^3 + \ln x}\, dx \approx 0.1197$$

Steps

- As $x \to \infty$ we compare the integrand with $\frac{1}{x^3}$ (limit comparison test): $\lim_{x\to\infty} \frac{\frac{1}{x^3+\ln x}}{\frac{1}{x^3}} = 1$. The limit is a positive number, so the integral behaves there like the integral of $\frac{1}{x^3}$, which converges for $\alpha = 3 > 1$
- The integral converges at every end; its value: $\int_2^{\infty} \frac{1}{x^3 + \ln x}\, dx \approx 0.1197$
- No exact form was found, so the value of the improper integral was computed numerically by adaptive Gauss–Kronrod quadrature.

Graph



$f(x) = \frac{1}{x^3 + \ln x}$

above the x-axis (+) below the x-axis (−)

The definite integral is the area between the graph and the x-axis – the part above the axis counts positively, the part below negatively.

Area under a curve and between curves

The app computes the area between a graph and the x-axis, or between two curves, as the integral of the absolute value of the difference – parts above and below both count as positive. You don't have to enter the limits: it finds them as the intersections of the curves.

How to enter it

`area(x^2, x)`	The area between two curves, limits from the intersections.
`area(sin x, 0, pi)`	The area between a curve and the x-axis on an interval.
`x^2`	A function with the “Integrate” chip → “Area between curves”; fill in the second curve and the limits in the fields.

Examples

The area between a parabola and a line.

Input: `x^2` Area between curves, second curve x

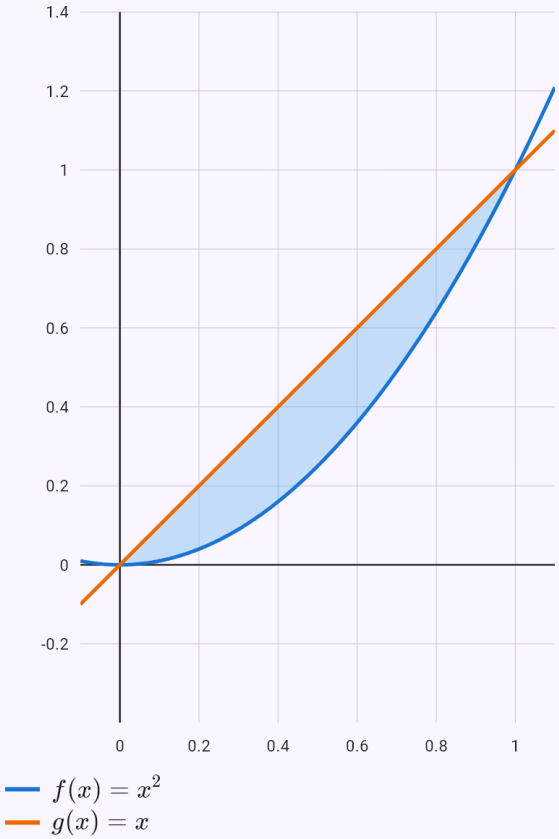
Result

$$S = \frac{1}{6}$$

Steps

- The limits are the curves' intersections:
  $x^2 = x \Rightarrow x = 0,\ x = 1$
- The area between the curves (parts above and below count positively), $S = \int_a^b |f(x) - g(x)|\,dx$:
  $S = \int_0^1 |x^2 - x|\,dx$
- The expression in absolute value doesn't change its sign on the interval – the absolute value removed by it:
  $\int_0^1 |x^2 - x|\,dx = \int_0^1 (-x^2 + x)\,dx$
- The integral of a sum is the sum of the integrals:
  $\int (-x^2 + x)\,dx = \int -x^2\,dx + \int x\,dx$
- The constant taken out of the integral:
  $\int -x^2\,dx = -\int x^2\,dx$
- From the table of basic integrals: $\int x^2\,dx = \frac{x^3}{3}$
- From the table of basic integrals: $\int x\,dx = \frac{x^2}{2}$
- The Newton–Leibniz formula, F(b) − F(a)
  $\int_0^1 (-x^2 + x)\,dx = \left[-\frac{x^3}{3} + \frac{x^2}{2}\right]_0^1 = \frac{1}{6} - 0 = \frac{1}{6}$

Graph



The shaded region – its area is the result (parts above and below the axis count positively).

Arc length

The app computes the length of a graph on an interval by the formula l = ∫√(1 + f′²) dx. It tries to simplify the root (for a semicircle it comes out as arcsine); where the integral has no elementary antiderivative, the result is numerical.

How to enter it

`length(x^(3/2), 0, 4)`	The arc length on an interval.

Examples

The length of a semicircle.

Input: `sqrt(1 - x^2)` Arc length, limits −1 and 1

Result

$$l = \pi \approx 3.1416$$

Steps

- The length of the curve

$$l = \int_a^b \sqrt{1 + (f'(x))^2}\, dx, \quad f'(x) = -\frac{x}{\sqrt{1 - x^2}}$$

$$l = \int_{-1}^{1} \sqrt{1 + \left(-\frac{x}{\sqrt{1 - x^2}}\right)^2}\, dx = \int_{-1}^{1} \frac{1}{\sqrt{1 - x^2}}\, dx$$

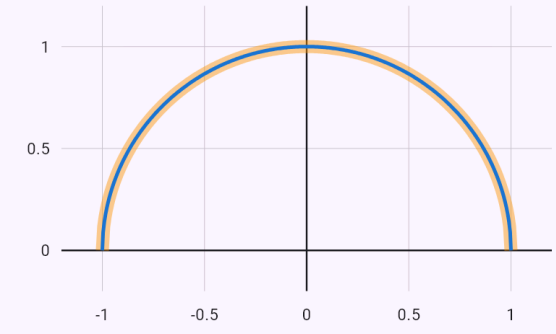
- From the table of basic integrals:

$$\int \frac{1}{\sqrt{1 - x^2}}\, dx = \arcsin x$$

- An improper integral – at an infinite limit and where the function is unbounded we take the limit of the antiderivative. The limits are finite, the integral converges

$$\int_{-1}^{1} \frac{1}{\sqrt{1 - x^2}}\, dx = \lim_{x \to 1^-} \arcsin x - \lim_{x \to -1^+} \arcsin x = \frac{\pi}{2}$$

Graph



$f(x) = \sqrt{1 - x^2}$

The highlighted arc – its length is the result.

Volume of a solid of revolution

The app computes the volume of a solid of revolution about the x-axis by the formula V = π∫f² dx (between two curves as washers), about the y-axis by the shell method. It also shows the solid in 3D – rotate it with your finger.

How to enter it

`volume(sqrt(x), 0, 4)`	The volume of the solid rotated about the x-axis.
`volume(x^2, 0, 2, y)`	Rotation about the y-axis.

Examples

A paraboloid of revolution.

Input: `sqrt(x)` Volume of a solid, limits 0 and 4

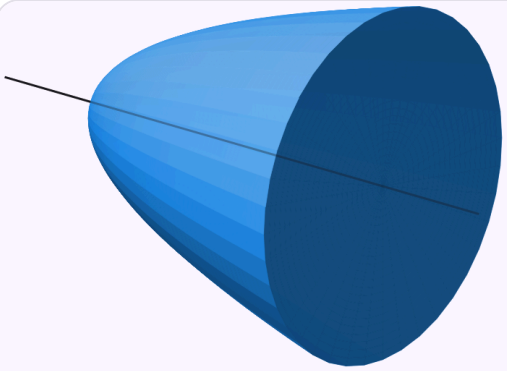
Result

$$V = 8\pi \approx 25.1327$$

Steps

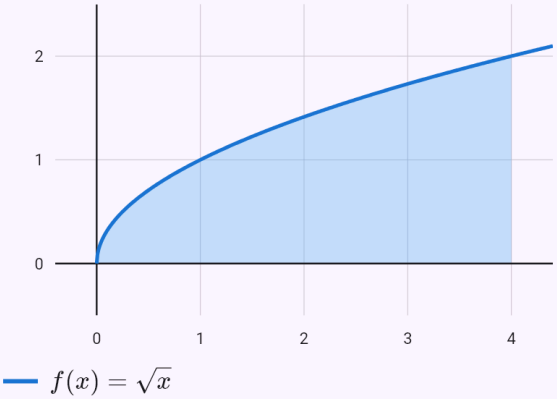
- The volume of the solid of revolution about the x-axis, $V = \pi \int_a^b f^2(x)\,dx$: $V = \pi \int_0^4 \left(\sqrt{x}\right)^2 dx = \pi \int_0^4 x\,dx$
- From the table of basic integrals: $\int x\,dx = \frac{x^2}{2}$
- The Newton–Leibniz formula, F(b) − F(a): $\int_0^4 x\,dx = \left[\frac{x^2}{2}\right]_0^4 = 8 - 0 = 8$
- The integral's value times the constant in front of it: $V = \pi \cdot 8 = 8\pi$

Solid of revolution



The shaded region rotates about the x-axis – giving the solid above; turn it with your finger.

Graph



Surface area of a solid of revolution

The app computes the lateral surface area by the formula S = 2π∫f√(1 + f′²) dx; the product under the root often simplifies (for a sphere it comes out constant).

How to enter it

`surface(sqrt(4 - x^2), -2, 2)`	The surface of the solid rotated about the x-axis.

Examples

The surface of a sphere of radius 2.

Input: `sqrt(4 - x^2)` Surface of a solid, limits −2 and 2

Result

$$S = 16\pi \approx 50.2655$$

Steps

- The lateral surface of the solid of revolution about the x-axis

$$S = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\,dx, \quad f'(x) = -\frac{x}{\sqrt{4 - x}}$$

$$S = 2\pi \int_{-2}^{2} \sqrt{4 - x^2}\sqrt{1 + \left(-\frac{x}{\sqrt{4 - x^2}}\right)^2}\,dx = 2\pi \int_{-2}^{2}$$

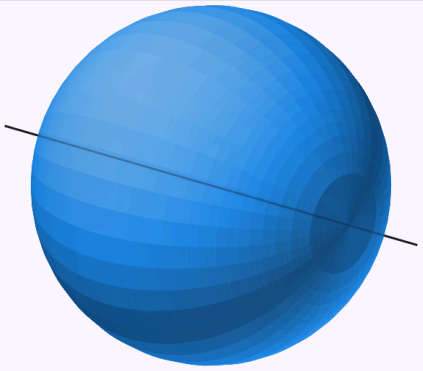
- From the table of basic integrals: $\int 2\,dx = 2x$
- The Newton–Leibniz formula, F(b) − F(a):

$$\int_{-2}^{2} 2\,dx = [2x]_{-2}^{2} = 4 - (-4) = 8$$

- The integral's value times the constant in front of it:

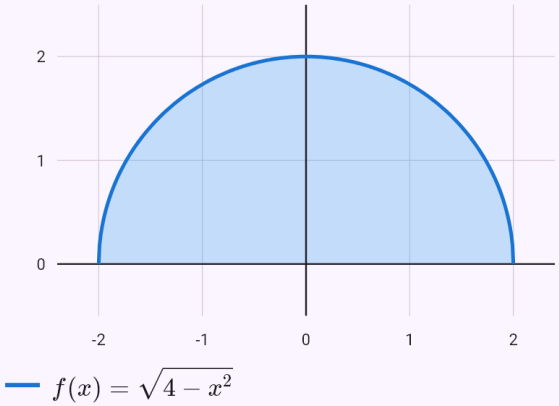
$$S = 2\pi \cdot 8 = 16\pi$$

Solid of revolution



The shaded region rotates about the x-axis – giving the solid above; turn it with your finger.

