Mathtrace – user guide

IV. Differential equations and advanced analysis

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34. Functional analysis

> *“Linear functional analysis is really linear algebra in infinite dimension.”*
>
> – Prof. RNDr. Ondřej Došlý, DrSc., in the opening lecture

The computational part of linear functional analysis. Write a vector's norm as ||(1; -2; 3)||_p (p = 1, 2, ∞ or another number; without the index the app computes all three), a sequence's norm in the space ℓ^p as ||1/n||_2 – the app sums the series $\Sigma|a_n|^p$, and if it diverges the sequence is not in the space. A function's L^p norm is ||x^2||_{L^2(0; 1)}: the app removes the absolute value by splitting the integral at the function's zeros. The supremum norm ||f||_{C⟨a; b⟩} is the largest value of |f| from the stationary points and the endpoints. dist(f, g, ⟨a; b⟩) gives the distance of two functions in the supremum norm and in L^2, holder(...) and minkowski(...) check Hölder's and Minkowski's inequality on given functions or vectors.

A sequence of functions convergence(x^n, ⟨0; 1⟩) (or f_n(x) = ..., x ∈ ⟨a; b⟩) is examined in three steps: the pointwise limit piece by piece (a power q^n goes to zero for |q| < 1), uniform convergence by $s_n = \sup |f_n - f|$ from the stationary point x_n (a discontinuous limit of continuous functions cannot be uniform) and convergence in L^2 by $\int (f_n - f)^2$. Where no formula comes out, the app estimates the supremum and the integral numerically. The graph has a slider over the terms f_1 to f_{20}.

The inner product in L^2 with a weight, inner(f, g, ⟨a; b⟩) (optionally w = ...), also gives the norms and the angle of the functions. gramschmidt(1, x, x^2; ⟨-1; 1⟩) runs the Gram–Schmidt process and recognizes the classical polynomials: Legendre, Chebyshev with the weight $1/\sqrt{(1 - x^2)}$ or $\sqrt{(1 - x^2)}$, Laguerre with e^(−x) on ⟨0; ∞⟩ and Hermite with e^(−x²); it works for vectors too. approx(e^x, 2, ⟨-1; 1⟩) finds the best approximation by a polynomial of degree n as the orthogonal projection, with the error and Bessel's inequality. Parseval's identity (and the derivation of $\Sigma\, 1/n^2 = \pi^2/6$) is in the Fourier series chapter.

Matrix norms are written ||[1 2; 3 4]||_1 (also _2, _oo and _F for Frobenius; without a subscript the app computes them all), the condition number cond([1 2; 3 4]) (optionally p = 1). opnorm(x t, ⟨0; 1⟩) computes the norm of an integral operator with the kernel K(x, t): on C⟨a; b⟩ as the maximum of $\int |K|\, dt$, in L^2 exactly for a degenerate kernel, otherwise bounded by the Hilbert–Schmidt norm. fnorm(g, ⟨a; b⟩) gives the norm of the functional $\varphi(u) = \int u \cdot g$ on C and on L^2 (Riesz's theorem), with p = ... on L^p.

Write an integral equation of the second kind as you see it: u(x) = x + λ∫_0^1 x t u(t) dt (Fredholm) or u(x) = 1 + ∫_0^x (x − t) u(t) dt (Volterra); a photo works too, or the commands fredholm(K; f; λ; ⟨a; b⟩) and volterra(K; f; λ; a). A degenerate kernel becomes a system of linear equations; with a parameter λ the app finds the characteristic values and the eigenfunctions and decides by Fredholm's alternative. Other kernels are solved by the Neumann series and Nyström's method, a Volterra equation with a kernel k(x − t) by the Laplace transform.

banach(cos x, ⟨0; 1⟩) checks the assumptions of Banach's fixed point theorem (g maps the interval into itself, q = max |g′| < 1), computes the a priori number of steps and iterates with the a posteriori error estimate (optionally eps = ..., x0 = ...); banach(A, b) does the same for the system x = Ax + b. picard(y' = x + y, y(0) = 1) gives Picard's successive approximations with the interval where the operator is a contraction and an error bound. Functional equations of Cauchy's type – f(x + y) = f(x) + f(y), f(x + y) = f(x)f(y), f(xy) = f(x) + f(y), f(xy) = f(x)f(y), Jensen's and d'Alembert's – are solved with the derivation.

How to enter it

`||(1; 2)||_1, ||1/n||_2, ||f||_{L^2(a; b)}, ||f||_{C⟨a; b⟩}`	Norms of a vector, a sequence (ℓᵖ) and a function (Lᵖ, C).
`dist(f, g, ⟨a; b⟩)`	The distance of two functions in the supremum norm and in L².
`holder(f, g, p, ⟨a; b⟩), minkowski(f, g, p, ⟨a; b⟩)`	Hölder's and Minkowski's inequality.
`convergence(f_n, ⟨a; b⟩)`	Pointwise, uniform and L² convergence of a sequence of functions.
`inner(f, g, ⟨a; b⟩, w = ...)`	The inner product, the norms and the angle of functions.
`gramschmidt(1, x, x^2; ⟨a; b⟩; w = ...)`	The Gram–Schmidt process for functions and vectors.
`approx(f, n, ⟨a; b⟩, w = ...)`	The best approximation by a polynomial of degree n.
`||[1 2; 3 4]||_1, ||A||_2, ||A||_oo, ||A||_F, cond(A)`	Matrix norms 1, 2, ∞, Frobenius and the condition number.
`opnorm(K(x, t), ⟨a; b⟩), fnorm(g, ⟨a; b⟩)`	The norm of an integral operator and of the functional ∫u·g.
`u(x) = f(x) + λ∫_a^b K(x, t) u(t) dt`	Fredholm and Volterra integral equations of the second kind.
`banach(g, ⟨a; b⟩), banach(A, b), picard(y' = f, y(x0) = y0)`	Banach's fixed point theorem and Picard's approximations.

`f(x + y) = f(x) + f(y)`	Functional equations of Cauchy's type.

Examples

The 1, 2 and ∞ norms of one vector.

Input: `||(1; -2; 3)||`

Result

$$\|\mathbf{x}\|_1 = 6,\ \|\mathbf{x}\|_2 = \sqrt{14},\ \|\mathbf{x}\|_\infty = 3$$

Steps

- The three most common norms of the vector $\mathbf{x} = (1;\ -2;\ 3)$: the sum of the components' absolute values ($\|\cdot\|_1$), the Euclidean length ($\|\cdot\|_2$) and the largest absolute value of a component ($\|\cdot\|_\infty$)

  $\|\mathbf{x}\|_1 = |1| + |-2| + |3| = 6$

  $\|\mathbf{x}\|_2 = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14} \doteq 3.7417$

  $\|\mathbf{x}\|_\infty = \max\{|1|,\ |-2|,\ |3|\} = 3$

||1/n||₂ = π/√6 – the Basel problem.

Input: `||1/n||_2`

Result

$$\|a\|_2 = \frac{\sqrt{6}\pi}{6} \doteq 1.2825$$

Steps

- The norm of the sequence $a_n = \frac{1}{n}$ in the space ℓ² for p = 2: we sum the series $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^2 = \frac{\pi^2}{6}$ and take the root

  $\|a\|_2 = \sqrt{\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^2} = \sqrt{\frac{\pi^2}{6}} = \frac{\sqrt{6}\pi}{6} \doteq 1.2825$

The harmonic series diverges – 1/n is not in ℓ^1.

Input: `||1/n||_1`

Result

The sequence (function) does not lie in this space – its norm is infinite.

Steps

- $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ – the sequence $a_n = \frac{1}{n}$ has no finite norm, it does not lie in the space ℓ^p for p = 1

The absolute value removed by splitting the integral.

Input: `||x - 1/2||_{L^3(0; 1)}`

Result

$$\|f\|_3 = \frac{1}{2\sqrt[3]{2^2}} \doteq 0.315$$

Steps

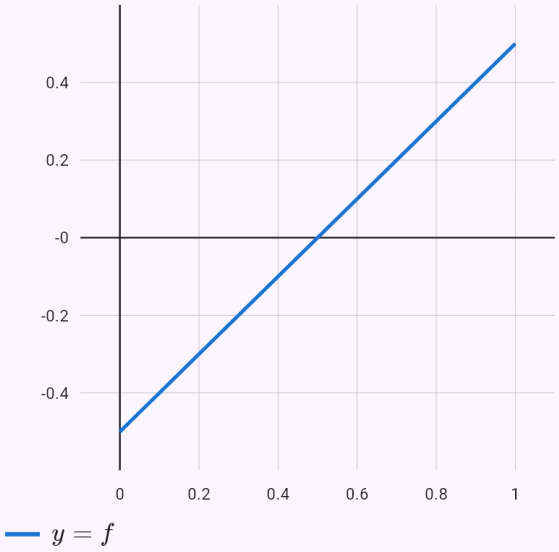
- We remove the absolute value by splitting the integral at the function's zeros $x = \frac{1}{2}$ – on each piece the function keeps its sign

  $$\int_0^{\frac{1}{2}} \left(\frac{1}{2} - x\right)^3 dx + \int_{\frac{1}{2}}^1 \left(x - \frac{1}{2}\right)^3 dx$$

- The norm in the space L^p for p = 3: the integral $\int_0^1 \left|x - \frac{1}{2}\right|^3 dx = \frac{1}{32}$, and so

  $$\|f\|_3 = \left(\int_0^1 \left|x - \frac{1}{2}\right|^3 dx\right)^{\frac{1}{3}} = \left(\frac{1}{32}\right)^{\frac{1}{3}} = \frac{1}{2\sqrt[3]{2^2}} \doteq 0.3$$

Graph



The function on the given interval.

The supremum norm from the stationary points and the endpoints.

Input: `||x^3 - 3x||_{C⟨-2; 3⟩}`

Result

$$\|f\|_\infty = 18$$

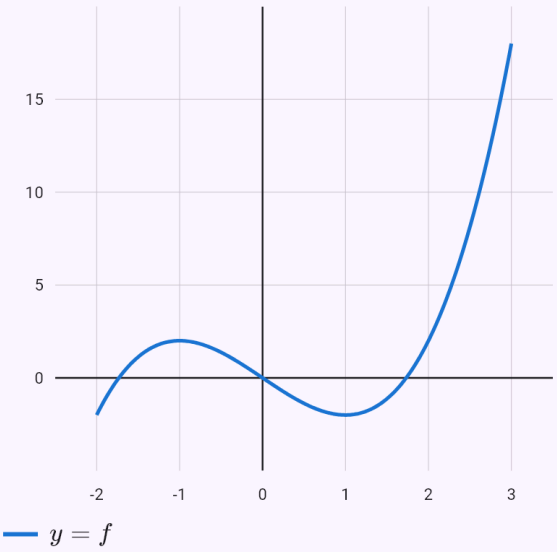
Steps

- The supremum norm of a continuous function on a closed interval is the largest value of |f|. We look for it at the endpoints and at the stationary points inside: the derivative $f'(x) = 3\left(x^2 - 1\right)$, the stationary points $x = -1,\ x = 1$
- The values of |f| at these points

x	-2	-1	1	3
$\lvert f(x)\rvert$	2	2	2	18

- The largest of them is the supremum norm $\|f\|_\infty = \max\limits_{x\in\langle -2;\,3\rangle} |f(x)| = |f(3)| = 18$

Graph



The function on the given interval.

The distance in the supremum norm and in L².

Input: `dist(x, x^2, ⟨0; 1⟩)`

Result

$$\rho_\infty(f,g) = \frac{1}{4}, \quad \rho_2(f,g) = \frac{\sqrt{30}}{30} \doteq 0.$$

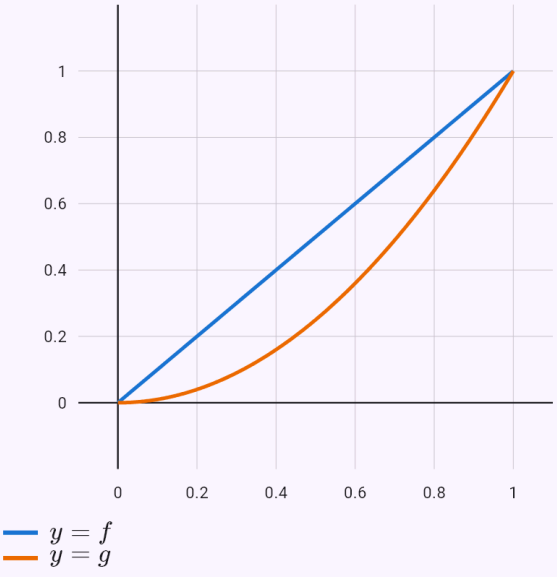
Steps

- The distance of two functions is the norm of their difference: $f - g = x(1 - x)$
- The supremum norm of a continuous function on a closed interval is the largest value of |f|. We look for it at the endpoints and at the stationary points inside: the derivative $(f - g)'(x) = 1 - 2x$, the stationary points $x = \frac{1}{2}$
- The values of |f| at these points

x	0	$\frac{1}{2}$	1
$\lvert(f-g)(x)\rvert$	0	$\frac{1}{4}$	0

- The largest of them is the supremum norm $\|(f-g)\|_\infty = \max\limits_{x\in\langle 0;\,1\rangle} |(f-g)(x)| = \left|(f-g)\left(\frac{1}{2}\right)\right| = \frac{1}{4}$
- The norm in the space L² for p = 2: the integral $\int_0^1 (x(1-x))^2\,dx = \frac{1}{30}$, and so $\|f-g\|_2 = \sqrt{\int_0^1 (x(1-x))^2\,dx} = \sqrt{\frac{1}{30}} = \frac{\sqrt{30}}{30} \doteq 0.$

Graph



Both functions on the given interval.

Hölder's inequality with the conjugate exponent q = 3/2.

Input: `holder(x, 1 - x, 3, ⟨0; 1⟩)`

Result

$$\left|\int_0^1 f g\,dx\right| = \frac{1}{6} \le \|f\|_3 \|g\|_{\frac{3}{2}} = \frac{1}{\sqrt[3]{5^2}} \doteq$$

Steps

- Hölder's inequality works with the conjugate exponent q: $\frac{1}{3} + \frac{1}{q} = 1 \;\Rightarrow\; q = \frac{3}{2}$
- The norm in the space L𝕀 for p = 3: the integral $\int_0^1 |x|^3\,dx = \frac{1}{4}$, and so
  $\|f\|_3 = \left(\int_0^1 |x|^3\,dx\right)^{\frac{1}{3}} = \left(\frac{1}{4}\right)^{\frac{1}{3}} = \frac{1}{\sqrt[3]{2^2}} \doteq 0.63$
- The norm in the space L𝕀 for p = $\frac{3}{2}$: the integral $\int_0^1 |1-x|^{\frac{3}{2}}\,dx = \frac{2}{5}$, and so
  $\|g\|_{\frac{3}{2}} = \left(\int_0^1 |1-x|^{\frac{3}{2}}\,dx\right)^{\frac{2}{3}} = \left(\frac{2}{5}\right)^{\frac{2}{3}} = \sqrt[3]{\left(\frac{2}{5}\right)^2} \doteq 0.5$
- The left side $\left|\int_0^1 f g\,dx\right| = \frac{1}{6}$, the right side $\|f\|_3 \|g\|_{\frac{3}{2}} = \frac{1}{\sqrt[3]{5^2}} \doteq 0.342$ – Hölder's inequality holds.

x^n – pointwise yes, uniformly no, in L^2 yes.

Input: `convergence(x^n, ⟨0; 1⟩)`

Result

The sequence of functions converges pointwise, but not uniformly.

Steps

- The limit is decided by the power q^n with the base $q = x$: for $|q| < 1$, $q^n \to 0$ (faster than any power of n grows), for $|q| > 1$ it grows without bound. The points where $|q| = 1$ are computed separately.
- The pointwise limit: for every fixed x, $f_n(x) = x^n$ is a sequence of numbers; we compute its limit as $n \to \infty$

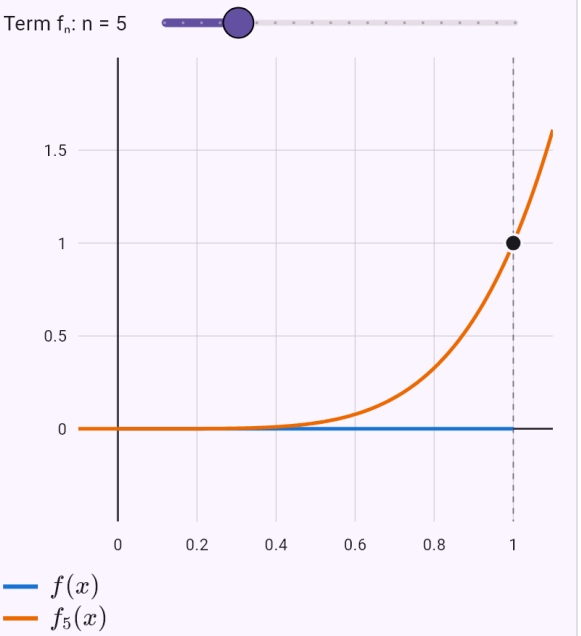
$$x = 0 : \lim_{n\to\infty} f_n(0) = 0$$
$$x \in (0;\, 1) : \lim_{n\to\infty} f_n(x) = 0$$
$$x = 1 : \lim_{n\to\infty} f_n(1) = 1$$

- The limit function

$$f(x) = \begin{cases} 0, & x \in \langle 0;\, 1) \\ 1, & x = 1 \end{cases}$$

- All the functions f_n are continuous, but the limit function is not continuous at $x = 1$. A uniform limit of continuous functions is continuous – so the convergence is not uniform.
- Convergence in L^2: $\|f_n - f\|_2^2 = \int_0^1 (x^n)^2 dx = \frac{1}{2n+1}$, $\lim_{n\to\infty} \|f_n - f\|_2^2 = 0$ – the sequence converges in L^2.

Graph



The limit function f and the chosen term f_n – move the slider to watch f_n approach f.

Uniform convergence: $s_n = 1/(2\sqrt{n}) \to 0$.

Input: `convergence(x/(1 + n x^2), ⟨-1; 1⟩)`

Result

The sequence of functions converges uniformly.

Steps

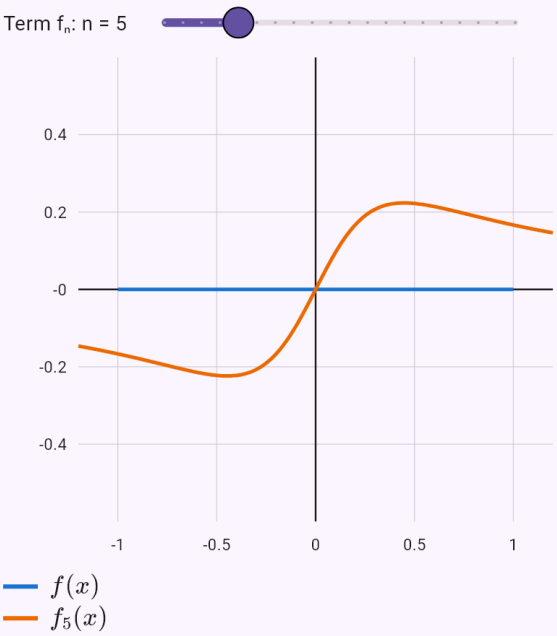
- The pointwise limit: for every fixed x, $f_n(x) = \dfrac{x}{nx^2+1}$ is a sequence of numbers; we compute its limit as n → ∞

  $x \in \langle -1;\ 1\rangle : \lim\limits_{n\to\infty} f_n(x) = 0$

- The limit function: $f(x) = 0$
- We judge uniformity by $s_n = \sup |f_n - f|$. The difference $f_n(x) - f(x) = \dfrac{x}{nx^2+1}$ has the derivative $-\dfrac{2nx^2}{(nx^2+1)^2} + \dfrac{1}{nx^2+1}$; the candidates for the largest value are $x = -1,\ x = 1,\ x_n = -\dfrac{1}{\sqrt{n}},\ x_n = \dfrac{1}{\sqrt{n}}$: $s_n = \sup\limits_x |f_n(x) - f(x)| = \dfrac{1}{2\sqrt{n}}$. Since $\lim\limits_{n\to\infty} s_n = 0$, the convergence is uniform.
- Convergence in L² – we compute the integral $\|f_n - f\|_2^2$ numerically. The values decrease to zero – the sequence converges in L².

n	10	100	1000
$\|\|f_n - f\|\|_2^2$	0.0309	0.0014	$4.767 \cdot 10^{-5}$

Graph



The limit function f and the chosen term f_n – move the slider to watch f_n approach f.

The travelling hump – pointwise to zero, but s_n and $\|f_n\|_2$ grow.

Input: `convergence(n x e^(-n x^2), ⟨0; 1⟩)`

Result

The sequence of functions converges pointwise, but not uniformly.

Steps

- The limit is decided by the power q^n with the base $q = e^{-x^2}$: for |q| < 1, $q^n \to 0$ (faster than any power of n grows), for |q| > 1 it grows without bound. The points where |q| = 1 are computed separately.
- The pointwise limit: for every fixed x, $f_n(x) = nxe^{-nx^2}$ is a sequence of numbers; we compute its limit as $n \to \infty$

$$x \in \langle 0;\ 1\rangle : \lim_{n\to\infty} f_n(x) = 0$$

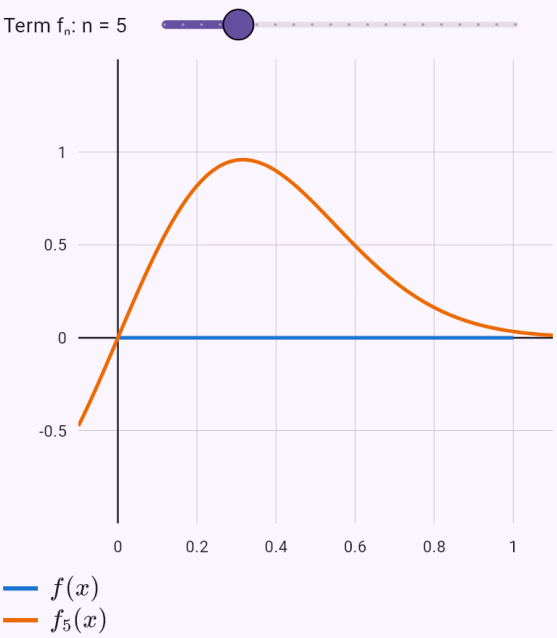
- The limit function: $f(x) = 0$
- We judge uniformity by $s_n = \sup |f_n - f|$. The difference $f_n(x) - f(x) = nxe^{-nx^2}$ has the derivative $ne^{-nx^2}\left(1 - 2nx^2\right)$; the candidates for the largest value are $x = 0,\ x = 1,\ x_n = \dfrac{\sqrt{2}}{2\sqrt{n}}$. Since $\lim_{n\to\infty} s_n = \infty$ (not 0), the convergence is not uniform.

$$s_n = \sup_x |f_n(x) - f(x)| = \frac{\sqrt{n}}{\sqrt{2e}}$$

- Convergence in L² – we compute the integral $\|f_n - f\|_2^2$ numerically. The values do not decrease to zero – the sequence does not converge in L².

n	10	100	1000
$\|f_n - f\|_2^2$	0.4954	1.5666	4.9542

Graph



The limit function f and the chosen term f_n – move the slider to watch f_n approach f.

The inner product, the norms and the angle of two functions.

Input: `inner(x, x^2, ⟨0; 1⟩)`

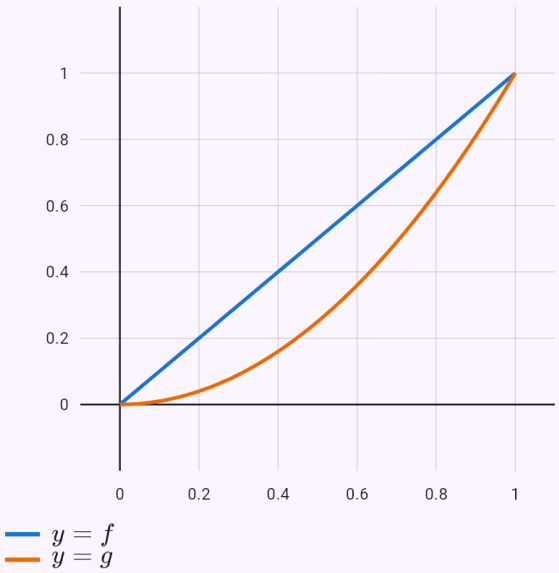
Result

$$\langle f, g \rangle = \frac{1}{4}$$

Steps

- The inner product in the space L^2:
  $$\langle f, g \rangle = \int_0^1 x \cdot x^2 \, dx = \frac{1}{4}$$
- The norms of both functions given by the inner product
  $$\|f\| = \sqrt{\langle f, f \rangle} = \frac{\sqrt{3}}{3} \doteq 0.5774, \quad \|g\| = \sqrt{\langle g, g \rangle} = \frac{\sqrt{}}{}$$
- The angle between the functions (the Cauchy–Schwarz inequality guarantees $|\cos \varphi| \le 1$)
  $$\cos \varphi = \frac{\langle f, g \rangle}{\|f\| \|g\|} = \frac{\sqrt{15}}{4} \doteq 0.9682, \quad \varphi \doteq 14.4775^\circ$$

Graph



Both functions on the given interval.

Legendre polynomials from 1, x, x², x³.

Input: `gramschmidt(1, x, x^2, x^3; ⟨-1; 1⟩)`

Result

$$e_1 = 1,\ e_2 = x,\ e_3 = x^2 - \frac{1}{3},\ e_4 = x^3$$

Steps

- The Gram–Schmidt process – we keep the first function: $e_1 = 1, \|e_1\|^2 = 2$
- Step 2: we subtract the function's projections onto the previous ones: $e_2 = x$, $\|e_2\|^2 = \frac{2}{3}$. The inner products needed: $\langle f_2, e_1 \rangle = 0$
- Step 3: we subtract the function's projections onto the previous ones: $e_3 = x^2 - \frac{1}{3}$, $\|e_3\|^2 = \frac{8}{45}$. The inner products needed: $\langle f_3, e_1 \rangle = \frac{2}{3},\ \langle f_3, e_2 \rangle = 0$
- Step 4: we subtract the function's projections onto the previous ones: $e_4 = x^3 - \frac{3}{5}x = x^3 - \frac{3x}{5}$, $\|e_4\|^2 = \frac{8}{175}$. The inner products needed

  $\langle f_4, e_1 \rangle = 0,\ \langle f_4, e_2 \rangle = \frac{2}{5},\ \langle f_4, e_3 \rangle = 0$
- Dividing by the norms gives an orthonormal system

  $\hat{e}_1 = \frac{e_1}{\sqrt{2}},\ \hat{e}_2 = \frac{e_2}{\frac{\sqrt{6}}{3}},\ \hat{e}_3 = \frac{e_3}{\frac{2\sqrt{10}}{15}},\ \hat{e}_4 = \frac{e_4}{\frac{2\sqrt{14}}{35}}$
- These are the Legendre polynomials – in their usual form with $P_k(1) = 1$

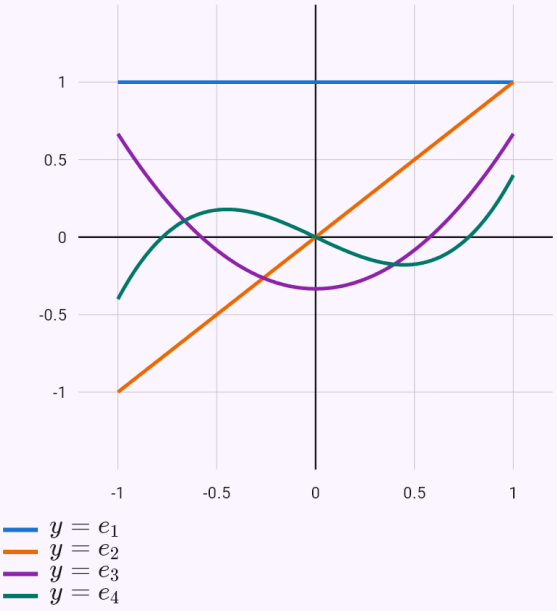
  $P_0(x) = 1$

  $P_1(x) = x$

  $P_2(x) = \frac{3}{2}e_3 = \frac{3x^2}{2} - \frac{1}{2}$

  $P_3(x) = \frac{5}{2}e_4 = \frac{5x^3}{2} - \frac{3x}{2}$

Graph



The orthogonal functions on the given interval.