Graph



Applications with parameters

Every letter except x is a positive parameter and the result is a formula, such as the sphere's volume 4πr³/3. Numeric checks run with values filled in. The graph and the 3D solid have a slider for each parameter.

How to enter it

`objem(sqrt(r^2-x^2), -r, r)`	The volume of a sphere of radius r.

Examples

The volume of a sphere of radius r.

Input: `objem(sqrt(r^2-x^2), -r, r)`

Result

$$V = \frac{4\pi r^3}{3}$$

Steps

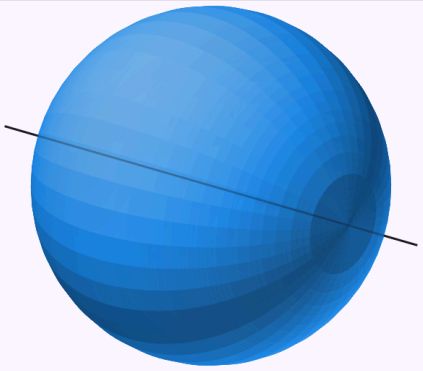
- Letters other than the variable are positive parameters: $r > 0$
- The volume of the solid of revolution about the x-axis,
  $V = \pi \int_a^b f^2(x)\,dx$
  $V = \pi \int_{-r}^{r} \left(\sqrt{r^2 - x^2}\right)^2 dx = \pi \int_{-r}^{r} \left(r^2 - x^2\right) dx$
- The integral of a sum is the sum of the integrals:
  $\int \left(r^2 - x^2\right) dx = \int -x^2\,dx + \int r^2\,dx$
- The constant taken out of the integral:
  $\int -x^2\,dx = -\int x^2\,dx$
- From the table of basic integrals: $\int x^2\,dx = \frac{x^3}{3}$
- From the table of basic integrals: $\int r^2\,dx = r^2 x$
- The Newton–Leibniz formula, F(b) − F(a)
  $\int_{-r}^{r} \left(r^2 - x^2\right) dx = \left[-\frac{x^3}{3} + r^2 x\right]_{-r}^{r} = \frac{2r^3}{3} - \left(-\frac{2r^3}{3}\right)$
- The integral's value times the constant in front of it:
  $V = \pi \cdot \frac{4r^3}{3} = \frac{4\pi r^3}{3}$

Values of the parameters

$r = 1$

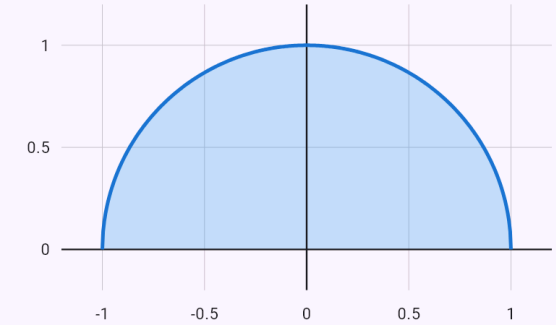
$V \approx 4.189$

Solid of revolution



The shaded region rotates about the x-axis – giving the solid above; turn it with your finger.

Graph



A frustum with radii a, b and height h.

Input: `objem(a+(b-a)*x/h, 0, h)`

Result

$$V = \frac{\pi h(a^2 + ab + b^2)}{3}$$

Steps

- Letters other than the variable are positive parameters: $a > 0,\ b > 0,\ h > 0$
- The volume of the solid of revolution about the x-axis, $V = \pi \int_a^b f^2(x)\,dx$: $V = \pi \int_0^h \left(\frac{(b-a)x}{h} + a\right)^2 dx$
- From the table – the inner function is linear, so the result is divided by its coefficient of the variable

$$\int \left(\frac{(b-a)x}{h} + a\right)^2 dx = \frac{h\left(\frac{(b-a)x}{h} + a\right)^3}{3(b-a)}$$

- The Newton–Leibniz formula, F(b) − F(a)

$$\int_0^h \left(\frac{(b-a)x}{h} + a\right)^2 dx = \left[\frac{(-ax + bx + ah)^3}{3h^2(b-a)}\right]_0^h = \frac{}{3}$$

- The integral's value times the constant in front of it

$$V = \pi \cdot \frac{h(a^2 + ab + b^2)}{3} = \frac{\pi h(a^2 + ab + b^2)}{3}$$

Values of the parameters

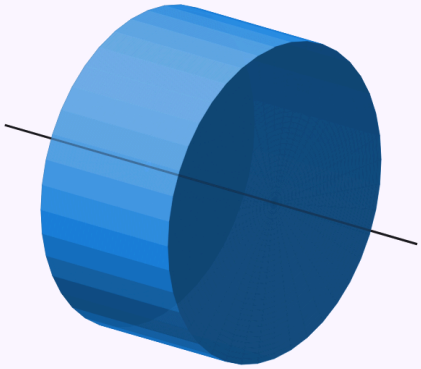
$a = 1$

$b = 1$

$h = 1$

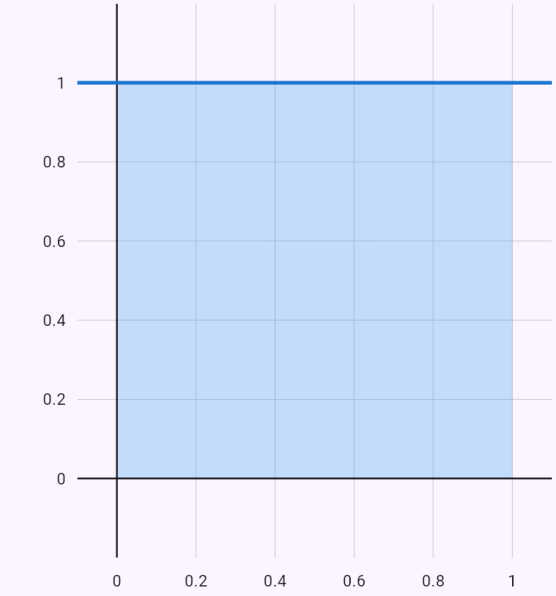
$V \approx 3.142$

Solid of revolution



The shaded region rotates about the x-axis – giving the solid above; turn it with your finger.

Graph



Centroid

The centroid of a region under or between curves, of an arc and of a solid of revolution comes from integrals with all the steps and is marked in the graph. Letters work as parameters.

How to enter it

`teziste(x^2, 0, 1)` — A region under the curve; a second function gives a region between curves.

`teziste(sqrt(r^2-x^2), -r, r, oblouk)`	The last word oblouk or teleso picks an arc or a solid of revolution.

Examples

The centroid of a half-disc, 4r/(3π).

Input: `teziste(sqrt(r^2-x^2), -r, r)`

Result

$$T\left[0;\ \frac{4r}{3\pi}\right]$$

Steps

- Letters other than the variable are positive parameters: $r > 0$
- The centroid comes from integrals
  $x_T = \frac{1}{S}\int_a^b x\, f(x)\, dx$
  $y_T = \frac{1}{2S}\int_a^b f^2(x)\, dx$
- We compute the integral: $S = \int_{-r}^{r} \sqrt{r^2 - x^2}\, dx$
- A root of a difference of squares – the trigonometric substitution $x = r\sin t$, $dx = r\cos t\, dt$, the root $\sqrt{r^2 - x^2} = r\cos t$: $\int \sqrt{r^2 - x^2}\, dx = \int r^2\cos^2 t\, dt$
- The constant taken out of the integral:
  $\int r^2\cos^2 t\, dt = r^2\int \cos^2 t\, dt$
- Using the formula $\cos^2 t = \frac{1+\cos 2t}{2}$:
  $\int \cos^2 t\, dt = \int \left(\frac{\cos(2t)}{2} + \frac{1}{2}\right) dt$
- The integral of a sum is the sum of the integrals
  $\int \left(\frac{\cos(2t)}{2} + \frac{1}{2}\right) dt = \int \frac{1}{2}\, dt + \int \frac{\cos(2t)}{2}\, dt$
- From the table of basic integrals: $\int \frac{1}{2}\, dt = \frac{t}{2}$
- The constant taken out of the integral:
  $\int \frac{\cos(2t)}{2}\, dt = \frac{1}{2}\int \cos(2t)\, dt$
- From the table – the inner function is linear, so the result is divided by its coefficient of the variable:
  $\int \cos(2t)\, dt = \frac{\sin(2t)}{2}$
- With the substitution, the limits are recomputed too – no need to go back to the original variable:
  $x = -r \Rightarrow t = -\frac{\pi}{2}, \quad x = r \Rightarrow t = \frac{\pi}{2}$
- The Newton–Leibniz formula, F(b) − F(a)
  $\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx = \left[r^2\left(\frac{t}{2} + \frac{\sin(2t)}{4}\right)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi r^2}{4} - \left($
- We compute the integral: $M_y = \int_{-r}^{r} x\sqrt{r^2 - x^2}\, dx$
- Substituting $t = r^2 - x^2$, $dt = -2x\, dx$:
  $\int x\sqrt{r^2 - x^2}\, dx = \int -\frac{\sqrt{t}}{2}\, dt$
- The constant taken out of the integral:
  $\int -\frac{\sqrt{t}}{2}\, dt = -\frac{1}{2}\int \sqrt{t}\, dt$
- From the table of basic integrals: $\int \sqrt{t}\, dt = \frac{2\sqrt{t^3}}{3}$
- With the substitution, the limits are recomputed too – no need to go back to the original variable:
  $x = -r \Rightarrow t = 0, \quad x = r \Rightarrow t = 0$
- The Newton–Leibniz formula, F(b) − F(a)

$$\int_{-r}^{r} x\sqrt{r^2-x^2}\,dx = \left[-\frac{\sqrt{t^3}}{3}\right]_0^0 = 0-0=0$$

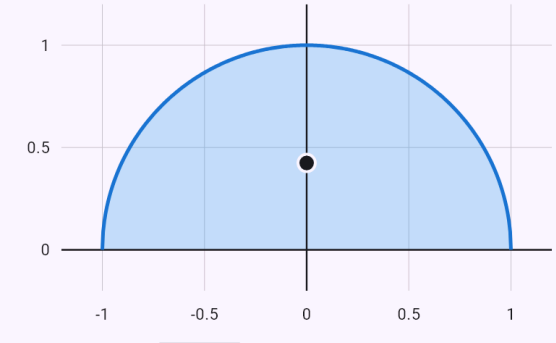
- We compute the integral: $M_x = \frac{1}{2}\int_{-r}^{r}\left(r^2-x^2\right)dx$
- The integral of a sum is the sum of the integrals: $\int\left(r^2-x^2\right)dx = \int -x^2\,dx + \int r^2\,dx$
- The constant taken out of the integral: $\int -x^2\,dx = -\int x^2\,dx$
- From the table of basic integrals: $\int x^2\,dx = \frac{x^3}{3}$
- From the table of basic integrals: $\int r^2\,dx = r^2x$
- The Newton–Leibniz formula, F(b) − F(a) $\int_{-r}^{r}\left(r^2-x^2\right)dx = \left[-\frac{x^3}{3}+r^2x\right]_{-r}^{r} = \frac{2r^3}{3}-\left(-\frac{2r^3}{3}\right)$
- The centroid's coordinates are quotients of the integrals $x_T = \frac{M_y}{S} = 0$ $y_T = \frac{M_x}{S} = \frac{4r}{3\pi}$

Values of the parameters

$r = 1$

$T \approx [0;\ 0.424]$

Graph



$f(x) = \sqrt{r^2-x^2}$

The point T is the centroid.

A cone's centroid, 3/4 of the height from the apex.

Input: `teziste(r*x/h, 0, h, teleso)`

Result

$$T\left[\frac{3h}{4};\ 0\right]$$

Steps

- Letters other than the variable are positive parameters: $h > 0,\ r > 0$
- The centroid comes from integrals

$$x_T = \frac{\int_a^b x\, f^2(x)\, dx}{\int_a^b f^2(x)\, dx}$$

$$y_T = 0$$

- We compute the integral: $J = \int_0^h \frac{r^2x^2}{h^2}\, dx$
- The constant taken out of the integral: $\int \frac{r^2x^2}{h^2}\, dx = \frac{r^2}{h^2}\int x^2\, dx$
- From the table of basic integrals: $\int x^2\, dx = \frac{x^3}{3}$
- The Newton–Leibniz formula, F(b) − F(a)

$$\int_0^h \frac{r^2x^2}{h^2}\, dx = \left[\frac{r^2x^3}{3h^2}\right]_0^h = \frac{hr^2}{3} - 0 = \frac{hr^2}{3}$$

- We compute the integral: $M_y = \int_0^h \frac{r^2x^3}{h^2}\, dx$
- The constant taken out of the integral: $\int \frac{r^2x^3}{h^2}\, dx = \frac{r^2}{h^2}\int x^3\, dx$
- From the table of basic integrals: $\int x^3\, dx = \frac{x^4}{4}$
- The Newton–Leibniz formula, F(b) − F(a)

$$\int_0^h \frac{r^2x^3}{h^2}\, dx = \left[\frac{r^2x^4}{4h^2}\right]_0^h = \frac{h^2r^2}{4} - 0 = \frac{h^2r^2}{4}$$

- The centroid's coordinates are quotients of the integrals

$$x_T = \frac{M_y}{J} = \frac{3h}{4}$$

$$y_T = 0$$

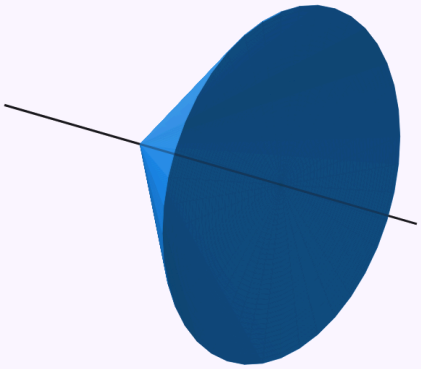
Values of the parameters

$h = 1$

$r = 1$

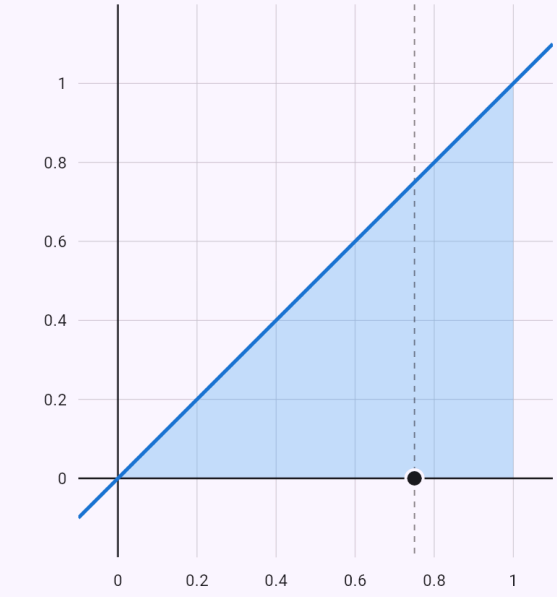
$T \approx [0.75;\ 0]$

Solid of revolution



The shaded region rotates about the x-axis – giving the solid above; turn it with your finger.

Graph



$f(x) = \frac{rx}{h}$

The point T is the centroid.

Parametric curves and curves in polar coordinates

Write a curve x = φ(t), y = ψ(t) on two lines, a curve in polar coordinates as r = ρ(φ). The “Integrate” chip then offers the area, length, volume and surface; the limits of the parameter are optional (without them from 0 to 2π). The graph shades the region, and volume and surface appear as a 3D solid. Rotation is about the x- or y-axis; for the y-axis add y as the last argument.

How to enter it

`x = t - sin(t)` `y = 1 - cos(t)`	A parametric curve on two lines.
`r = 1 + cos(phi)`	A curve in polar coordinates (also r^2 = … for a lemniscate).
`length(x=cos(t)^3, y=sin(t)^3, t, 0, 2pi)`	The written form with the parameter and the limits.

Examples

The area under an arch of a cycloid.

Input: `x = t - sin(t)`

`y = 1 - cos(t)` Area

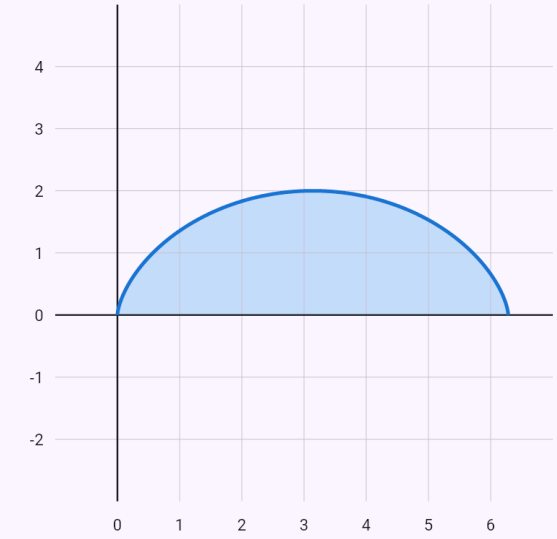
Result

$$S = 3\pi \approx 9.4248$$

Steps

- The derivatives by the parameter:
  $x'(t) = 1 - \cos t, \quad y'(t) = \sin t$
- The area under a parametric curve (inside a closed one)
  for $t \in [0,\ 2\pi]$, $S = \left|\int_{\alpha}^{\beta} y(t)\, x'(t)\, dt\right|$:
  $S = \left|\int_{0}^{2\pi} \left(-2\cos t + \cos^2 t + 1\right) dt\right|$
- The integral of a sum is the sum of the integrals
  $\int \left(-2\cos t + \cos^2 t + 1\right) dt = \int 1\, dt + \int -2\cos t\, dt$ -
- From the table of basic integrals: $\int 1\, dt = t$
- The constant taken out of the integral:
  $\int -2\cos t\, dt = -2\int \cos t\, dt$
- From the table of basic integrals: $\int \cos t\, dt = \sin t$
- Using the formula $\cos^2 t = \frac{1 + \cos 2t}{2}$:
  $\int \cos^2 t\, dt = \int \left(\frac{\cos(2t)}{2} + \frac{1}{2}\right) dt$
- The integral of a sum is the sum of the integrals
  $\int \left(\frac{\cos(2t)}{2} + \frac{1}{2}\right) dt = \int \frac{1}{2}\, dt + \int \frac{\cos(2t)}{2}\, dt$
- From the table of basic integrals: $\int \frac{1}{2}\, dt = \frac{t}{2}$
- The constant taken out of the integral:
  $\int \frac{\cos(2t)}{2}\, dt = \frac{1}{2}\int \cos(2t)\, dt$
- From the table – the inner function is linear, so the result is divided by its coefficient of the variable:
  $\int \cos(2t)\, dt = \frac{\sin(2t)}{2}$
- The Newton–Leibniz formula, F(b) − F(a)
  $\int_{0}^{2\pi} \left(-2\cos t + \cos^2 t + 1\right) dt = \left[\frac{3t}{2} - 2\sin t + \frac{\sin(2t}{4}\right.$

Graph



$x = t - \sin t,\ y = 1 - \cos t$

The curve given parametrically (or in polar coordinates), drawn for the chosen limits of the parameter.

The length of a cardioid.