Chebyshev polynomials – the integrals from the weight's moments.

Input: `gramschmidt(1, x, x^2, x^3; ⟨-1; 1⟩; w = 1/sqrt(1 - x^2))`

Result

$$e_1 = 1,\ e_2 = x,\ e_3 = x^2 - \frac{1}{2},\ e_4 = x^3$$

Steps

- The inner products of polynomials with the weight $w(x) = \frac{1}{\sqrt{1-x^2}}$ come from the weight's moments $\mu_k = \int_{-1}^{1} x^k\, w(x)dx$

  $\mu_0 = \pi,\ \mu_2 = \frac{\pi}{2},\ \mu_4 = \frac{3\pi}{8},\ \mu_6 = \frac{5\pi}{16}$

- The Gram–Schmidt process – we keep the first function: $e_1 = 1$, $\|e_1\|^2 = \pi \doteq 3.1416$

- Step 2: we subtract the function's projections onto the previous ones: $e_2 = x$, $\|e_2\|^2 = \frac{\pi}{2} \doteq 1.5708$. The inner products needed: $\langle f_2, e_1\rangle = 0$

- Step 3: we subtract the function's projections onto the previous ones: $e_3 = x^2 - \frac{1}{2}$, $\|e_3\|^2 = \frac{\pi}{8} \doteq 0.3927$. The inner products needed

  $\langle f_3, e_1\rangle = \frac{\pi}{2} \doteq 1.5708,\ \langle f_3, e_2\rangle = 0$

- Step 4: we subtract the function's projections onto the previous ones: $e_4 = x^3 - \frac{3}{4}x = x^3 - \frac{3x}{4}$, $\|e_4\|^2 = \frac{\pi}{32} \doteq 0.0982$. The inner products needed

  $\langle f_4, e_1\rangle = 0,\ \langle f_4, e_2\rangle = \frac{3\pi}{8} \doteq 1.1781,\ \langle f_4, e_3\rangle = 0$

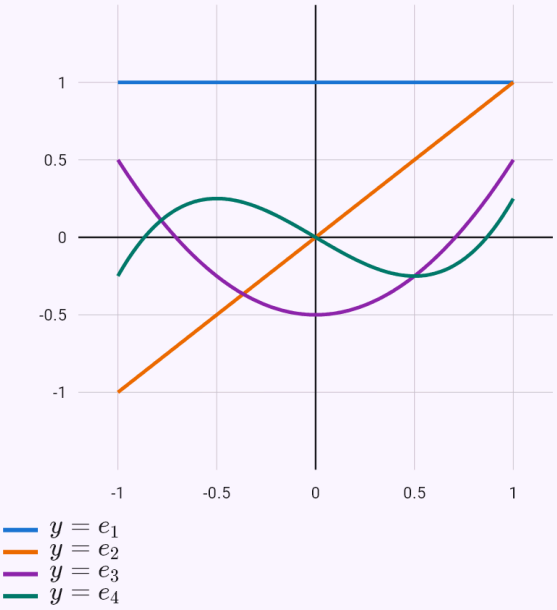
- Dividing by the norms gives an orthonormal system

  $\hat{e}_1 = \frac{e_1}{\sqrt{\pi}},\ \hat{e}_2 = \frac{e_2}{\sqrt{\frac{\pi}{2}}},\ \hat{e}_3 = \frac{e_3}{\sqrt{\frac{\pi}{8}}},\ \hat{e}_4 = \frac{e_4}{\sqrt{\frac{\pi}{32}}}$

- These are the Chebyshev polynomials of the first kind, $T_k(\cos t) = \cos kt$

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_2(x) = 2e_3 = 2x^2 - 1$$
$$T_3(x) = 4e_4 = 4x^3 - 3x$$

Graph



The orthogonal functions on the given interval.

The best quadratic approximation of e^x on $\langle -1; 1 \rangle$.

Input: `approx(e^x, 2, ⟨-1; 1⟩)`

Result

$$p(x) = \left(\frac{15(e^2-7)}{4e}\right)x^2 + \frac{3}{e}x + \frac{3(11}{}$$

Steps

- The best approximation in L² is the orthogonal projection onto the polynomials of degree at most n. The Gram–Schmidt process applied to 1, x, x², … gives an orthogonal basis: $e_1 = 1,\ e_2 = x,\ e_3 = x^2 - \frac{1}{3}$; $\|e_1\|^2 = 2,\ \|e_2\|^2 = \frac{2}{3},\ \|e_3\|^2 = \frac{8}{45}$
- The projection's coefficients

$$c_1 = \frac{\langle f, e_1\rangle}{\|e_1\|^2} = \frac{e - \frac{1}{e}}{2} \doteq 1.1752$$

$$c_2 = \frac{\langle f, e_2\rangle}{\|e_2\|^2} = \frac{3}{e} \doteq 1.1036$$

$$c_3 = \frac{\langle f, e_3\rangle}{\|e_3\|^2} = \frac{15\left(e - \frac{7}{e}\right)}{4} \doteq 0.5367$$

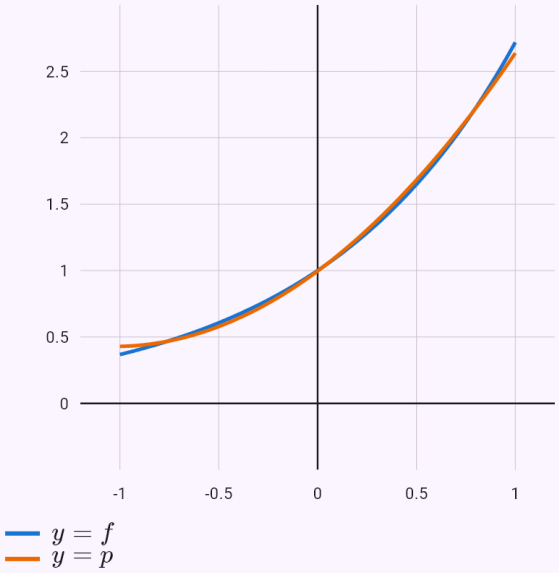
- The best approximation

$$p(x) = \sum c_k e_k = \left(\frac{15(e^2-7)}{4e}\right)x^2 + \frac{3}{e}x + \frac{3(11-e^2}{4e}$$

- The error by Pythagoras' theorem: $\|f\|^2 = \sinh 2 \doteq 3.6269$. Bessel's inequality: $\sum c_k^2 \|e_k\|^2 \doteq 3.6254 \le \|f\|^2 \doteq 3.6269$. For a complete system (n → ∞) it becomes an equality – Parseval's identity.

$$\|f - p\|^2 = \|f\|^2 - \sum c_k^2 \|e_k\|^2 = 36 - 132\cosh 2 + 12$$

Graph



The function f and its best approximation p.

The norms 1, 2, ∞ and Frobenius of one matrix.

Input: `||[1 2; 3 4]||`

Result

$$\|A\|_1 = 6,\ \|A\|_2 = \sqrt{15+\sqrt{221}} \doteq 5.$$

Steps

- The matrix
  $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$
- The column-sum norm is the largest sum of the absolute values in a column: $s_1 = 1+3 = 4, \quad s_2 = 2+4 = 6,$ so $\|A\|_1 = \max_j \sum_i |a_{ij}| = 6$
- The spectral norm is the square root of the largest eigenvalue of. The eigenvalues:
  $\lambda_1 = 15+\sqrt{221} \doteq 29.8661,\ \lambda_2 = 15-\sqrt{221} \doteq 0.1339$
  , so $\|A\|_2 = \sqrt{\lambda_{\max}\left(A^T A\right)} = \sqrt{15+\sqrt{221}} \doteq 5.465$
  $A^T A = \begin{pmatrix} 10 & 14 \\ 14 & 20 \end{pmatrix}$
- The row-sum norm is the largest sum of the absolute values in a row: $r_1 = 1+2 = 3, \quad r_2 = 3+4 = 7,$ so $\|A\|_\infty = \max_i \sum_j |a_{ij}| = 7$
- The Frobenius norm is the square root of the sum of the squares of all entries
  $\|A\|_F = \sqrt{\sum_{i,j} a_{ij}^2} = \sqrt{1^2+2^2+3^2+4^2} = \sqrt{30} \doteq 5.4'$

A nearly singular matrix – κ ≈ 4·10⁴.

Input: `cond([1 1; 1 1.0001])`

Result

$$\kappa_2(A) \doteq 40002.0014$$

Steps

- The matrix
  $A = \begin{pmatrix} 1 & 1 \\ 1 & \frac{10001}{10000} \end{pmatrix}$
- The spectral norm is the square root of the largest eigenvalue of. The eigenvalues so
  $A^T A = \begin{pmatrix} 2 & \frac{20001}{10000} \\ \frac{20001}{10000} & \frac{200020001}{100000000} \end{pmatrix}$
  $\lambda_1 = \frac{400020001+20001\sqrt{400000001}}{200000000} \doteq 4.0002,\ \lambda_2 =$
  $\sigma_{\max} = \sqrt{\frac{400020001+20001\sqrt{400000001}}{200000000}}, \quad \sigma_{\min} =$
- The condition number. It tells how many times the relative error of the input may grow; solving a system may lose about 4.6 significant digits – the matrix is ill-conditioned.
  $\kappa_2(A) = \|A\|_2 \|A^{-1}\|_2 = \frac{\sigma_{\max}}{\sigma_{\min}} = \sqrt{\frac{\frac{400020001+20001\sqrt{400}}{200000000}}{\frac{400020001-20001\sqrt{400}}{200000000}}}$

The kernel changes its sign – the integral split at t = x.

Input: `opnorm(x - t, ⟨0; 1⟩)`

Result

$$\|K\|_C = \frac{1}{2}, \quad \|K\|_{L^2} = \frac{\sqrt{3}}{6} \doteq 0.2887$$

Steps

- The norm of an integral operator on C⟨a; b⟩ is the largest value of φ(x) = ∫|K(x, t)| dt. The kernel $K(x,t) = x - t$ changes its sign where K = 0 – the integral is split there

$$\varphi(x) = \int_0^1 |K(x,t)|dt = \int_0^x (x-t)\,dt + \int_x^1 (t-x)\,dt$$

- The supremum norm of a continuous function on a closed interval is the largest value of |f|. We look for it at the endpoints and at the stationary points inside: the derivative $\varphi'(x) = 2x - 1$, the stationary points $x = \frac{1}{2}$
- The values of |f| at these points

x	0	$\frac{1}{2}$	1
$\lvert\varphi(x)\rvert$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$

- The largest of them is the supremum norm

$$\|\varphi\|_\infty = \max_{x\in\langle 0;\,1\rangle} |\varphi(x)| = |\varphi(0)| = \frac{1}{2}$$

- The kernel is degenerate, $K(x,t) = -t + x$. Its L² norm then comes from the Gram matrices of the functions a⃗(x) and b⃗(t). For comparison, the Hilbert–Schmidt norm is never smaller.

$$G_a = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{3} \end{pmatrix}, \quad G_b = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$$

$$\|K\|_{L^2} = \sqrt{\lambda_{\max}(G_b G_a)} = \sqrt{\frac{1}{12}} = \frac{\sqrt{3}}{6} \doteq 0.2887$$

$$\|K\|_{HS} = \sqrt{\int_0^1 \int_0^1 K^2(x,t)\,dt\,dx} = \frac{\sqrt{6}}{6} \doteq 0.4082$$

A functional's norm on C and on L² (Riesz's theorem).

Input: `fnorm(x^2, ⟨0; 1⟩)`

Result

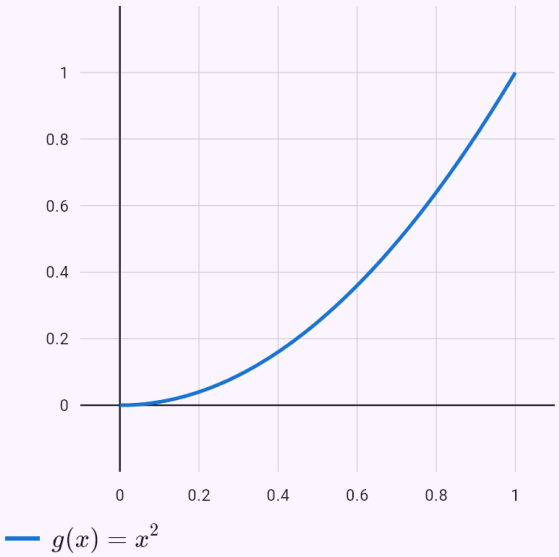
$$\|\varphi\|_{(C)^*} = \frac{1}{3}, \quad \|\varphi\|_{(L^2)^*} = \frac{\sqrt{5}}{5} \doteq 0.44$$

Steps

- The functional φ(u) = ∫u·g on $C\langle 0;\,1\rangle$ with the supremum norm: its norm is the integral of |g|, approached by continuous u ≈ sgn g: $\|\varphi\| = \sup_{\|u\|\le 1} \left|\int_0^1 u\,g\,dx\right| = \|g\|_1$
- The norm in the space L⃗ for p = 1: the integral $\int_0^1 \left|x^2\right| dx = \frac{1}{3}$, and so $\|g\|_1 = \int_0^1 \left|x^2\right| dx = \frac{1}{3}$
- On $L^2\langle 0;\,1\rangle$ the functional φ(u) = ∫u·g is the inner product with g (Riesz's theorem). By the Cauchy–Schwarz inequality its norm is ‖g‖₂, attained for u = g/‖g‖₂: $\|\varphi\| = \sup_{\|u\|\le 1} \left|\int_0^1 u\,g\,dx\right| = \|g\|_2$
- The norm in the space L⃗ for p = 2: the integral $\int_0^1 \left(x^2\right)^2 dx = \frac{1}{5}$, and so

$$\|g\|_2 = \sqrt{\int_0^1 \left(x^2\right)^2 dx} = \sqrt{\frac{1}{5}} = \frac{\sqrt{5}}{5} \doteq 0.4472$$

Graph



The function on the given interval.