Input: `r = 1 + cos(phi)` Arc length

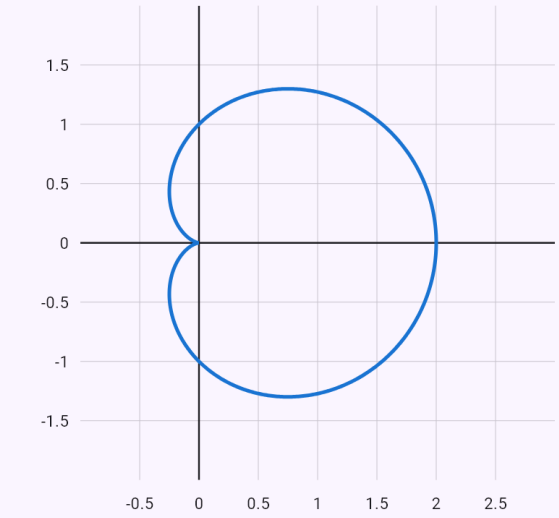
Result

$$l = 8$$

Steps

- The derivatives by the parameter: $\rho'(\varphi) = -\sin\varphi$
- The length of a polar curve for $\varphi \in [0,\, 2\pi]$,
  $$l = \int_{\alpha}^{\beta} \sqrt{\rho^2(\varphi) + \rho'^2(\varphi)}\, d\varphi$$
  $$l = \int_{0}^{2\pi} \sqrt{(\cos\varphi + 1)^2 + (-\sin\varphi)^2}\, d\varphi = \int_{0}^{2\pi} 2\left|\cos\left($$
- We split the interval at the zero points of the expressions in absolute value (π) and remove the absolute value in each part by its sign
  $$\int_{0}^{2\pi} 2\left|\cos\left(\frac{\varphi}{2}\right)\right| d\varphi = \int_{0}^{\pi} 2\cos\left(\frac{\varphi}{2}\right) d\varphi + \int_{\pi}^{2\pi} -2\cos$$
- The constant taken out of the integral
  $$\int 2\cos\left(\frac{\varphi}{2}\right) d\varphi = 2\int \cos\left(\frac{\varphi}{2}\right) d\varphi$$
- From the table – the inner function is linear, so the result is divided by its coefficient of the variable:
  $$\int \cos\left(\frac{\varphi}{2}\right) d\varphi = 2\sin\left(\frac{\varphi}{2}\right)$$
- The Newton–Leibniz formula, F(b) − F(a)
  $$\int_{0}^{\pi} 2\cos\left(\frac{\varphi}{2}\right) d\varphi = \left[4\sin\left(\frac{\varphi}{2}\right)\right]_{0}^{\pi} = 4 - 0 = 4$$
- The constant taken out of the integral
  $$\int -2\cos\left(\frac{\varphi}{2}\right) d\varphi = -2\int \cos\left(\frac{\varphi}{2}\right) d\varphi$$
- From the table – the inner function is linear, so the result is divided by its coefficient of the variable:
  $$\int \cos\left(\frac{\varphi}{2}\right) d\varphi = 2\sin\left(\frac{\varphi}{2}\right)$$
- The Newton–Leibniz formula, F(b) − F(a)
  $$\int_{\pi}^{2\pi} -2\cos\left(\frac{\varphi}{2}\right) d\varphi = \left[-4\sin\left(\frac{\varphi}{2}\right)\right]_{\pi}^{2\pi} = 0 - (-4) = 4$$
- The parts' values added up:
  $$\int_{0}^{2\pi} 2\left|\cos\left(\frac{\varphi}{2}\right)\right| d\varphi = 4 + 4 = 8$$

Graph



$r = \cos\varphi + 1$

The curve given parametrically (or in polar coordinates), drawn for the chosen limits of the parameter.

An astroid rotated about the y-axis.

Input: `x = cos(t)^3`

```
y = sin(t)^3
```

Result

$$V = \frac{32\pi}{105} \approx 0.9574$$

Steps

- The derivatives by the parameter:
  $x'(t) = -3\cos^2 t \sin t, \quad y'(t) = 3\sin^2 t \cos t$
- The solid comes from the part of the curve to the right of the y-axis (x ≥ 0): $t \in \left[0,\ \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2},\ 2\pi\right]$
- The volume of the solid about the y-axis for $t \in [0,\ 2\pi]$,
  $V = \pi \left| \int_{\alpha}^{\beta} x^2(t)\, y'(t)\, dt \right|$
  $V = \pi \left(\left| \int_{0}^{\frac{\pi}{2}} 3\cos^7 t \sin^2 t\, dt \right| + \left| \int_{\frac{3\pi}{2}}^{2\pi} 3\cos^7 t \sin^2 t\, dt \right| \right)$
- The constant taken out of the integral:
  $\int 3\cos^7 t \sin^2 t\, dt = 3 \int \cos^7 t \sin^2 t\, dt$
- An odd power – the substitution $w = \sin t$,
  $\cos^2 t = 1 - \sin^2 t$
  $dw = \cos t\, dt$
  $\int \cos^7 t \sin^2 t\, dt = \int \left(-w^8 + 3w^6 - 3w^4 + w^2\right) dw$
- The integral of a sum is the sum of the integrals
  $\int \left(-w^8 + 3w^6 - 3w^4 + w^2\right) dw = \int -w^8\, dw + \int 3w$
- The constant taken out of the integral:
  $\int -w^8\, dw = -\int w^8\, dw$
- From the table of basic integrals: $\int w^8\, dw = \frac{w^9}{9}$
- The constant taken out of the integral:
  $\int 3w^6\, dw = 3 \int w^6\, dw$
- From the table of basic integrals: $\int w^6\, dw = \frac{w^7}{7}$
- The constant taken out of the integral:
  $\int -3w^4\, dw = -3 \int w^4\, dw$
- From the table of basic integrals: $\int w^4\, dw = \frac{w^5}{5}$
- From the table of basic integrals: $\int w^2\, dw = \frac{w^3}{3}$
- With the substitution, the limits are recomputed too – no need to go back to the original variable:
  $t = 0 \Rightarrow w = 0, \quad t = \frac{\pi}{2} \Rightarrow w = 1$
- The Newton–Leibniz formula, F(b) – F(a)
  $\int_{0}^{\frac{\pi}{2}} 3\cos^7 t \sin^2 t\, dt = \left[3\left(-\frac{w^9}{9} + \frac{3w^7}{7} - \frac{3w^5}{5} + \frac{w^3}{3}\right)\right.$
- The constant taken out of the integral:
  $\int 3\cos^7 t \sin^2 t\, dt = 3 \int \cos^7 t \sin^2 t\, dt$
- An odd power – the substitution $w = \sin t$,
  $\cos^2 t = 1 - \sin^2 t$
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- From the table of basic integrals: $\int w^8\, dw = \frac{w^9}{9}$
- The constant taken out of the integral:
  $\int 3w^6\, dw = 3 \int w^6\, dw$

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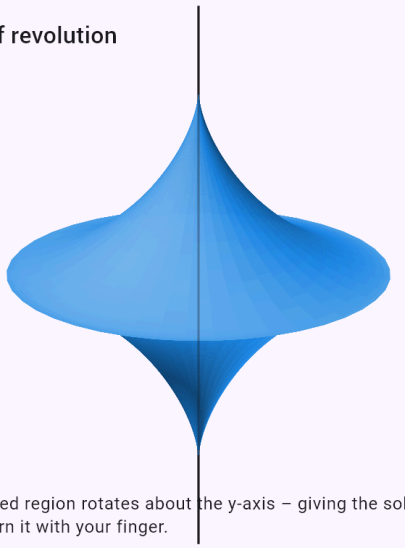
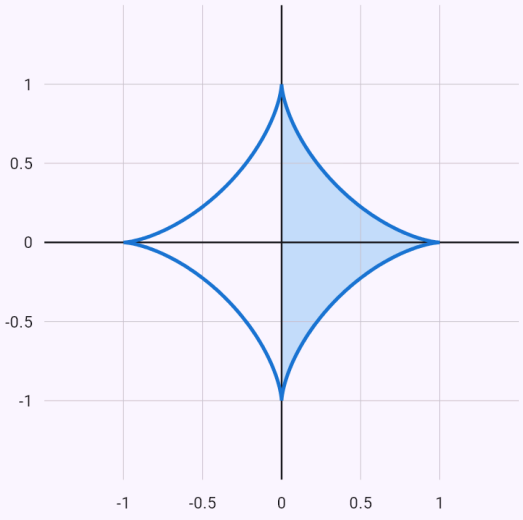
- The constant taken out of the integral: $\int -3w^4\,dw = -3\int w^4\,dw$
- From the table of basic integrals: $\int w^4\,dw = \frac{w^5}{5}$
- From the table of basic integrals: $\int w^2\,dw = \frac{w^3}{3}$
- With the substitution, the limits are recomputed too – no need to go back to the original variable: $t = \frac{3\pi}{2} \Rightarrow w = -1, \quad t = 2\pi \Rightarrow w = 0$
- The Newton–Leibniz formula, F(b) − F(a) $\int_{\frac{3\pi}{2}}^{2\pi} 3\cos^7 t \sin^2 t\,dt = \left[3\left(-\frac{w^9}{9} + \frac{3w^7}{7} - \frac{3w^5}{5} + \frac{w^3}{3}\right)\right.$
- The integral's value times the constant in front of it: $V = \pi \cdot \frac{32}{105} = \frac{32\pi}{105}$

Solid of revolution



The shaded region rotates about the y-axis – giving the solid above; turn it with your finger.

Graph



$x = \cos^3 t,\ y = \sin^3 t$

The curve given parametrically (or in polar coordinates), drawn for the chosen limits of the parameter.

Curves given by an equation

For an equation in x and y (x³ + y³ = 3xy) the "Integrate" chip offers the area, the length, the volume and the surface. Without limits it takes the whole closed curve: a circle or an ellipse in parametric form by cos and sin, a loop through the origin (the folium of Descartes) by the lines y = tx – the loop's area then exactly from the sectors, the length, volume and surface numerically. With limits in x it uses the upper branch y = f(x).

How to enter it

`x^3 + y^3 = 3xy`	A closed curve – the "Integrate" chip without limits.

`y^2 = x^3`	A curve that isn't closed – the "Integrate" chip with limits in x (the upper branch y = x^(3/2)).

Examples

The area of the folium's loop – by the lines y = tx and the sectors, S = 3/2.

Input: `x^3 + y^3 = 3xy` Integrate → Area

Result

$$S = \frac{3}{2}$$

Steps

- The loop through the origin in parametric form by the lines y = tx – substituted into the equation, x expressed

$$y = tx \;\Rightarrow\; x = \frac{3t}{t^3+1}, \quad y = \frac{3t^2}{t^3+1}, \quad t \in \langle 0;\, \infty)$$

- The loop's area from the sectors from the origin: S = ½∫(x·y' − y·x') dt, and on the line y = tx, x·y' − y·x' = x²

$$S = \frac{1}{2}\int_0^\infty x^2(t)\,dt = \int_0^\infty \frac{9t^2}{2\left(t^3+1\right)^2}\,dt$$

- The constant taken out of the integral

$$\int \frac{9t^2}{2\left(t^3+1\right)^2}\,dt = \frac{9}{2}\int \frac{t^2}{\left(t^3+1\right)^2}\,dt$$

- Substituting $u = t^3+1,\ du = 3t^2\,dt$:

$$\int \frac{t^2}{\left(t^3+1\right)^2}\,dt = \int \frac{1}{3u^2}\,du$$

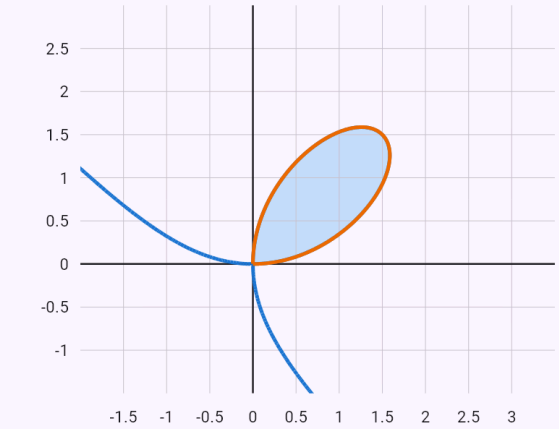
- The constant taken out of the integral:

$$\int \frac{1}{3u^2}\,du = \frac{1}{3}\int \frac{1}{u^2}\,du$$

- From the table of basic integrals: $\int \frac{1}{u^2}\,du = -\frac{1}{u}$
- Back to the original variable: $-\frac{1}{3u} = -\frac{1}{3\left(t^3+1\right)}$
- An improper integral – at an infinite limit and where the function is unbounded we take the limit of the antiderivative. The limits are finite, the integral converges

$$\int_0^\infty \frac{9t^2}{2\left(t^3+1\right)^2}\,dt = \lim_{t\to\infty}\left(-\frac{3}{2\left(t^3+1\right)}\right) - \left(-\frac{3}{2}\right) = 0$$

Graph



$x^3 + y^3 = 3xy$

$x = \frac{3t}{t^3+1},\ y = \frac{3t^2}{t^3+1}$

The whole curve given by the equation; the part the computation uses is highlighted.