A degenerate kernel with the parameter λ.

Input: `u(x) = x + λ∫_0^1 x t u(t) dt`

Result

$$u(x) = \frac{3x}{3-\lambda}$$

Conditions: $\lambda \neq 3$

Steps

- A Fredholm integral equation of the second kind (fixed limits): $u(x) = x + \lambda \int_0^1 xt\,u(t)\,dt$, the kernel $K(x,t) = xt$, $\lambda \in \mathbb{R}$
- The kernel is degenerate – a product of a function of x and a function of t: $K(x,t) = x \cdot t$
- The integrals with the unknown function are numbers – call them unknown constants $c = \int_0^1 t\,u(t)\,dt$. The solution therefore has the form $u(x) = x + \lambda(c \cdot x)$
- Substituting this form into the constants' definitions gives a system of linear equations with the coefficients

  $a_{11} = \int_0^1 t \cdot t\,dt = \frac{1}{3}, \quad \beta_1 = \int_0^1 t \cdot t\,dt = \frac{1}{3}$

  $c - \frac{c\lambda}{3} = \frac{1}{3}$
- The system has exactly one solution where its determinant is non-zero: $D(\lambda) = \det(I - \lambda A) = 1 - \frac{\lambda}{3}$. Its roots are the kernel's characteristic values: $\lambda_1 = 3$
- For $\lambda \neq 3$ the system's solution is $c = \frac{1}{3-\lambda}$, hence $u(x) = \frac{3x}{3-\lambda}$
- For $\lambda_1 = 3$ the homogeneous equation has non-zero solutions – the eigenfunctions $\varphi_1(x) = x$
- Fredholm's alternative for $\lambda_1 = 3$: the system is inconsistent ($h(I - \lambda A) = 0 < h(I - \lambda A \mid \beta) = 1$), so the equation has no solution for this λ (the eigenfunction $\varphi(x) = x$).

Characteristic values and eigenfunctions.

Input: `u(x) = λ∫_0^1 (x + t) u(t) dt`

Result

$$\lambda_1 = -6 - 4\sqrt{3}: \; u(x) = C\left(x - \frac{\sqrt{3}}{3}\right.$$

Steps

- A Fredholm integral equation of the second kind (fixed limits): $u(x) = \lambda \int_0^1 (x+t)\,u(t)\,dt$, the kernel $K(x,t) = x + t$, $\lambda \in \mathbb{R}$
- The kernel is degenerate – a sum of 2 products of a function of x and a function of t: $K(x,t) = t + x$
- The integrals with the unknown function are numbers – call them unknown constants $c_1 = \int_0^1 t\,u(t)\,dt, \quad c_2 = \int_0^1 u(t)\,dt$. The solution therefore has the form $u(x) = \lambda(c_1 + c_2 \cdot x)$
- Substituting this form into the constants' definitions gives a system of linear equations with the coefficients

  $a_{11} = \int_0^1 t \cdot 1\,dt = \frac{1}{2}, \quad a_{12} = \int_0^1 t \cdot t\,dt = \frac{1}{3}, \quad a_{21} =$

  $c_1 - \frac{c_1\lambda}{2} - \frac{c_2\lambda}{3} = 0$

  $c_2 - c_1\lambda - \frac{c_2\lambda}{2} = 0$
- The system has exactly one solution where its determinant is non-zero: $D(\lambda) = \det(I - \lambda A) = 1 - \lambda - \frac{\lambda^2}{12}$. Its roots are the kernel's characteristic values: $\lambda_1 = -6 - 4\sqrt{3} \doteq -12.9282, \; \lambda_2 = -6 + 4\sqrt{3} \doteq 0.9282$
- For $\lambda_1 = -6 - 4\sqrt{3}$ the homogeneous equation has non-zero solutions – the eigenfunctions $\varphi_1(x) = x - \frac{\sqrt{3}}{3}$
- For $\lambda_2 = -6 + 4\sqrt{3}$ the homogeneous equation has non-zero solutions – the eigenfunctions $\varphi_2(x) = x + \frac{\sqrt{3}}{3}$

A non-degenerate kernel – the Neumann series and Nyström's method.

Input: `u(x) = 1 + 1/2 ∫_0^1 e^(x t) u(t) dt`

Result

$$u(0) \doteq 2.517,\ u(0.5) \doteq 2.9997,\ u(1)$$

Steps

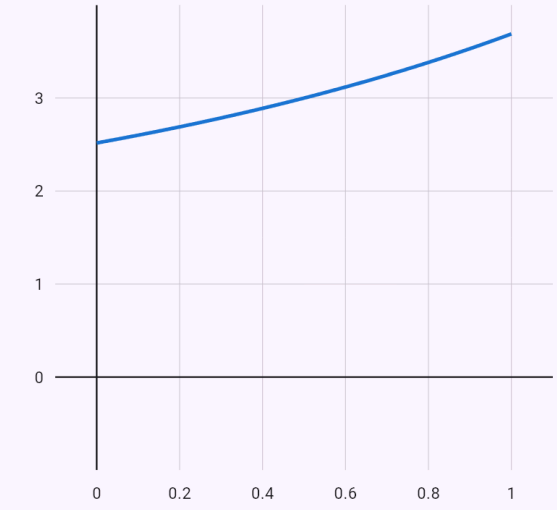
- A Fredholm integral equation of the second kind (fixed limits): $u(x) = 1 + \frac{1}{2}\int_0^1 \left(e^{xt}\right) u(t)\, dt$, the kernel $K(x,t) = e^{xt}$, $\lambda = \frac{1}{2}$
- The kernel isn't degenerate. The Neumann series – the successive approximations $u_{k+1} = f + \lambda \int K\, u_k\, dt$ – converges, because

$$|\lambda| \max_x \int_0^1 |K(x,t)| dt \doteq 0.5 \cdot 1.7183 \doteq 0.8591 < 1$$

- Nyström's method: the integral becomes a Gaussian quadrature with 32 nodes, the equation at the nodes a linear system, and the values between the nodes follow from the equation itself.Compared with half as many nodes, the accuracy is about 10^{-8}. The solution's values

x	0	0.25	0.5	0.75	1
$u(x)$	2.517	2.7372	2.9997	3.3134	3.6894

Graph



The solution of the integral equation.

A Volterra equation by the Laplace transform – cos x.

Input: `u(x) = 1 - ∫_0^x (x - t) u(t) dt`

Result

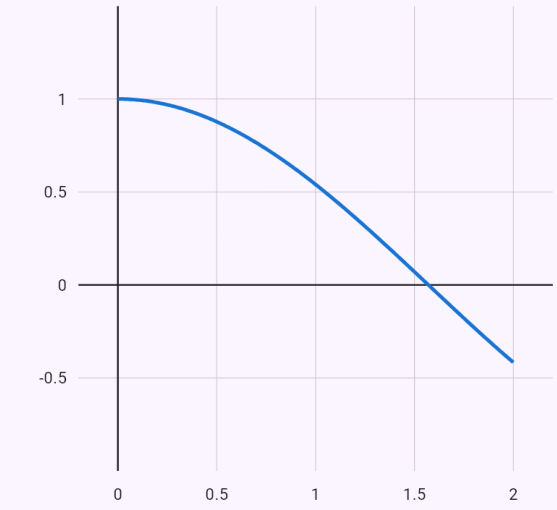
$$u(x) = \cos x$$

Steps

- A Volterra integral equation of the second kind (the upper limit is x): $u(x) = 1 - \int_0^x (x-t)\, u(t)\, dt$, the kernel $K(x,t) = x - t$, $\lambda = -1$
- The kernel depends on x − t only ($K(x,t) = k(x-t), \quad k(x) = x$) – the integral is a convolution, which the Laplace transform turns into a product. The transforms. The equation for the transforms $U(s) = F(s) - \hat{k}(s)\, U(s)$, hence $U(s) = \frac{F(s)}{1+\hat{k}(s)} = \frac{s}{s^2+1}$. Transforming back, $u(x) = \mathcal{L}^{-1}\{U(s)\} = \cos x$

$$F(s) = \mathcal{L}\{1\} = \frac{1}{s}, \quad \hat{k}(s) = \mathcal{L}\{x\} = \frac{1}{s^2}$$

Graph



The solution of the integral equation.

The fixed point of cos x with the a priori and a posteriori estimates.

Input: `banach(cos x, ⟨0; 1⟩)`

Result

$$x^* \doteq 0.7390851$$

Steps

- Banach's fixed point theorem needs $g(x) = \cos x$ to map the interval $I = \langle 0;\ 1\rangle$ into itself. Its range (from the ends and the stationary points) is $g(I) = \langle \cos 1 \doteq 0.5403;\ 1\rangle$ – it lies in I.
- The supremum norm of a continuous function on a closed interval is the largest value of |f|. The derivative $g''(x) = -\cos x$ has no zero inside the interval – the endpoints are enough.
- The values of |f| at these points

x	0	1
$\lvert g'(x)\rvert$	0	$\sin 1 \doteq 0.8415$

- The largest of them is the supremum norm

$$\|g'\|_\infty = \max_{x\in\langle 0;\,1\rangle} |g'(x)| = |g'(1)| = \sin 1 \doteq 0.8415$$

- By the mean value theorem |g(x) − g(y)| ≤ q·|x − y| with q the largest value of |g'|: $q = \max\limits_{x\in I} |g'(x)| = \sin 1 \doteq 0.8415 < 1$ – g is a contraction.
- The a priori estimate: start from $x_0 = 0.5, \quad x_1 = g(x_0) \doteq 0.8776$. From with $\varepsilon = 10^{-6}$ and $q = \sin 1$ we need n ≥ 86 steps.

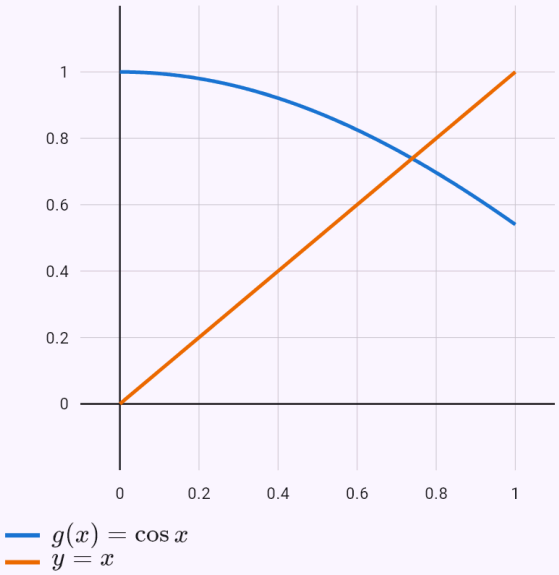
$$|x_n - x^*| \le \frac{q^n}{1-q}|x_1 - x_0| \le \varepsilon \ \Rightarrow\ n \ge \frac{\ln \frac{\varepsilon(1-q)}{|x_1 - x_0|}}{\ln q}$$

- Iterate $x_{n+1} = g(x_n)$; the last column is the a posteriori estimate of the error, and the computation stops once it drops below ε

n	x_n	$\lvert x_n - x_{n-1}\rvert$	$\frac{q}{1-q}\lvert x_n - x_{n-1}\rvert$
0	0.5		
1	0.8775826	0.3776	2.0042
2	0.6390125	0.2386	1.2663
3	0.8026851	0.1637	0.8688
4	0.694778	0.1079	0.5728
5	0.7681958	0.0734	0.3897
6	0.7191654	0.049	0.2603
7	0.7523558	0.0332	0.1762
⋮			
33	0.7390856	$1.145 \cdot 10^{-6}$	$6.075 \cdot 10^{-6}$
34	0.7390848	$7.710 \cdot 10^{-7}$	$4.092 \cdot 10^{-6}$
35	0.7390853	$5.193 \cdot 10^{-7}$	$2.757 \cdot 10^{-6}$
36	0.739085	$3.498 \cdot 10^{-7}$	$1.857 \cdot 10^{-6}$
37	0.7390852	$2.356 \cdot 10^{-7}$	$1.251 \cdot 10^{-6}$
38	0.7390851	$1.587 \cdot 10^{-7}$	$8.426 \cdot 10^{-7}$

- The fixed point: $x^* \doteq 0.7390851, \quad |x_{38} - x^*| \le 8.426 \cdot 10^{-7}$. It took 38 steps instead of the a priori 86 – the a posteriori estimate is usually much sharper.

Graph



The function g and the line y = x – the fixed point is where they meet.

The system x = Ax + b, a contraction in the ∞ norm.

Input: `banach([0.5 0.1; 0.2 0.3], (1; 2))`

Result

$$\mathbf{x}^* = \left(\frac{30}{11}; \frac{40}{11}\right)$$

Steps

- The system in the form for iteration

  $\mathbf{x} = A\mathbf{x} + \mathbf{b}, \quad A = \begin{pmatrix} \frac{1}{2} & \frac{1}{10} \\ \frac{1}{5} & \frac{3}{10} \end{pmatrix}, \quad \mathbf{b} = (1; 2)$

- The column-sum norm is the largest sum of the absolute values in a column so $\|A\|_1 = \max_j \sum_i |a_{ij}| = \frac{7}{10}$

  $s_1 = \frac{1}{2} + \frac{1}{5} = \frac{7}{10}, \quad s_2 = \frac{1}{10} + \frac{3}{10} = \frac{2}{5}$

- The row-sum norm is the largest sum of the absolute values in a row so $\|A\|_\infty = \max_i \sum_j |a_{ij}| = \frac{3}{5}$

  $r_1 = \frac{1}{2} + \frac{1}{10} = \frac{3}{5}, \quad r_2 = \frac{1}{5} + \frac{3}{10} = \frac{1}{2}$

- The map x ↦ Ax + b is a contraction when some norm of A is below 1. We take the smallest: $q = \|A\|_\infty = \frac{3}{5} < 1$

- The a priori estimate: start from $\mathbf{x}_0 = \mathbf{0}, \quad \mathbf{x}_1 = \mathbf{b}$. From with $\varepsilon = 10^{-6}$ and $q = 0.6$ we need n ≥ 31 steps.

  $\|\mathbf{x}_n - \mathbf{x}^*\| \le \frac{q^n}{1-q}\|\mathbf{x}_1 - \mathbf{x}_0\| \le \varepsilon \;\Rightarrow\; n \ge \frac{\ln \frac{\varepsilon(1-q)}{\|\mathbf{b}\|}}{\ln q}$

- Iterate $\mathbf{x}_{n+1} = A\mathbf{x}_n + \mathbf{b}$; the last column is the a posteriori estimate of the error, and the computation stops once it drops below ε

n	x_1	x_2	$\|\mathbf{x}_n - \mathbf{x}_{n-1}\|_\infty$
0	0	0	
1	1	2	2
2	1.7	2.8	0.8
3	2.13	3.18	0.43
4	2.383	3.38	0.253
5	2.5295	3.4906	0.1465
6	2.61381	3.55308	0.0843
7	2.662213	3.588686	0.0484
⋮			
23	2.7272639	3.6363572	$6.581 \cdot 10^{-6}$
24	2.7272677	3.6363599	$3.772 \cdot 10^{-6}$
25	2.7272698	3.6363615	$2.162 \cdot 10^{-6}$
26	2.7272711	3.6363624	$1.239 \cdot 10^{-6}$
27	2.7272718	3.6363629	$7.104 \cdot 10^{-7}$
28	2.7272722	3.6363632	$4.072 \cdot 10^{-7}$

- The fixed point. It took 28 steps instead of the a priori 31 – the a posteriori estimate is usually much sharper.

  $\mathbf{x}^* \doteq (2.7272722; 3.6363632), \quad \|\mathbf{x}_{28} - \mathbf{x}^*\| \le 6.108 \cdot 1$

- For comparison, the exact fixed point from the linear system

  $(I - A)\mathbf{x}^* = \mathbf{b} \;\Rightarrow\; \mathbf{x}^* = \left(\frac{30}{11}; \frac{40}{11}\right)$

Picard's approximations and the exact solution.

Input: `picard(y' = x + y, y(0) = 1)`

Result

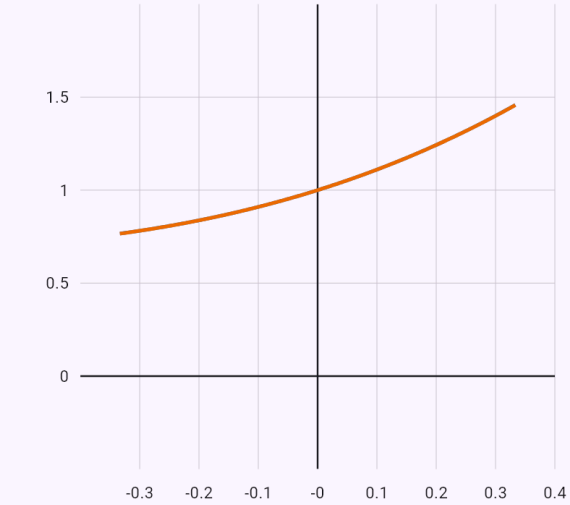
$$y_3(x) = \frac{x^4}{24} + \frac{x^3}{3} + x^2 + x + 1$$

Steps

- The problem $y' = x + y, \quad y(0) = 1$ becomes the integral equation y = T(y); Picard's successive approximations iterate the operator T:
  $y_{k+1}(x) = 1 + \int_0^x f(t, y_k(t))dt, \quad y_0(x) = 1$
- Approximation 1:
  $y_1(x) = 1 + \int_0^x (t+1)dt = \frac{x^2}{2} + x + 1$
- Approximation 2
  $y_2(x) = 1 + \int_0^x \left(\frac{t^2}{2} + 2t + 1\right)dt = \frac{x^3}{6} + x^2 + x + 1$
- Approximation 3
  $y_3(x) = 1 + \int_0^x \left(\frac{t^3}{6} + t^2 + 2t + 1\right)dt = \frac{x^4}{24} + \frac{x^3}{3} + x^$
- Where T is a contraction (the Picard–Lindelöf theorem): on the rectangle $R: \ |x| \le 1, \ |y-1| \le 1$ we have $M = \max_R |f| \doteq 3$ and. On the interval with $h = \min\left(1, \frac{1}{M}\right) \doteq 0.3333$ the approximations stay in the rectangle and $q = Lh \doteq 0.3333 < 1$
  $\frac{\partial f}{\partial y} = 1, \quad L = \max_R \left|\frac{\partial f}{\partial y}\right| \doteq 1$
- The last approximation's error bound for $|x| \le 0.3333$:
  $|y_3(x) - y(x)| \le \frac{ML^3h^4}{4!} \doteq 0.0015$

- The exact solution for comparison:
  $y(x) = -x + 2e^x - 1$; the actual error at the interval's end $|y_3(0.3333) - y(0.3333)| \doteq 0.0006$

Graph



Picard's approximation and the exact solution (where known).

Cauchy's equation – f(x) = cx.

Input: `f(x + y) = f(x) + f(y)`

Result

$$f(x) = c\,x$$

Steps

- Cauchy's functional equation of an additive function:
  $f(x+y) = f(x) + f(y), x, y \in \mathbb{R}$
- We assume the function f is continuous. Without that, Cauchy's equation also has discontinuous “pathological” solutions (built from a Hamel basis).
- Substitute: $x = y = 0:\quad f(0) = 2f(0) \;\Rightarrow\; f(0) = 0$
- Substituting y = −x shows the function is odd:
  $y = -x:\quad 0 = f(x) + f(-x) \;\Rightarrow\; f(-x) = -f(x)$
- By induction for natural n:
  $f(nx) = f((n-1)x) + f(x) = \cdots = n\,f(x)$
- For rational r = m/n
  $$n\,f\left(\frac{m}{n}\right) = f(m) = m\,f(1) \;\Rightarrow\; f(r) = c\,r,\; c = f(1),$$
- The rationals are dense and the function is continuous, so this holds for every real x:
  $f(x) = \lim_{r \to x} f(r) = \lim_{r \to x} c\,r = c\,x$
- All continuous solutions (a, b, c arbitrary constants):
  $f(x) = c\,x$
- Check by substitution: $c(x+y) = c\,x + c\,y$

d'Alembert's equation – cos and cosh.

Input: `f(x + y) + f(x - y) = 2 f(x) f(y)`

Result

$$f(x) = 0, \quad f(x) = \cos(cx), \quad f(x) =$$

Steps

- d'Alembert's functional equation:
  $f(x+y) + f(x-y) = 2f(x)\,f(y),\ x, y \in \mathbb{R}$
- We assume the function f is continuous. Without that, Cauchy's equation also has discontinuous "pathological" solutions (built from a Hamel basis).
- Substitute
  $x = y = 0: \quad 2f(0) = 2f(0)^2 \ \Rightarrow\ f(0) \in \{0, 1\}; \quad f($
- Substituting x = 0 shows the function is even:
  $x = 0: \quad f(y) + f(-y) = 2f(y) \ \Rightarrow\ f(-y) = f(y)$
- Continuous solutions are smooth; differentiating twice in y at y = 0 gives a differential equation
  $\frac{\partial^2}{\partial y^2}|_{y=0}: \quad 2f''(x) = 2f''(0)\,f(x) \ \Rightarrow\ f'' = k\,f,\ f(0)$
- By the sign of the constant k:
  $k = -c^2:\ \cos(cx), \qquad k = 0:\ 1, \qquad k = c^2:\ \cosh(cx)$
- All continuous solutions (a, b, c arbitrary constants):
  $f(x) = 0, \quad f(x) = \cos(cx), \quad f(x) = \cosh(cx)$
- Check by substitution:
  $\cos(cx + cy) + \cos(cx - cy) = 2\cos(cx)\cos(cy)$

Known limitations

- The app finds the pointwise limit for sequences whose limit is decided by a power q^n or a rational expression in n; more complicated ones (sin nx without dividing by n) it may decline.
- Convergence is examined on a closed bounded interval; an infinite interval not yet.
- The Gram–Schmidt process for at most 6 functions, the approximation by a polynomial of degree at most 5.
- A degenerate kernel with 4 terms at most; with a parameter λ degenerate kernels only, other kernels are solved numerically for a numerical λ. A Volterra equation exactly only with a kernel k(x − t) from zero or a(x)·b(t), otherwise by successive approximations and numerically.
- Matrices up to 6 × 6, at most 6 Picard approximations. Functional equations of the listed types only (and Cauchy's with an extra polynomial term), always with continuous solutions.

35. Functions of a complex variable

The Cauchy–Riemann equations of f(z) are checked by cr(z^2 + 3z); the function may be written in z, $\bar{z}$ (conj(z)), |z|, Re z, Im z or directly in x and y. The app splits f = u + iv, computes the partial derivatives and substitutes into u_x = v_y, u_y = −v_x. If they hold everywhere, the function is analytic and the result is the derivative f'(z) written in z; otherwise the app says where they hold (points or a curve) and the function is not analytic. For a harmonic function harmonic(u = x^2 - y^2) it finds the harmonic conjugate by integrating the equations and the whole f(z) by putting x = z, y = 0; the imaginary part can be given too, harmonic(v = 2xy).

Values of the elementary functions in $\mathbb{C}$: the principal logarithm ln(1 + i) = ln|w| + i Arg w with all values differing by 2kπi, a power with a complex exponent i^i = e^(i Log i), sin, cos, sinh and cosh of a complex number by the addition formulas. The argument is in the same interval as for complex numbers (the “Principal argument” chip). The equations sin z = 2, cos z = 3, sinh z = i, cosh z = 2 are solved by the substitution w = e^(iz) (e^z) and a quadratic equation, e^z = 1 + i by the logarithm; the result is the general solution with k ∈ $\mathbb{Z}$ and several solutions in the Gauss plane. An equation in x without a real solution (sin x = 2) offers the complex solution button.

Singularities and residues: singularities(f) finds the roots of the denominator and the essential singularities (e^(1/z), sin(1/z)), tells their kind – removable, a pole of order m, essential – and computes the residues: at a simple pole g(a)/q'(a), at a higher pole by the derivative of (z − a)^m f(z), at an essential singularity from the Laurent series. Res(f, z = a) computes one residue. The Laurent series of a rational function in an annulus laurent(f, 1 < |z| < 2) (or |z| > 1, 0 < |z − a| < r, around a point laurent(f, z = a)) is built from the partial fractions over $\mathbb{C}$ and the geometric series of each fraction – by whether its pole lies inside or outside the annulus. The integral over a circle oint(f, |z − a| = r) (from a photo \oint_{|z−a|=r}) is computed by the residue theorem; with no singularity inside it is zero, a singularity on the circle is reported.

Residues in real analysis: an improper integral of a rational function over the whole axis, with cos ax or sin ax (Jordan's lemma) and from 0 to ∞ of an even function; an integral of a rational function of cos t and sin t over ⟨0; 2π⟩ by z = e^(it). Integrals with a cosine or sine and over ⟨0; 2π⟩ go by residues at once, for the others choose the “By residues” chip (back by “The usual way”). The same chip on a series sums Σ R(n) from 1 (from 0 or another whole number) by the kernel π cot πz, an alternating series by π/sin πz; a sum over all whole numbers from −∞ goes at once. The inverse Laplace transform of a rational function invlaplace(F) (from a photo \mathcal{L}^{-1}\{F\}) is computed as the sum of the residues of F(s)e^(st).

Rouché's theorem rouche(p, |z| < r) counts the roots of a polynomial in a disc by the term that dominates the sum of the others on the circle, in an annulus r < |z| < R as the difference of two discs; the roots are shown in the Gauss plane. A conformal map w = f(z) is split into u and v, the app tells where it is

conformal ($f' \neq 0$) and for a Möbius transformation $(\alpha z + \beta)/(\gamma z + \delta)$ finds the fixed points and the images of the real and imaginary axis and of the unit circle. The graph shows the image of the grid of lines x = c and y = c.

The Fourier transform of a rational function ftransform(1/(1 + t^2)) (from a photo \mathcal{F}\{f\}) is computed by residues - for $\omega > 0$ the path is closed in the lower half-plane, for $\omega < 0$ in the upper one - and for an even (odd) function the result is written with $|\omega|$. For a density in x it gives the distribution's characteristic function up to the sign of ω, for the symmetric Cauchy distribution e^(−|t|) directly. The inverse Z-transform invz(F) (\mathcal{Z}^{-1}\{F\}) is the sum of the residues of $F(z)z^{n-1}$ with the initial value x[0] = lim F(z) for $z \to \infty$. Their use in filters, control and optics is in the chapter Applications of complex analysis.