The area of a lemniscate, by polar coordinates.

Input: `(x^2+y^2)^2 = x^2-y^2`

Result

$$S = 1$$

Steps

- We rewrite the curve in polar coordinates

  $x = \rho\cos\varphi,\ y = \rho\sin\varphi \ \Rightarrow\ \rho^2 = \cos(2\varphi)$

- The area bounded by a polar curve (the sectors from the origin added up) for $\varphi \in \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]$,

  $$S = \frac{1}{2}\int_{\alpha}^{\beta} \rho^2(\varphi)\,d\varphi$$

  $$S = \frac{1}{2}\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \left(\sqrt{\cos(2\varphi)}\right)^2 d\varphi = \frac{1}{2}\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \cos(2\varphi)\,d\varphi$$

- From the table – the inner function is linear, so the result is divided by its coefficient of the variable:

  $$\int \cos(2\varphi)\,d\varphi = \frac{\sin(2\varphi)}{2}$$

- The Newton–Leibniz formula, F(b) − F(a)

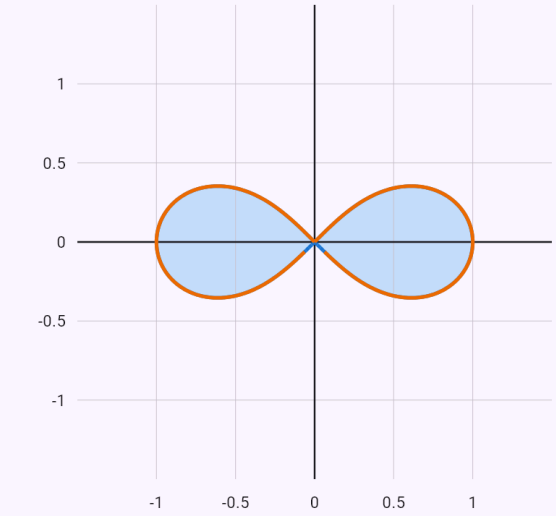
  $$\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \cos(2\varphi)\,d\varphi = \left[\frac{\sin(2\varphi)}{2}\right]_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1$$

- The curve has congruent loops (2); one loop's value times their number: $2 \cdot 1$

- The integral's value times the constant in front of it:

  $$S = \frac{1}{2}\cdot 2 = 1$$

Graph



$(x^2+y^2)^2 = x^2 - y^2$

$r = \sqrt{\cos(2\varphi)}$

The whole curve given by the equation; the part the computation uses is highlighted.

A rotated ellipse with an xy term.

Input: `x^2+x*y+y^2 = 1`

Result

$$S = \frac{2\sqrt{3}\pi}{3} \approx 3.6276$$

Steps

- The curve is a rotated ellipse – its axes follow from the angle of rotation; parametrically

  $\tan 2\theta = \frac{B}{A-C}, \quad x = \frac{\sqrt{3}\cos t}{3} - \sin t, \quad y = \frac{\sqrt{3}\cos$

- The derivatives by the parameter:

  $x'(t) = -\cos t - \frac{\sqrt{3}\sin t}{3}, \quad y'(t) = \cos t - \frac{\sqrt{3}\sin t}{3}$

- The area under a parametric curve (inside a closed one)

  for $t \in [0,\, 2\pi]$, $S = \left|\int_{\alpha}^{\beta} y(t)\, x'(t)\, dt\right|$

  $S = \left|\int_0^{2\pi} \left(-\frac{\sqrt{3}\cos^2 t}{3} - \frac{4\cos t \cdot \sin t}{3} - \frac{\sqrt{3}\sin^2 t}{3}\right) d$

- The integral of a sum is the sum of the integrals

  $\int \left(-\frac{\sqrt{3}\cos^2 t}{3} - \frac{4\cos t \cdot \sin t}{3} - \frac{\sqrt{3}\sin^2 t}{3}\right) dt = \int -$

- The constant taken out of the integral:

  $\int -\frac{\sqrt{3}\cos^2 t}{3}\, dt = -\frac{\sqrt{3}}{3}\int \cos^2 t\, dt$

- Using the formula $\cos^2 t = \frac{1+\cos 2t}{2}$:

  $\int \cos^2 t\, dt = \int \left(\frac{\cos(2t)}{2} + \frac{1}{2}\right) dt$

- The integral of a sum is the sum of the integrals

  $\int \left(\frac{\cos(2t)}{2} + \frac{1}{2}\right) dt = \int \frac{1}{2}\, dt + \int \frac{\cos(2t)}{2}\, dt$

- From the table of basic integrals: $\int \frac{1}{2}\, dt = \frac{t}{2}$

- The constant taken out of the integral:

  $\int \frac{\cos(2t)}{2}\, dt = \frac{1}{2}\int \cos(2t)\, dt$

- From the table – the inner function is linear, so the result is divided by its coefficient of the variable:

  $\int \cos(2t)\, dt = \frac{\sin(2t)}{2}$

- The constant taken out of the integral:

  $\int -\frac{4\cos t \cdot \sin t}{3}\, dt = -\frac{4}{3}\int \cos t \cdot \sin t\, dt$

- Using the formula $\sin\alpha\cos\alpha = \frac{1}{2}\sin 2\alpha$:

  $\int \cos t \cdot \sin t\, dt = \int \frac{\sin(2t)}{2}\, dt$

- The constant taken out of the integral:

  $\int \frac{\sin(2t)}{2}\, dt = \frac{1}{2}\int \sin(2t)\, dt$

- From the table – the inner function is linear, so the result is divided by its coefficient of the variable:

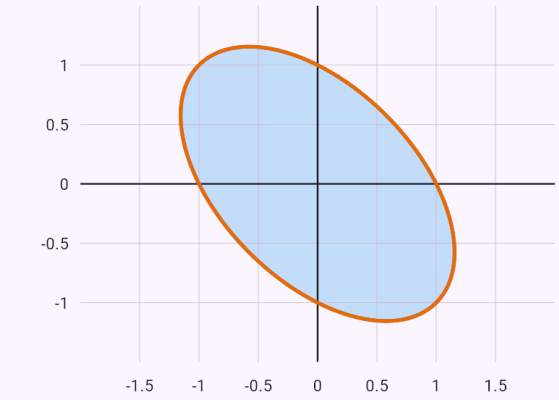
  $\int \sin(2t)\, dt = -\frac{\cos(2t)}{2}$

- The constant taken out of the integral:

  $\int -\frac{\sqrt{3}\sin^2 t}{3}\, dt = -\frac{\sqrt{3}}{3}\int \sin^2 t\, dt$

- Using the formula $\sin^2 t = \frac{1-\cos 2t}{2}$:
$$\int \sin^2 t\, dt = \int \left(\frac{1}{2} - \frac{\cos(2t)}{2}\right) dt$$
- The integral of a sum is the sum of the integrals
$$\int \left(\frac{1}{2} - \frac{\cos(2t)}{2}\right) dt = \int \frac{1}{2}\, dt + \int -\frac{\cos(2t)}{2}\, dt$$
- From the table of basic integrals: $\int \frac{1}{2}\, dt = \frac{t}{2}$
- The constant taken out of the integral:
$$\int -\frac{\cos(2t)}{2}\, dt = -\frac{1}{2}\int \cos(2t)\, dt$$
- From the table – the inner function is linear, so the result is divided by its coefficient of the variable:
$$\int \cos(2t)\, dt = \frac{\sin(2t)}{2}$$
- The Newton–Leibniz formula, F(b) − F(a)
$$\int_0^{2\pi} \left(-\frac{\sqrt{3}\cos^2 t}{3} - \frac{4\cos t \cdot \sin t}{3} - \frac{\sqrt{3}\sin^2 t}{3}\right) dt = \left[$$
- The integral came out negative (the curve is traced right to left) – we take its absolute value:
$$S = \left|-\frac{2\sqrt{3}\pi}{3}\right| = \frac{2\sqrt{3}\pi}{3}$$

Graph



$x^2 + xy + y^2 = 1$

$x = \frac{\sqrt{3}\cos t}{3} - \sin t,\ y = \frac{\sqrt{3}\cos t}{3} + \sin t$

The whole curve given by the equation; the part the computation uses is highlighted.

Known limitations

- The arc length of y = ∛x (and similar curves) is only computed numerically – it leads to an elliptic integral, which has no elementary antiderivative.
- Rotation about the y-axis only for a region on one side of the axis.
- A formula with parameters is given only where the result doesn't depend on how they compare; if the sign changes with their values, no formula is given.
- For implicit curves without limits: conics (even with an xy term), a loop through the origin and curves of two homogeneous parts (a lemniscate).
- A definite integral with a parameter only with finite limits so far.
- When the limit comparison test doesn't decide (the integrand oscillates at infinity, say cos x/(x + ln x)), the app gives no value without a proof of convergence and reports the integral as not solved yet.

32. Infinite series

For a numerical series the app first checks the necessary condition for convergence (the term test) and then picks one test by the form of the series – geometric series, $\Sigma\ 1/n^p$, the ratio, root, limit comparison, integral or alternating series (Leibniz) test; for a series with alternating signs it distinguishes absolute and conditional convergence. Where possible it sums the series. The graph shows the terms and the partial sums.

Numerical series

The app finds the sum of geometric and telescoping series, series $\Sigma\ P(n)q^n$ and well-known series ($\Sigma\ 1/n^2 = \pi^2/6$); otherwise it computes it numerically. A finite sum is added up the school way – telescoping, arithmetic, geometric, otherwise term by term. An equation with the sum of a geometric series is solved too, with the condition of convergence $|q| < 1$ checked.

A sum up to a letter (sum(k^2, k, 1, n)) by the power sums' formulas, a geometric one by its formula and a rational term by partial fractions (telescoping); the result is checked on the first terms.