How to enter it

`cr(z^2 + 3z), cr(conj(z))`	The Cauchy–Riemann equations and the derivative.
`harmonic(u = x^2 - y^2), harmonic(v = 2xy)`	The harmonic conjugate of a real (imaginary) part.
`ln(1 + i), i^i, sin(1 + i), sin z = 2`	The logarithm, powers, trigonometric functions and equations in $\mathbb{C}$.
`Res(f, z = a), singularities(f)`	A residue at a point, all singularities.
`laurent(f, 1 < |z| < 2), laurent(f, z = a)`	A Laurent series in an annulus or around a point.
`oint(f, |z - a| = r)`	An integral over a circle by the residue theorem.
`int(cos(x)/(x^2 + 1), x, -oo, oo), int(1/(2 + cos(t)), t, 0, 2pi)`	Real integrals by residues.
`residues(sum(1/n^2, n, 1, oo))`	A sum of a series by residues.
`rouche(z^5 + 3z + 1, |z| < 1)`	The number of roots of a polynomial in a disc or annulus.
`w = (z - i)/(z + i)`	A conformal map, a Möbius transformation.
`invlaplace(s/(s^2 + 4))`	The inverse Laplace transform by residues.
`ftransform(1/(1 + t^2))`	The Fourier transform by residues.

`invz(z/(z - 1/2))`	The inverse Z-transform by residues.

Examples

An analytic function – the equations hold everywhere, the derivative in z.

Input: `cr(z^2 + 3z)`

Result

$$f'(z) = 2z + 3$$

Steps

- We split the function into its real and imaginary part, f = u + iv with z = x + iy:
  $u = x^2 - y^2 + 3x, \quad v = y(2x + 3)$
- The partial derivatives of u and v with respect to x and y
  $u_x = 2x + 3,\ u_y = -2y$
  $v_x = 2y,\ v_y = 2x + 3$
- We substitute into the Cauchy–Riemann equations
  $u_x = v_y,\ u_y = -v_x$
  $u_x = v_y:\ 2x + 3 = 2x + 3$
  $u_y = -v_x:\ -2y = -2y$
- Both equations hold in the whole plane and the partial derivatives are continuous – the function is analytic in all of $\mathbb{C}$.
- The derivative of the analytic function, written in z:
  $f'(z) = u_x + i\,v_x = 2z + 3$

The equations hold only on the line y = x – the function is not analytic.

Input: `cr(x^2 + i y^2)`

Result

The function is not analytic – the Cauchy–Riemann equations hold on no region.

Steps

- We split the function into its real and imaginary part, f = u + iv with z = x + iy: $u = x^2, \quad v = y^2$
- The partial derivatives of u and v with respect to x and y
  $u_x = 2x,\ u_y = 0$
  $v_x = 0,\ v_y = 2y$
- We substitute into the Cauchy–Riemann equations $u_x = v_y, u_y = -v_x$
  $u_x = v_y:\ 2x = 2y$
  $u_y = -v_x:\ 0 = 0$
- Both equations hold only where $2(x - y) = 0$. The function has a derivative there, but no neighbourhood of a point satisfies them – the function is analytic nowhere.

The harmonic conjugate and f(z) = z² + z.

Input: `harmonic(u = x^2 - y^2 + x)`

Result

$$v = 2xy + y + C, \quad f(z) = z^2 + z +$$

Steps

- The real part u is given: $u = x^2 - y^2 + x$
- We check that it is harmonic – its Laplacian is zero: $\Delta u = u_{xx} + u_{yy} = 2 + (-2) = 0$
- The first Cauchy–Riemann equation $v_y = u_x$ gives v by integrating with respect to y – the "constant" may depend on x:
  $v_y = u_x = 2x + 1 \ \Rightarrow\ v = \int 2x + 1\, dy = 2xy + y + C(x)$
- We find C(x) from the second equation $v_x = -u_y$:
  $v_x = 2y + C'(x) = -u_y = 2y \ \Rightarrow\ C'(x) = 0,\ C(x) = 0 + C$
- The harmonic conjugate: $v = 2xy + y + C$
- We write f(z) = u + iv in z – it is enough to put x = z and y = 0: $f = u + i\,v,\ x = z,\ y = 0:\ f(z) = z^2 + z + iC$

The principal value of the logarithm and all its values.

Input: `ln(1 + i)`

Result

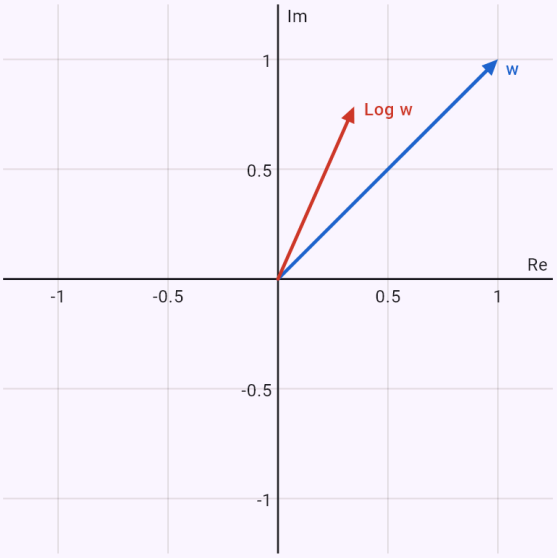
$$\operatorname{Log}(1+i) = \frac{\ln 2}{2} + \frac{\pi}{4}i, \quad \log(1+i)$$

Steps

- The principal value of the logarithm is $\operatorname{Log} w = \ln|w| + i \operatorname{Arg} w$, the argument in $\langle 0; 2\pi)$: $\operatorname{Log} w = \ln|w| + i \operatorname{Arg} w$
- The modulus and the principal argument of the number: $|1+i| = \sqrt{2}, \ \operatorname{Arg}(1+i) = \frac{\pi}{4}$
- The exponential is periodic with period 2πi, so the logarithm has infinitely many values – they differ by multiples of 2πi

  $\log(1+i) = \frac{\ln 2}{2} + i\left(\frac{\pi}{4} + 2k\pi\right), \ k \in \mathbb{Z}$

Graph



The complex plane – the given numbers blue, the result red, the argument φ as an arc from the positive real axis.

i^i is the real number e^(−π/2).

Input: `i^i`

Result

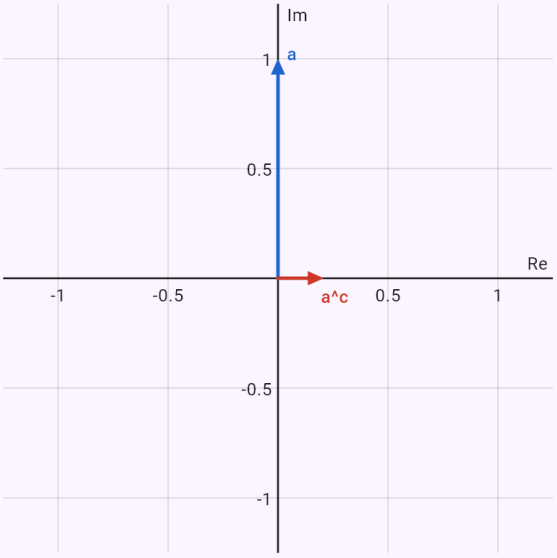
$$i^i = e^{-\frac{\pi}{2}} \approx 0.208$$

Steps

- A power with a complex exponent is defined by the principal logarithm (the argument in $\langle 0; 2\pi)$)

  $a^c = e^{c \operatorname{Log} a}, \quad \operatorname{Log}(i) = \frac{\pi}{2}i$
- We split the exponent into its real and imaginary part: $c \operatorname{Log} a = i\left(\frac{\pi}{2}i\right) = -\frac{\pi}{2}$
- The principal value of the power, $e^{X+iY} = e^X(\cos Y + i \sin Y)$: $i^i = e^{-\frac{\pi}{2}} \approx 0.208$
- The other values of the logarithm give all values of the power: $i^i = e^{-\frac{\pi}{2} - 2k\pi}, \ k \in \mathbb{Z}$

Graph



The complex plane – the given numbers blue, the result red, the argument φ as an arc from the positive real axis.

The equation sin z = 2 – the substitution w = e^(iz) and a quadratic equation.

Input: `sin z = 2`

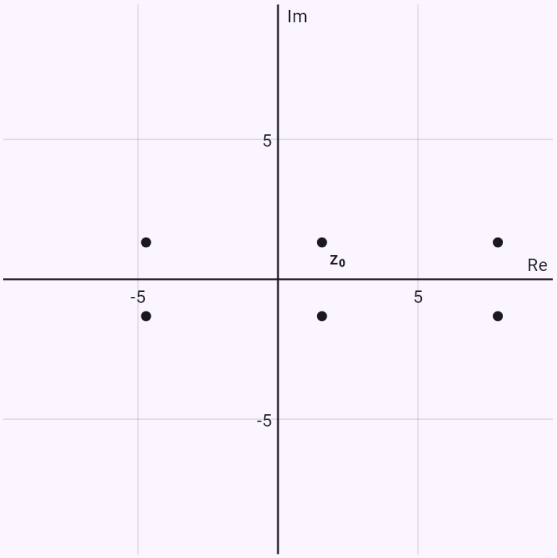
Result

$$z_k = \frac{\pi}{2} + 2k\pi \pm i \ln\left(2+\sqrt{3}\right),\ k \in \mathbb{Z}$$

Steps

- We write the function by exponentials and substitute: $\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad w = e^{iz}$
- Multiplying the equation by w gives a quadratic equation: $w^2 - 4iw - 1 = 0$
- Its roots: $D = -12,\ w_1 = \left(2+\sqrt{3}\right)i,\ w_2 = \left(2-\sqrt{3}\right)i$
- Back to z: $iz = \log w$, so $z = -i \log w = \operatorname{Arg} w + 2k\pi - i \ln|w|$ (the argument in $\langle 0; 2\pi)$): $z_k = \frac{\pi}{2} + 2k\pi \pm i \ln\left(2+\sqrt{3}\right),\ k \in \mathbb{Z}$

Graph



The complex (Gauss) plane – the solutions as points: the real part on the horizontal axis, the imaginary part on the vertical one.

A pole of a lower order thanks to a zero of the numerator, and a simple pole.

Input: `singularities(sin(z)/(z^3 (z - 1)))`

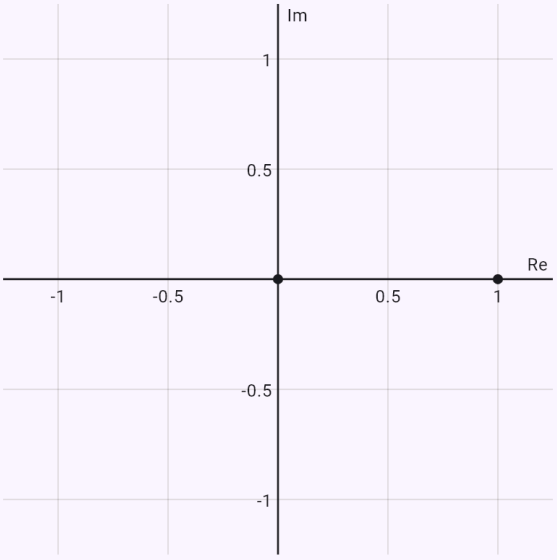
Result

$$\operatorname{Res}_{z=0} f = 0, \quad \operatorname{Res}_{z=1} f = \sin 1$$

Steps

- The singular points – the roots of the denominator: $z^4 - z^3 = 0:\ z_1 = 0,\ z_2 = 1$
- At 0 the denominator has a root of multiplicity 3, the numerator of multiplicity 1 – it is a pole of order 2
- The residue at a pole of order 2 $\operatorname{Res}_{z=0} f = \frac{1}{1!} \lim_{z\to 0} \frac{d}{dz}\left[z^2 f(z)\right], \quad z^2 f(z) = \frac{\sin z}{z(z-1)},$ I
- At 1 the denominator has a root of multiplicity 1, the numerator of multiplicity 0 – it is a pole of order 1
- The residue at a simple pole, f = g/q $\operatorname{Res}_{z=1} f = \frac{g(1)}{q'(1)} = \frac{\sin 1}{1} = \sin 1, \quad g = \sin z,\ q' = 4z^$

Graph



Gauss plane – the singular points of the function.

An essential singularity – the residue from the Laurent series.

Input: `Res(z^2 e^(1/z), z = 0)`

Result

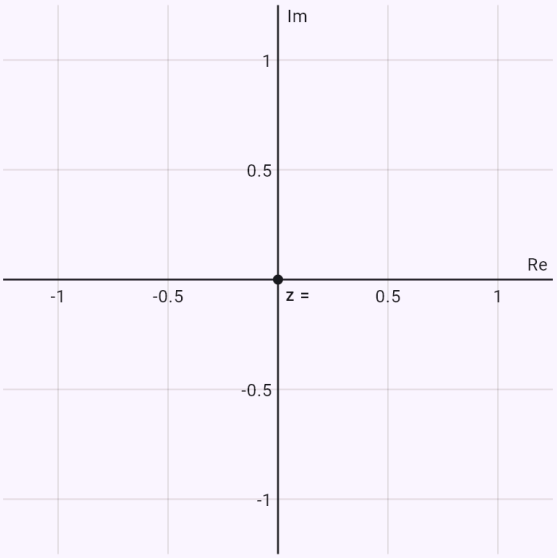
$$\operatorname{Res}_{z=0} f = \frac{1}{6}$$

Steps

- At 0 there is an essential singularity – the exponent (argument) has a pole there and the principal part of the Laurent series is infinite.
- The residue is the coefficient c_{-1} of $\frac{1}{z-a}$ in the Laurent series around 0

$$f(z) = \cdots + \frac{1}{120z^3} + \frac{1}{24z^2} + \frac{1}{6z} + \frac{1}{2} + z + z^2, \quad c_{-1}$$

Graph



Gauss plane – the singular points of the function.

The Laurent series in the annulus 1 < |z| < 2 – one fraction in powers of 1/z, the other in powers of z.

Input: `laurent(1/((z - 1)(z - 2)), 1 < |z| < 2)`

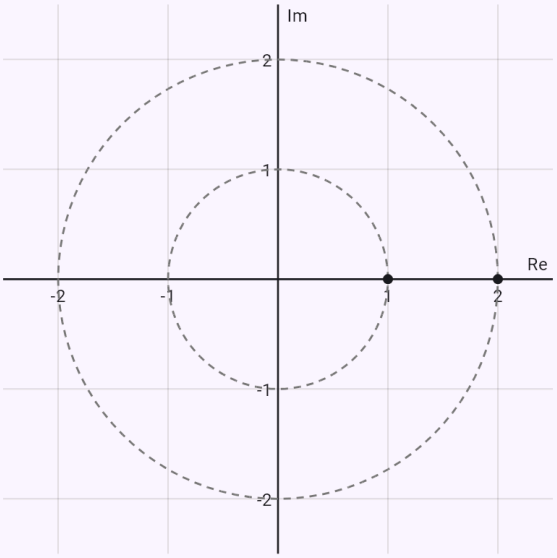
Result

$$f(z) = \cdots - \frac{1}{z^3} - \frac{1}{z^2} - \frac{1}{z} - \frac{1}{2} - \frac{1}{4}z$$

Steps

- Singular points: $z_1 = 1,\ z_2 = 2$. We look for the Laurent series in the annulus $1 < |z| < 2$ – no singular point lies in it.
- We split the function into partial fractions (over $\mathbb{C}$):
  $f(z) = -\frac{1}{z-1} + \frac{1}{z-2}$
- The fraction's pole lies inside the inner circle – we take out $\frac{1}{z-c}$ and use the geometric series in powers of $\frac{1}{z-c}$
  $$-\frac{1}{z-1} = -\frac{1}{z} \cdot \frac{1}{1-\frac{1}{z}} = -\sum_{n=1}^{\infty} \frac{1}{z^n}, \quad \left|\frac{1}{z}\right| < 1$$
- The fraction's pole lies outside the annulus – we take out a constant and use the geometric series in powers of z − c
  $$\frac{1}{z-2} = -\frac{1}{2} \cdot \frac{1}{1-\frac{z}{2}} = -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}, \quad \left|\frac{z}{2}\right| < 1$$
- We add the series term by term
  $f(z) = \cdots - \frac{1}{z^3} - \frac{1}{z^2} - \frac{1}{z} - \frac{1}{2} - \frac{1}{4}z - \frac{1}{8}z^2 - \frac{1}{16}z^3 +$
- The general term: $f(z) = -\sum_{n=1}^{\infty} \frac{1}{z^n} - \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n$

Graph



Gauss plane – the annulus of the Laurent series (dashed) and the singular points.

The residue theorem at a pole of order 2.

Input: `oint(e^z/(z - i)^2, |z - i| = 1)`

Result

$$\oint_{|z-i|=1} \frac{e^z}{(z-i)^2}\,dz = 2\pi i\, e^i$$

Steps

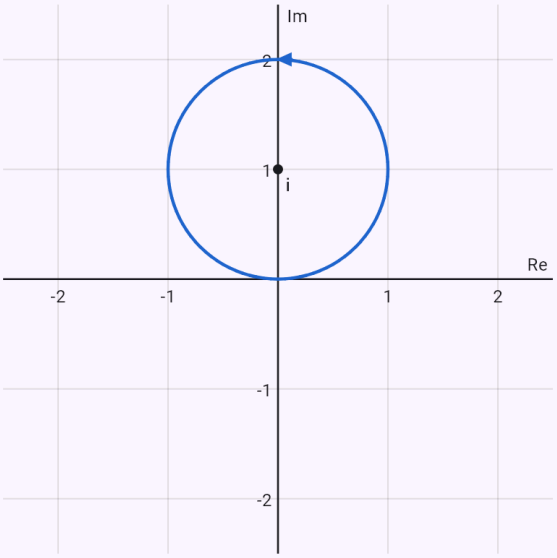
- We integrate over the circle with centre i and radius 1, counterclockwise: $|z-i| = 1$
- Inside the circle lie the singular points $z_0 = i$, outside $\emptyset$
- At i the denominator has a root of multiplicity 2, the numerator of multiplicity 0 – it is a pole of order 2
- The residue at a pole of order 2

$$\operatorname{Res}_{z=i} f = \frac{1}{1!}\lim_{z\to i}\frac{d}{dz}\Big[(z-i)^2 f(z)\Big], \quad (z-i)^2 f(z) = \,$$

- By the residue theorem

$$\oint_C f\,dz = 2\pi i \sum \operatorname{Res} f = 2\pi i\big(e^i\big) = 2\pi i\, e^i$$

Graph



Gauss plane – the circle we integrate over (the arrow shows the direction) and the singular points.

An improper integral with a cosine – Jordan's lemma.

Input: `int(cos(x)/(x^2 + 1), x, -oo, oo)`

Result

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx = \frac{\pi}{e} \approx 1.1557$$

Steps

- With a cosine or sine, $\deg Q \geq \deg P + 1$ and Q without real roots is enough – the integral over the half-circle then tends to zero by Jordan's lemma:

$$R(x) = \frac{1}{x^2+1},\ \deg Q = 2,\ \deg P = 0$$

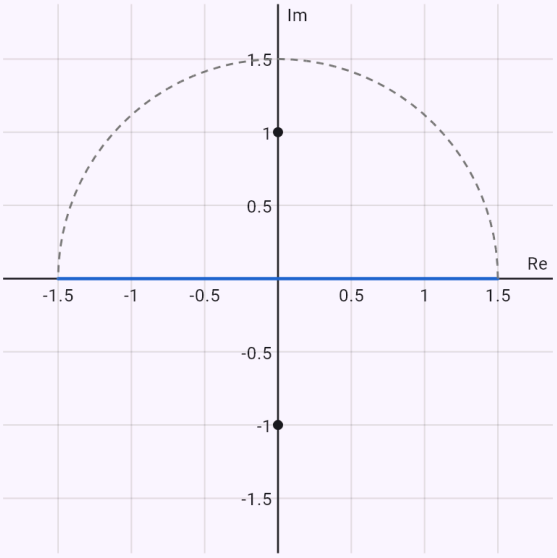
- The cosine is the real part of e^{iax}
- We close the real axis by a large half-circle in the upper half-plane. Inside lie the poles $z_0 = i$, the integral over the half-circle tends to zero.
- At i the denominator has a root of multiplicity 1, the numerator of multiplicity 0 – it is a pole of order 1
- The residue at a simple pole, f = g/q

$$\operatorname{Res}_{z=i} f = \frac{g(i)}{q'(i)} = \frac{\frac{1}{e}}{2i} = \frac{-\frac{1}{2}i}{e}, \quad g = e^{iz},\ q' = 2z$$

- By the residue theorem the integral is 2πi times the sum of the residues in the upper half-plane:

$$2\pi i\left(\frac{-\frac{1}{2}i}{e}\right) \Rightarrow \frac{\pi}{e}$$

Graph



Gauss plane – the real axis closed by a half-circle in the upper half-plane (dashed) and the poles of the integrand.

An integral over ⟨0; 2π⟩ as an integral over the unit circle.

Input: `int(1/(2 + cos(t)), t, 0, 2pi)`

Result

$$\int_0^{2\pi} \frac{1}{\cos t + 2}\, dt = \frac{2\sqrt{3}\pi}{3} \approx 3.6276$$

Steps

- We turn the integral over ⟨0; 2π⟩ into an integral over the unit circle by the substitution

  $z = e^{it},\ \cos t = \frac{z + z^{-1}}{2},\ \sin t = \frac{z - z^{-1}}{2i},\ dt = \frac{dz}{iz}$

- After substituting and simplifying we get an integral of a rational function over |z| = 1: $\oint_{|z|=1} \frac{-2i}{z^2 + 4z + 1}\, dz$

- Inside the circle lie the singular points $z_0 = -2 + \sqrt{3}$, outside $z_0 = -2 - \sqrt{3}$

- At $-2 + \sqrt{3}$ the denominator has a root of multiplicity 1, the numerator of multiplicity 0 – it is a pole of order 1

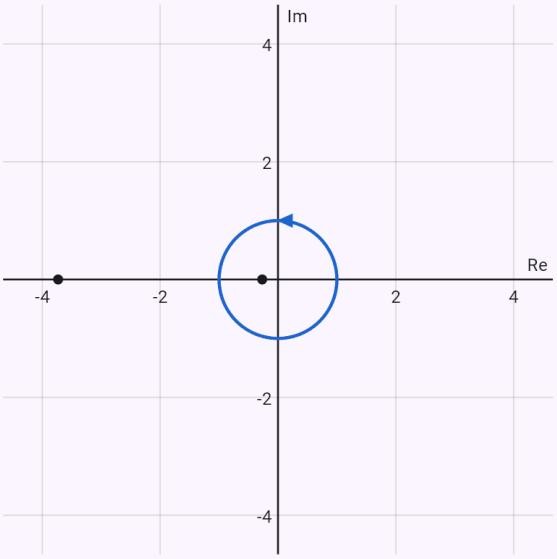
- The residue at a simple pole, f = g/q

  $\operatorname{Res}_{z=-2+\sqrt{3}} f = \frac{g(-2+\sqrt{3})}{q'(-2+\sqrt{3})} = \frac{-2i}{2\sqrt{3}} = \frac{-i\sqrt{3}}{3}, \quad g =$

- By the residue theorem

  $\oint_C f\, dz = 2\pi i \sum \operatorname{Res} f = 2\pi i \left(\frac{-i\sqrt{3}}{3}\right) = \frac{2\sqrt{3}\pi}{3}$

Graph



Gauss plane – the unit circle z = e^(it) and the poles of the integrand.

Σ 1/n² = π²/6 – the residue at zero, where the kernel has a pole too.

Input: `residues(sum(1/n^2, n, 1, oo))` By residues

Result

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.6449$$

Computing the integral or sum

The usual way | ✓ By residues

Steps

- The function π cot πz has a simple pole with residue 1 at every whole number n. The integral over a large square tends to zero, so the sum of the series is minus the sum of the residues at the poles of R

$$\sum_{n=-\infty}^{\infty} R(n) = -\sum \operatorname{Res}\left[\pi \cot \pi z \cdot R(z)\right], \quad R(z) = \frac{1}{z^2}$$

- The condition $\deg Q \geq \deg P + 2$ holds:
$\deg Q = 2,\ \deg P = 0$
- The singular points – the roots of the denominator:
$z^2 = 0:\ z_0 = 0$
- At 0 both R and the kernel have a pole – the residue is the coefficient of 1/z of their product

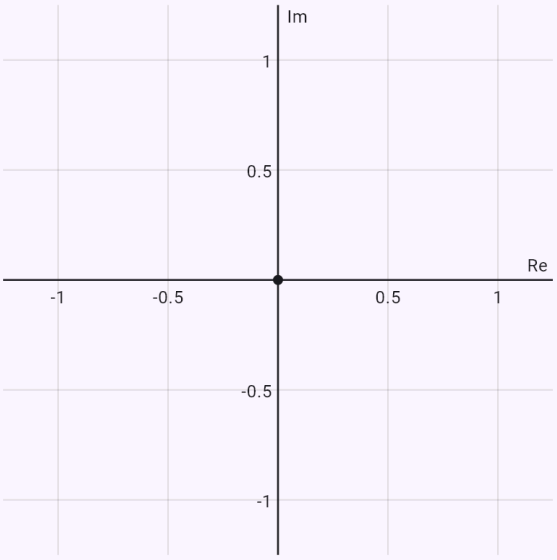
$$\pi \cot \pi z \cdot R(z) = \frac{1}{z^3} - \frac{\pi^2}{3z} - \frac{\pi^4 z}{45} + \cdots, \quad \operatorname{Res}_{z=0} = -$$

- The sum over all whole numbers:

$$\sum_{n \neq 0} R(n) = -\left(-\frac{\pi^2}{3}\right) = \frac{\pi^2}{3}$$

- R is even, R(−n) = R(n), so the sum from 1 is half of it (without the term n = 0): $\sum_{n=1}^{\infty} R(n) = \frac{1}{2} \sum_{n \neq 0} = \frac{\pi^2}{6}$

Graph



Gauss plane – the poles of R(z); the kernel π cot πz also has poles at all whole numbers.

Rouché's theorem – four roots in the annulus 1 < |z| < 2.

Input: `rouche(z^5 + 3z + 1, 1 < |z| < 2)`

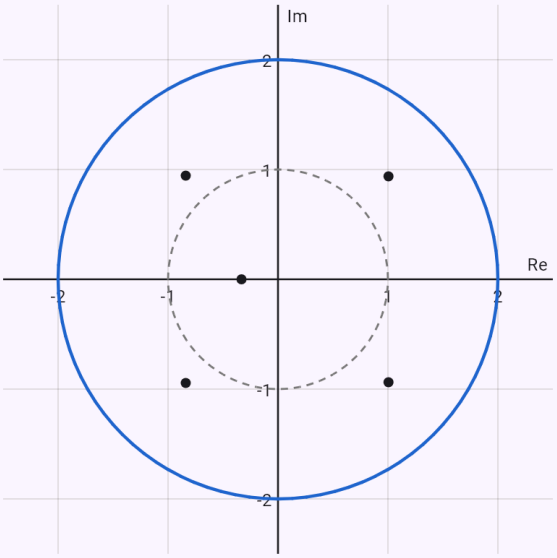
Result

$$N(1 < |z| < 2) = 4$$

Steps

- On the circle |z| = 2 the term z^5 dominates the sum of the others – by Rouché's theorem the polynomial has as many roots in the disc as z^5, that is 5:
  $|1| = 1,\ |3z| = 6,\ |z^5| = 32,\quad 32 > 7$
- On the circle |z| = 1 the term $3z$ dominates the sum of the others – by Rouché's theorem the polynomial has as many roots in the disc as $3z$, that is 1:
  $|1| = 1,\ |3z| = 3,\ |z^5| = 1,\quad 3 > 2$
- In the annulus it is the difference of the counts in the larger and the smaller disc: $5 - 1 = 4$
- The roots numerically, as a check:
  $-0.839 - 0.944i,\ -0.839 + 0.944i,\ -0.332,\ 1.005 + 0.937i,\ 1.005 - 0.937i$

Graph



Gauss plane – the circles and the roots of the polynomial (computed numerically).