How to enter it

`sum(1/n^2)`	The general term; the index n from 1.
`sum(1/n^2, n, 1, oo)`	With the index and the starting value.
`\sum_{n=1}^{\infty} \frac{1}{n^2}`	From a photo or in LaTeX.
`1/n^2`	The general term with the "Series" chip – fill in the starting index in the field.
`sum(2k+1, k, 1, 100)`	A finite sum – with the index, the lower and the upper bound.
`sum(k^2, k, 1, n)`	A sum up to n – the sum's formula.

Examples

The series Σ 1/n² – converges, the sum is π²/6.

Input: `sum(1/n^2)`

Result

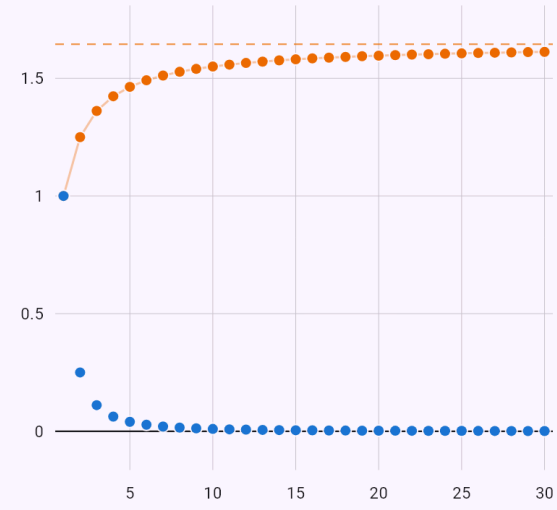
The series converges.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.6449$$

Steps

- The necessary condition for convergence holds (it doesn't decide by itself): $\lim_{n\to\infty} \frac{1}{n^2} = 0$
- The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is Σ 1/n^p with p = 2 > 1 – it converges.
- A known series' sum: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

Graph



The points are the terms a_n and the partial sums s_n = a_1 + … + a_n. If the series converges, the terms approach zero and the partial sums its sum.

The alternating series test – converges conditionally.

Input: `sum((-1)^(n+1)/n)`

Result

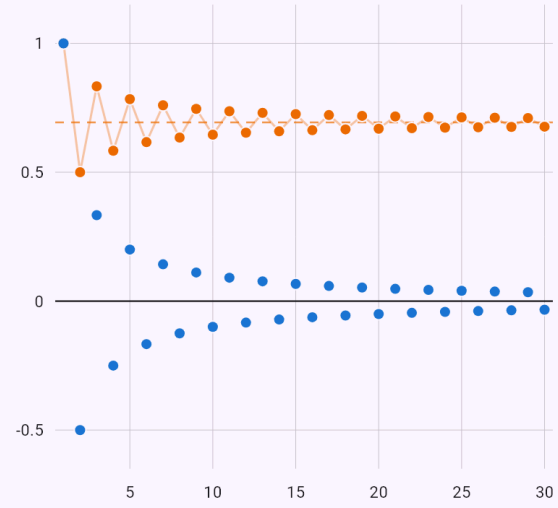
The series converges, but not absolutely.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2 \approx 0.6931$$

Steps

- The necessary condition for convergence holds (it doesn't decide by itself): $\lim_{n\to\infty} \frac{(-1)^{n+1}}{n} = 0$
- The terms have both signs – absolute convergence first $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$
- The harmonic series diverges – the terms grouped by powers of two, each group at least 1/2 $1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots > 1 + \frac{1}{2}$
- Leibniz's test: |a_n| = $\frac{1}{n}$ decreases and $\lim_{n\to\infty} \frac{1}{n} = 0$ – the series converges.
- Conclusion: the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges, but not absolutely (by Leibniz's test).
- A known series' sum: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$

Graph



The points are the terms a_n and the partial sums s_n = a_1 + … + a_n. If the series converges, the terms approach zero and the partial sums its sum.

The ratio test and the sum.

Input: `sum(n/2^n)`

Result

The series converges.

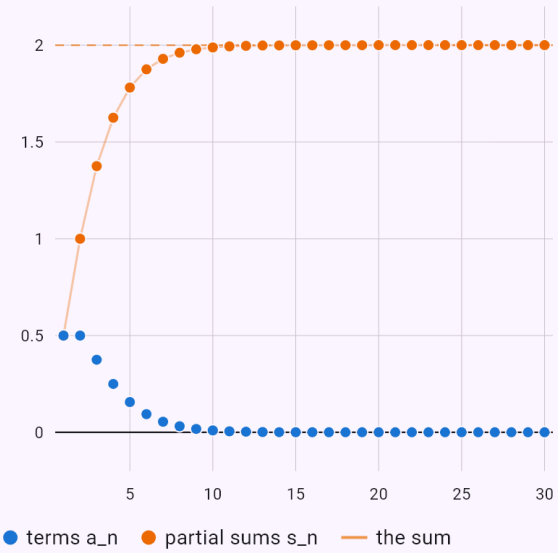
$$\sum_{n=1}^{\infty} \frac{n}{2^n} = 2$$

Steps

- The necessary condition for convergence holds (it doesn't decide by itself): $\lim_{n\to\infty} \frac{n}{2^n} = 0$
- The ratio (d'Alembert's) test – the series converges.

  $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2} < 1$
- A series Σ P(n)·q^n summed by differentiating the geometric series: $\sum_{n=0}^{\infty} n\,q^n = \frac{q}{(1-q)^2}, \quad s = 2$

Graph



The points are the terms a_n and the partial sums s_n = a_1 + … + a_n. If the series converges, the terms approach zero and the partial sums its sum.

The necessary condition fails – the series diverges.

Input: `sum(n/(n+1))`

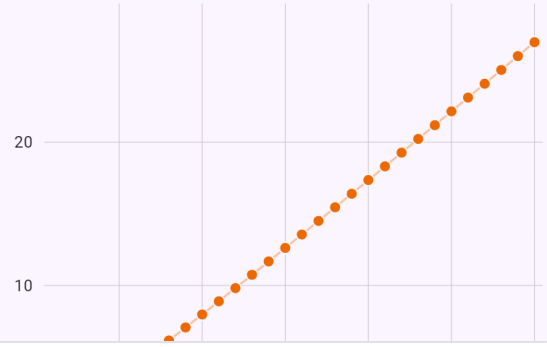
Result

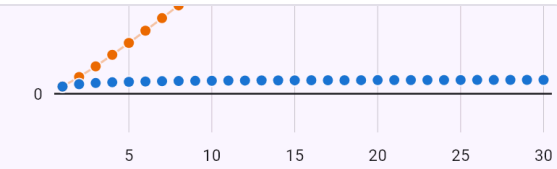
The series diverges.

Steps

- The necessary condition for convergence fails: $\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$

Graph





The points are the terms a_n and the partial sums s_n = a_1 + … + a_n. If the series converges, the terms approach zero and the partial sums its sum.

A telescoping sum – the first and the last term remain.

Input: `sum(sqrt(2i+1) - sqrt(2i-1), i, 1, 60)`

Result

$$\sum_{i=1}^{60} \left(\sqrt{2i+1} - \sqrt{2i-1}\right) = 10$$

Steps

- A telescoping sum – neighbouring terms cancel, only the first and the last remain:
  $\sum_{i=1}^{60} \left(\sqrt{2i+1} - \sqrt{2i-1}\right) = 11 - 1 = 10$

An equation with the sum of a geometric series.

Input: `sum((2^x)^n, n, 1, oo) = 1`

Result

$$x = -1$$

Steps

- The series is geometric with the ratio q = 2^x and converges for $|2^x| < 1$; its sum:
  $\sum_{n=1}^{\infty} (2^x)^n = \frac{a_1}{1-q} = \frac{2^x}{1-2^x}$
- Solve: $\frac{2^x}{1-2^x} = 1 \Rightarrow x = -1$
- $x = -1$ meets the condition of convergence: $\left|\frac{1}{2}\right| < 1$

The sum of squares by the formula.

Input: `sum(k^2, k, 1, n)`

Result

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Steps

- We use the formulas for sums of powers:
  $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
- The sum: $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

A telescoping sum up to n – partial fractions.

Input: `sum(1/(k(k+1)), k, 1, n)`

Result

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$$

Steps

- We split the term into partial fractions – their parts cancel one by one in the sum (a telescoping sum): $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$
- The sum: $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$

Power series

For a power series the app finds the radius of convergence by the ratio test, examines the endpoints as numerical series and states the interval of convergence; where possible also the sum. The graph shows the partial sums with a slider for the number of terms.

Examples

The interval of convergence [−1, 1).

Input: `sum(x^n/n)`

Result

$$x \in [-1,\ 1)$$

Steps

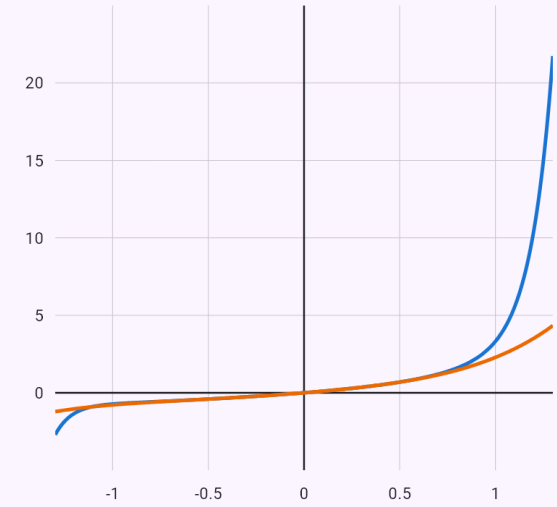
- The radius of convergence R = 1 by the ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{xn}{n+1}\right| = |x| < 1$$

- The endpoint $x = -1$: the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges, but not absolutely (by Leibniz's test).
- The endpoint $x = 1$: the series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges by grouping the harmonic series' terms.
- The interval of convergence: $x \in [-1,\ 1)$

Graph

Terms: 5



$$s_{15}(x) = \frac{x^{15}}{15} + \frac{x^{14}}{14} + \frac{x^{13}}{13} + \frac{x^{12}}{12} + \frac{x^{11}}{11} + \frac{x^{10}}{10}$$

$$s_5(x) = \frac{x^5}{5} + \frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} + x$$

Pick the number of terms with the slider – the partial sums approach the series' sum only inside the interval of convergence.

Taylor series

The app derives the Taylor series of a function from known expansions (e^x, sin x, ln(1 + x), …) by substitution and determines its interval of convergence.

How to enter it

`taylor(e^x, 0, oo)`	The function, the point and oo instead of the degree; or the "Taylor polynomial" chip → "Series".

Examples

The series of ln(1 + x) and its interval of convergence.

Input: `taylor(ln(1 + x), 0, oo)`

Result

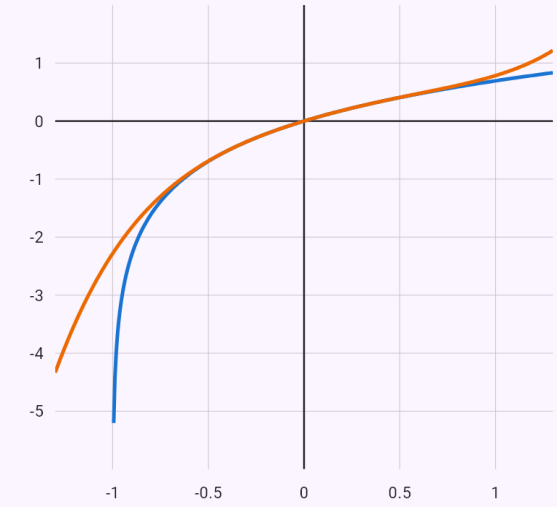
$$\ln(x+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}, \quad x \in (-$$

Steps

- From the known expansion $\ln(1+u) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} u^n}{n}$ with $u = x$: $\ln(x+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$
- The radius of convergence R = 1 by the ratio test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|-\frac{xn}{n+1}\right| = |x| < 1$
- The endpoint $x = -1$: the series $\sum_{n=1}^{\infty} -\frac{1}{n}$ diverges by grouping the harmonic series' terms.
- The endpoint $x = 1$: the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges, but not absolutely (by Leibniz's test).
- The interval of convergence: $x \in (-1, 1]$

Graph

Terms: 5



$s(x) = \ln(x+1)$

$s_5(x) = \frac{x^5}{5} - \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} + x$

Pick the number of terms with the slider – the partial sums approach the series' sum only inside the interval of convergence.

33. Fourier series

The app builds the Fourier series of a function on an interval taken as one period (without one on [−π, π]), or the sine or cosine series of a function on [0, L]. The function can also be given piecewise.

It computes the coefficients by integrals with the full working (integration by parts with the parameter n), substitutes sin nπ = 0 and cos nπ = $(-1)^n$ and uses symmetry for even and odd functions. At discontinuities it determines the sum of the series by Dirichlet's theorem, and by substituting a point and by Parseval's identity it derives sums of numerical series (such as Σ $1/n^2 = \pi^2/6$). The graph shows the periodic extension of the function and the partial sums with a slider.

The chip “Complex” (or fourierc) also shows the series in the complex form: the coefficients c_n, their relation to a_n and b_n and the amplitude and phase spectrum.

How to enter it

`fourier(x^2, -pi, pi)`	The function and the interval.
`fouriersin(x, 0, pi)`	The sine series (fouriercos – the cosine series).
`-1, -pi < x < 0` `1, 0 < x < pi`	A piecewise function line by line, with the “Fourier series” chip.
`x^2`	A function with the “Fourier series” chip – choose the kind of series and the interval there.

Examples

An even function – cosines only; the sums Σ 1/n² and Σ 1/n⁴.

Input: `fourier(x^2)`

Result

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx), \quad$$

Steps

- The interval $[-\pi, \pi]$ is one period: $T = 2\pi$, half the period $l = \pi$. We look for the series with the coefficients

  $$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

  $$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$
  $$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$$
  $$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

- The function is even, $f(-x) = x^2 = f(x)$, so the sine coefficients are zero and the others can be computed over half the interval

  $$b_n = 0$$
  $$a_0 = \frac{2}{\pi} \int_{0}^{\pi} f(x)\, dx$$
  $$a_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos(nx)\, dx$$

- The coefficient a_0: $a_0 = \frac{2}{\pi} \int_0^{\pi} x^2\, dx$
- From the table of basic integrals: $\int x^2\, dx = \frac{x^3}{3}$
- Substituting the limits: $a_0 = \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi} = \frac{2\pi^2}{3}$
- The coefficient a_n: $a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cos(nx)\, dx$
- By parts (∫u·v' = u·v − ∫u'·v), u = x^2, v' = $\cos(nx)$, so u' = $2x$, v = $\frac{\sin(nx)}{n}$

  $$\int x^2 \cos(nx)\, dx = \frac{x^2 \sin(nx)}{n} - \int \frac{2x \sin(nx)}{n}\, dx$$

- The constant taken out of the integral:

  $$\int \frac{2x \sin(nx)}{n}\, dx = \frac{2}{n} \int x \sin(nx)\, dx$$

- By parts (∫u·v' = u·v − ∫u'·v), u = x, v' = $\sin(nx)$, so u' = 1, v = $-\frac{\cos(nx)}{n}$

  $$\int x \sin(nx)\, dx = -\frac{x \cos(nx)}{n} - \int -\frac{\cos(nx)}{n}\, dx$$

- The constant taken out of the integral:

  $$\int -\frac{\cos(nx)}{n}\, dx = -\frac{1}{n} \int \cos(nx)\, dx$$

- From the table – the inner function is linear, so the result is divided by its coefficient of the variable:

  $$\int \cos(nx)\, dx = \frac{\sin(nx)}{n}$$

- For a whole number n, $\sin n\pi = 0, \quad \cos n\pi = (-1)^n$
- Substituting the limits

  $$a_n = \frac{2}{\pi} \left[\frac{x^2 \sin(nx)}{n} + \frac{2x \cos(nx)}{n^2} - \frac{2 \sin(nx)}{n^3} \right]_0^{\pi} = \frac{4}{-}$$

- The coefficients substituted into the series

$$s(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx)$$

- The periodic extension of the function is continuous, so by Dirichlet's theorem the sum of the series equals f(x) for $x \in [-\pi,\, \pi]$
- Substituting $x = 0$ gives the sum of a series of numbers

$$s(0) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} = 0 \;\Rightarrow\; \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

- Substituting $x = \pi$ gives the sum of a series of numbers

$$s(\pi) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} = \pi^2 \;\Rightarrow\; \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

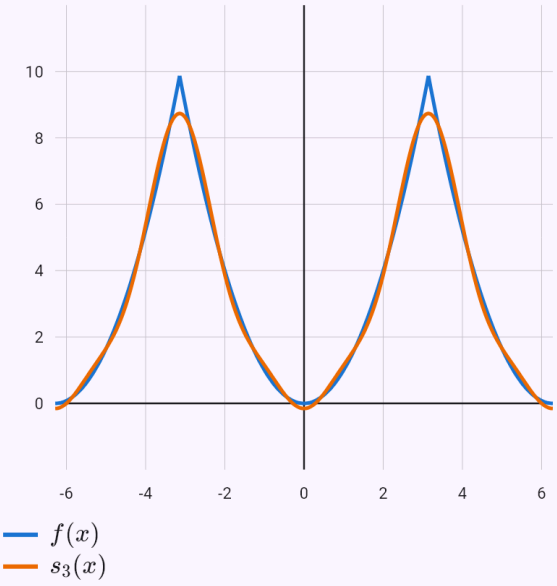
- From Parseval's identity

$$\frac{2}{\pi} \int_0^{\pi} f^2(x)\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right)$$

$$\frac{2\pi^4}{5} = \frac{2\pi^4}{9} + \sum_{n=1}^{\infty} \frac{16}{n^4} \;\Rightarrow\; \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

Graph

Terms up to n = 3



The function is extended periodically. The slider adds terms of the series – the partial sums approach the function; at a jump they overshoot (the Gibbs phenomenon) and pass through the average of both sides, marked by a point.

A piecewise function – jumps and the Gibbs phenomenon.

Input: `-1, -pi < x < 0`

`1, 0 < x < pi` Fourier series

Result

$$f(x) = \sum_{k=1}^{\infty} \frac{4}{\pi(2k-1)} \sin\left((2k-1)x\right)$$

Steps

- The interval $[-\pi,\ \pi]$ is one period: $T = 2\pi$, half the period $l = \pi$. We look for the series with the coefficients

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

- The function is odd, $f(-x) = -f(x)$, so the constant term and the cosine coefficients are zero and the sine coefficients can be computed over half the interval

$$a_0 = a_n = 0$$

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(nx)\,dx$$

- The coefficient b_n: $b_n = \frac{2}{\pi}\int_0^{\pi} \sin(nx)\,dx$
- From the table – the inner function is linear, so the result is divided by its coefficient of the variable:

$$\int \sin(nx)\,dx = -\frac{\cos(nx)}{n}$$

- For a whole number n, $\sin n\pi = 0, \quad \cos n\pi = (-1)^n$
- Substituting the limits

$$b_n = \frac{2}{\pi}\left[-\frac{\cos(nx)}{n}\right]_0^{\pi} = \frac{2-2(-1)^n}{n\pi}$$

- The coefficients substituted into the series:

$$s(x) = \sum_{n=1}^{\infty} \frac{2-2(-1)^n}{n\pi}\sin(nx)$$

- The terms with an even n are zero – only the odd n = 2k − 1 remain

$$s(x) = \sum_{k=1}^{\infty} \frac{4}{\pi(2k-1)}\sin\left((2k-1)x\right)$$

- By Dirichlet's theorem the sum of the series equals f(x) for $x \in (-\pi,\ 0) \cup (0,\ \pi)$; at a point of discontinuity it is the average of the one-sided limits

$$s(-\pi) = s(\pi) = \frac{f(\pi^-) + f(-\pi^+)}{2} = \frac{1+(-1)}{2} = 0$$

$$s(0) = \frac{f(0^-) + f(0^+)}{2} = \frac{-1+1}{2} = 0$$

- Substituting $x = \frac{\pi}{2}$ gives the sum of a series of numbers

$$s\left(\frac{\pi}{2}\right) = \sum_{k=1}^{\infty} \frac{4(-1)^{k+1}}{\pi(2k-1)} = 1 \ \Rightarrow\ \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} = \frac{\pi}{4}$$

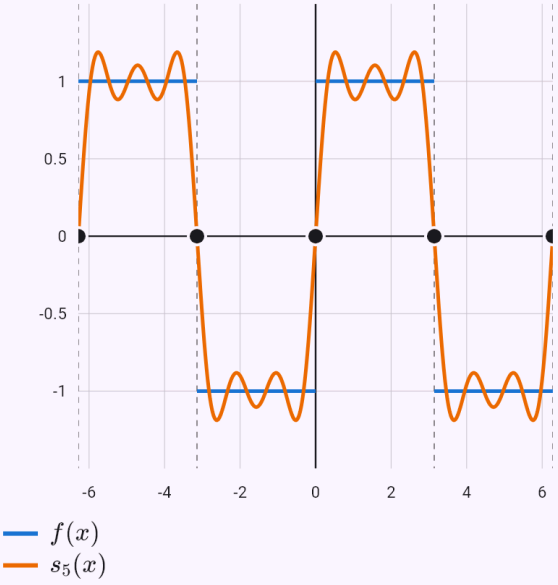
- From Parseval's identity

$$\frac{2}{\pi}\int_0^{\pi} f^2(x)\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

$$2 = \sum_{k=1}^{\infty} \frac{16}{\pi^2(2k-1)^2} \ \Rightarrow\ \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$$

Graph

Terms up to n = 5



The function is extended periodically. The slider adds terms of the series – the partial sums approach the function; at a jump they overshoot (the Gibbs phenomenon) and pass through the average of both sides, marked by a point.

A sine series and the sum 1 − 1/3³ + 1/5³ − … = π³/32.

Input: `fouriersin(x*(pi - x), 0, pi)`

Result

$$f(x) = \sum_{k=1}^{\infty} \frac{8}{\pi(2k-1)^3} \sin\left((2k-1)x\right.$$

Steps

- We extend the function from $[0,\ \pi]$ as an odd function, then periodically ($T = 2\pi$) – a sine series

  $f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx)$, where

  $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$

- The coefficient b_n: $b_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin(nx)\,dx$

- By parts (∫u·v' = u·v − ∫u'·v), u = $x(\pi - x)$, v' = $\sin(nx)$, so u' = $\pi - 2x$, v = $-\frac{\cos(nx)}{n}$

  $\int x(\pi - x)\sin(nx)\,dx = \frac{x(x-\pi)\cos(nx)}{n} - \int \frac{(2x$

- The constant taken out of the integral

  $\int \frac{(2x-\pi)\cos(nx)}{n}\,dx = \frac{1}{n}\int (2x-\pi)\cos(nx)\,dx$

- By parts (∫u·v' = u·v − ∫u'·v), u = $2x - \pi$, v' = $\cos(nx)$, so u' = 2, v = $\frac{\sin(nx)}{n}$

  $\int (2x-\pi)\cos(nx)\,dx = \frac{(2x-\pi)\sin(nx)}{n} - \int \frac{2\sin$

- The constant taken out of the integral:

  $\int \frac{2\sin(nx)}{n}\,dx = \frac{2}{n}\int \sin(nx)\,dx$

- From the table – the inner function is linear, so the result is divided by its coefficient of the variable:

  $\int \sin(nx)\,dx = -\frac{\cos(nx)}{n}$

- For a whole number n, $\sin n\pi = 0, \quad \cos n\pi = (-1)^n$

- Substituting the limits

  $b_n = \frac{2}{\pi}\left[\frac{x^2\cos(nx)}{n} - \frac{\pi x\cos(nx)}{n} + \frac{(\pi - 2x)\sin(nx$

- The coefficients substituted into the series:

  $s(x) = \sum_{n=1}^{\infty} \frac{4 - 4(-1)^n}{n^3\pi}\sin(nx)$

- The terms with an even n are zero – only the odd n = 2k − 1 remain

  $s(x) = \sum_{k=1}^{\infty} \frac{8}{\pi(2k-1)^3}\sin\left((2k-1)x\right)$

- The periodic extension of the function is continuous, so by Dirichlet's theorem the sum of the series equals f(x) for $x \in [0,\ \pi]$

- Substituting $x = \frac{\pi}{2}$ gives the sum of a series of numbers

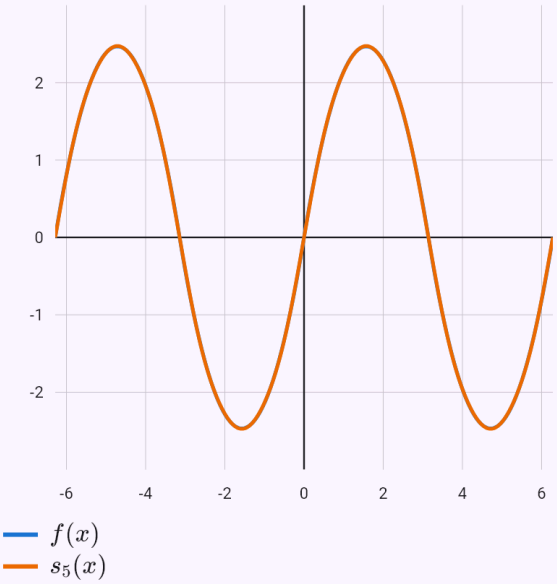
  $s(\frac{\pi}{2}) = \sum_{k=1}^{\infty} \frac{8(-1)^{k+1}}{\pi(2k-1)^3} = \frac{\pi^2}{4} \Rightarrow \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^3} = \frac{\pi^3}{32}$

- From Parseval's identity $\frac{2}{\pi}\int_0^{\pi} f^2(x)\,dx = \sum_{n=1}^{\infty} b_n^2$

  $\frac{\pi^4}{15} = \sum_{k=1}^{\infty} \frac{64}{\pi^2(2k-1)^6} \Rightarrow \sum_{k=1}^{\infty} \frac{1}{(2k-1)^6} = \frac{\pi^6}{960}$

Graph

Terms up to n = 5



The function is extended periodically. The slider adds terms of the series – the partial sums approach the function; at a jump they overshoot (the Gibbs phenomenon) and pass through the average of both sides, marked by a point.

The complex form with its spectrum.

Input: `fourierc(x, -pi, pi)`

Result

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{inx}$$

$$c_0 = 0, \quad c_n = -i \, \frac{(-1)^{n+1}}{n}$$

Steps

- The interval $[-\pi, \pi]$ is one period: $T = 2\pi$, half the period $l = \pi$. We look for the series with the coefficients

  $$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

  $$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$
  $$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$$
  $$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

- The function is odd, $f(-x) = -x = -f(x)$, so the constant term and the cosine coefficients are zero and the sine coefficients can be computed over half the interval

  $$a_0 = a_n = 0$$
  $$b_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin(nx)\, dx$$

- The coefficient b_n: $b_n = \frac{2}{\pi} \int_0^{\pi} x \sin(nx)\, dx$

- By parts (∫u·v' = u·v − ∫u'·v), u = x, v' = $\sin(nx)$, so u' = 1, v = $-\frac{\cos(nx)}{n}$

  $$\int x \sin(nx)\, dx = -\frac{x \cos(nx)}{n} - \int -\frac{\cos(nx)}{n}\, dx$$

- The constant taken out of the integral:
  $$\int -\frac{\cos(nx)}{n}\, dx = -\frac{1}{n} \int \cos(nx)\, dx$$

- From the table – the inner function is linear, so the result is divided by its coefficient of the variable:
  $$\int \cos(nx)\, dx = \frac{\sin(nx)}{n}$$

- For a whole number n, $\sin n\pi = 0, \quad \cos n\pi = (-1)^n$

- Substituting the limits

  $$b_n = \frac{2}{\pi} \left[\frac{\sin(nx)}{n^2} - \frac{x \cos(nx)}{n} \right]_0^{\pi} = \frac{2(-1)^{n+1}}{n}$$

- The coefficients substituted into the series:
  $$s(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx)$$

- By Dirichlet's theorem the sum of the series equals f(x) for $x \in (-\pi, \pi)$; at a point of discontinuity it is the average of the one-sided limits:
  $$s(-\pi) = s(\pi) = \frac{f(\pi^-) + f(-\pi^+)}{2} = \frac{\pi + (-\pi)}{2} = 0$$

- Substituting $x = \frac{\pi}{2}$ gives the sum of a series of numbers

  $$s(\frac{\pi}{2}) = \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{2k-1} = \frac{\pi}{2} \;\Rightarrow\; \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} = \frac{\pi}{4}$$

- From Parseval's identity

  $$\frac{2}{\pi} \int_0^{\pi} f^2(x)\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

  $$\frac{2\pi^2}{3} = \sum_{n=1}^{\infty} \frac{4}{n^2} \;\Rightarrow\; \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

- The complex form of the series: $f(x) \sim \sum_{n=-\infty}^{\infty} c_n \, e^{inx}$, where $c_n = \frac{1}{2l} \int_{-l}^{l} f(x)\, e^{-inx}\, dx$. Relation to the real form

$$c_0 = \frac{a_0}{2}, \quad c_n = \frac{a_n - i\,b_n}{2}, \quad c_{-n} = \frac{a_n + i\,b_n}{2} = \overline{c_n}$$

- We substitute the coefficients of the real form

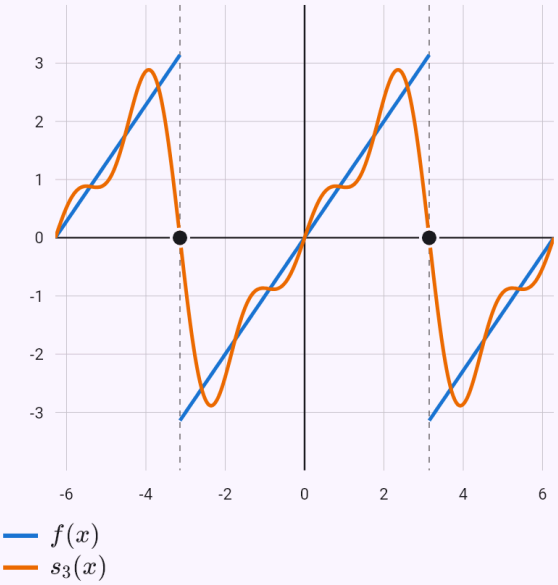
$$c_0 = 0$$

$$c_n = -i\,\frac{(-1)^{n+1}}{n}, \quad n \neq 0$$

$$c_{-n} = i\,\frac{(-1)^{n+1}}{n}$$

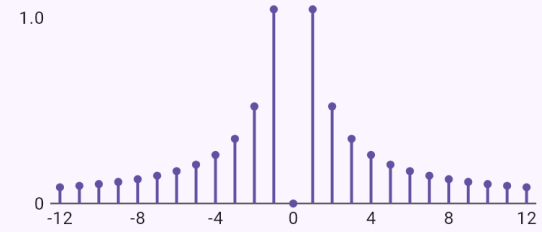
Graph

Terms up to n = 3

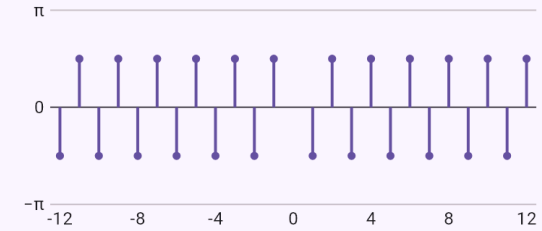


The function is extended periodically. The slider adds terms of the series – the partial sums approach the function; at a jump they overshoot (the Gibbs phenomenon) and pass through the average of both sides, marked by a point.

Amplitude spectrum $|c_n|$



Phase spectrum arg c_n



The horizontal axis is n. The height shows the size or the angle of c_n; zero coefficients have no phase.

Known limitations

- A function unbounded at an end of the interval (ln x on [0, 2π]) is expanded with numerically computed coefficients; Dirichlet's theorem does not hold at the point where it grows without bound. A function with a pole (1/x) is not expanded – it is not integrable.
- When the coefficients lead to an integral without an elementary antiderivative (ln x on [1, 3]), they are expressed with the sine and cosine integrals.

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