A Möbius transformation – the upper half-plane onto the unit disc.

Input: `w = (z - i)/(z + i)`

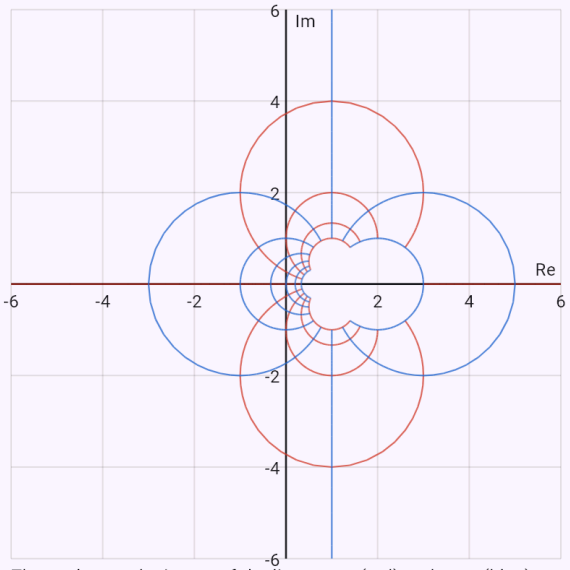
Result

$$w = \frac{z - i}{z + i}$$

Steps

- We write the map by its components
  $w = u + iv,\quad u = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 2y + 1},\ v = -\frac{2x}{x^2 + y^2 +}$
- The map is conformal everywhere except at the pole $z_0 = -i$: $f'(z) = \frac{2i}{z^2 + 2iz - 1}$
- It is a Möbius transformation $w = \frac{\alpha z + \beta}{\gamma z + \delta}$
  $\alpha = 1,\ \beta = -i,\ \gamma = 1,\ \delta = i,\quad \alpha\delta - \beta\gamma = 2i \neq 0$
- The fixed points solve the equation f(z) = z
  $z^2 + (-1 + i)z + i = 0:\ z_1 = \frac{1 - \sqrt{3}}{2} + \frac{-1 + \sqrt{3}}{2}i,\ z$
- The real axis is mapped onto $|w| = 1$
- The imaginary axis is mapped onto $\mathrm{Im}\, w = 0$
- The unit circle is mapped onto $\mathrm{Re}\, w = 0$
- The graph shows the image of a grid – the lines x = c (red) and y = c (blue) for |c| ≤ 2 in steps of 0.5

Graph



The w-plane – the image of the lines x = c (red) and y = c (blue) under w = f(z).

The inverse Laplace transform as a sum of residues.

Input: `invlaplace(s/((s + 1)(s^2 + 4)))`

Result

$$f(t) = -\frac{e^{-t}}{5} + \frac{\cos(2t)}{5} + \frac{2\sin(2t)}{5}$$

Steps

- The inverse Laplace transform of a rational function is the sum of the residues of $F(s)\,e^{st}$ at all poles of F

$$f(t) = \sum \mathrm{Res}\left[F(s)\,e^{st}\right], \quad F(s) = \frac{s}{s^3 + s^2 + 4s + 4}$$

- The singular points – the roots of the denominator: $s^3 + s^2 + 4s + 4 = 0:\ s_1 = -1,\ s_2 = -2i,\ s_3 = 2i$
- At -1 the denominator has a root of multiplicity 1, the numerator of multiplicity 0 – it is a pole of order 1
- The residue at a simple pole, f = g/q

$$\mathrm{Res}_{s=-1} f = \frac{g(-1)}{q'(-1)} = \frac{\sinh t - \cosh t}{5} = -\frac{e^{-t}}{5}, \quad g =$$

- At $-2i$ the denominator has a root of multiplicity 1, the numerator of multiplicity 0 – it is a pole of order 1
- The residue at a simple pole, f = g/q

$$\mathrm{Res}_{s=-2i} f = \frac{g(-2i)}{q'(-2i)} = \frac{-2ie^{-2it}}{-8 - 4i} = \left(\frac{1}{10} + \frac{1}{5}i\right)e^{-2it},$$

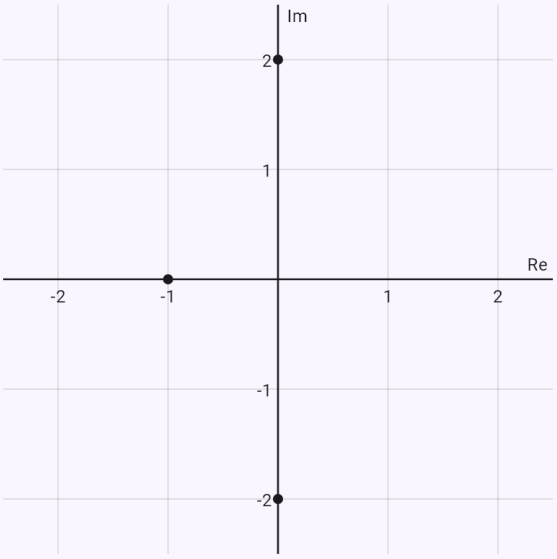
- At $2i$ the denominator has a root of multiplicity 1, the numerator of multiplicity 0 – it is a pole of order 1
- The residue at a simple pole, f = g/q

$$\mathrm{Res}_{s=2i} f = \frac{g(2i)}{q'(2i)} = \frac{2ie^{2it}}{-8 + 4i} = \left(\frac{1}{10} - \frac{1}{5}i\right)e^{2it}, \quad g =$$

- We add the residues and write the result in real form:

$$f(t) = -\frac{e^{-t}}{5} + \frac{\cos(2t)}{5} + \frac{2\sin(2t)}{5}$$

Graph



Gauss plane – the poles of F(s).

The Fourier transform of the Lorentz function: π·e^(−|ω|).

Input: `ftransform(1/(1 + t^2))`

Result

$$F(\omega) = \pi e^{-|\omega|}$$

Steps

- The Fourier transform of a rational function: we close the integral by a large semicircle (Jordan's lemma) and add the residues

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt, \quad f(t) = \frac{1}{t^2+1}$$

- The singular points – the roots of the denominator: $t^2 + 1 = 0:\ t_1 = -i,\ t_2 = i$
- At $-i$ the denominator has a root of multiplicity 1, the numerator of multiplicity 0 – it is a pole of order 1
- The residue at a simple pole, f = g/q

$$\mathrm{Res}_{t=-i} f = \frac{g(-i)}{q'(-i)} = \frac{e^{-w}}{-2i} = \frac{1}{2} i e^{-w}, \quad g = e^{-itw},\ q' =$$

- At i the denominator has a root of multiplicity 1, the numerator of multiplicity 0 – it is a pole of order 1
- The residue at a simple pole, f = g/q

$$\mathrm{Res}_{t=i} f = \frac{g(i)}{q'(i)} = \frac{e^{w}}{2i} = -\frac{1}{2} i e^{w}, \quad g = e^{-itw},\ q' = 2t$$

- For ω > 0 we close the path in the lower half-plane (negative direction, −2πi)

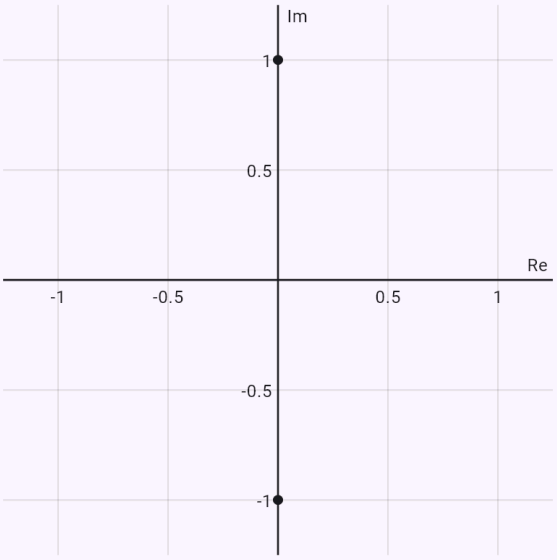
$$\omega > 0: \quad F(\omega) = -2\pi i \sum_{\mathrm{Im}\, t_k < 0} \mathrm{Res} = \pi e^{-\omega}$$

- For ω < 0 we close the path in the upper half-plane (+2πi)

$$\omega < 0: \quad F(\omega) = 2\pi i \sum_{\mathrm{Im}\, t_k > 0} \mathrm{Res} = \pi e^{\omega}$$

- The result - the Fourier transform: $F(\omega) = \pi e^{-|\omega|}$

Graph



Gauss plane – the poles of F(s).

The Cauchy distribution's characteristic function e^(−|t|).

Input: `ftransform(1/(pi (1 + x^2)))`

Result

$$F(\omega) = e^{-|\omega|}$$

Steps

- The Fourier transform of a rational function: we close the integral by a large semicircle (Jordan's lemma) and add the residues

$$F(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx, \quad f(x) = \frac{1}{\pi x^2 + \pi}$$

- The singular points – the roots of the denominator: $\pi x^2 + \pi = 0:\ x_1 = -i,\ x_2 = i$
- At $-i$ the denominator has a root of multiplicity 1, the numerator of multiplicity 0 – it is a pole of order 1
- The residue at a simple pole, f = g/q

$$\mathrm{Res}_{x=-i} f = \frac{g(-i)}{q'(-i)} = \frac{e^{-w}}{-2\pi i} = \frac{\frac{1}{2} i e^{-w}}{\pi}, \quad g = e^{-iwx},\ q$$

- At i the denominator has a root of multiplicity 1, the numerator of multiplicity 0 – it is a pole of order 1
- The residue at a simple pole, f = g/q

$$\mathrm{Res}_{x=i} f = \frac{g(i)}{q'(i)} = \frac{e^{w}}{2\pi i} = \frac{-\frac{1}{2} i e^{w}}{\pi}, \quad g = e^{-iwx},\ q' = 2$$

- For ω > 0 we close the path in the lower half-plane (negative direction, −2πi)

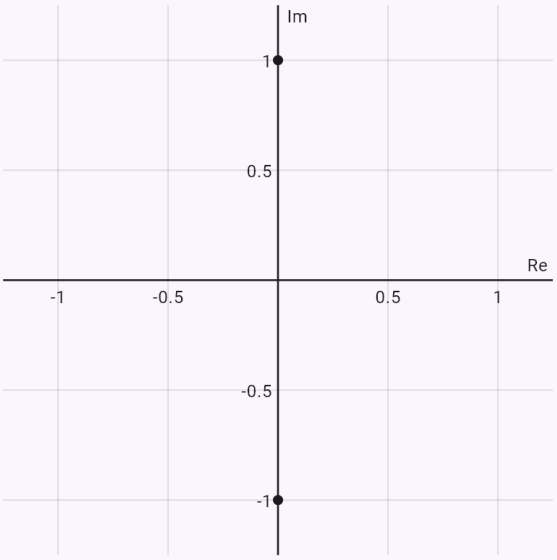
$$\omega > 0: \quad F(\omega) = -2\pi i \sum_{\mathrm{Im}\, x_k < 0} \mathrm{Res} = e^{-\omega}$$

- For ω < 0 we close the path in the upper half-plane (+2πi)

$$\omega < 0: \quad F(\omega) = 2\pi i \sum_{\mathrm{Im}\, x_k > 0} \mathrm{Res} = e^{\omega}$$

- The result - the Fourier transform: $F(\omega) = e^{-|\omega|}$

Graph



Gauss plane – the poles of F(s).

The inverse Z-transform with two poles: x[n] = 2 − 2^{-n}.

Input: `invz(z^2/((z - 1)(z - 1/2)))`

Result

$$x[n] = 2 - 2^{-n}, \quad n \geq 0$$

Steps

- The inverse Z-transform is the sum of the residues of $F(z)z^{n-1}$ at the poles of F

$$x[n] = \sum \text{Res}\left[F(z)\, z^{n-1}\right], \quad F(z) = \frac{2z^2}{2z^2 - 3z + 1}$$

- The singular points – the roots of the denominator: $2z^2 - 3z + 1 = 0 : \; z_1 = \frac{1}{2}, \; z_2 = 1$
- At $\frac{1}{2}$ the denominator has a root of multiplicity 1, the numerator of multiplicity 0 – it is a pole of order 1
- The residue at a simple pole, f = g/q

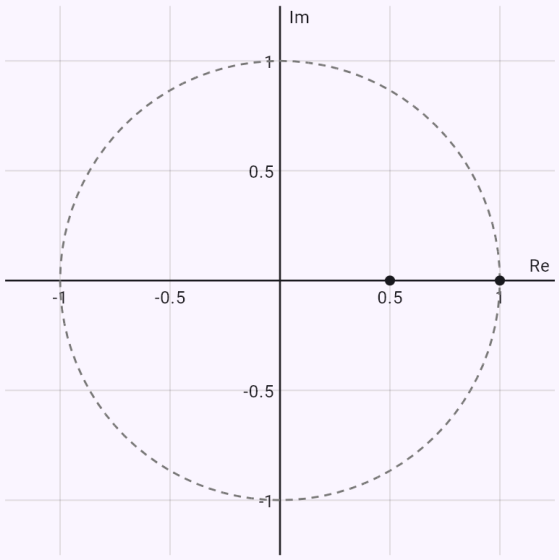
$$\text{Res}_{z=\frac{1}{2}} f = \frac{g\left(\frac{1}{2}\right)}{q'\left(\frac{1}{2}\right)} = \frac{2^{-n}}{-1} = -\frac{1}{2^n}, \quad g = 2z^{n+1}, \; q' = 4$$

- At 1 the denominator has a root of multiplicity 1, the numerator of multiplicity 0 – it is a pole of order 1
- The residue at a simple pole, f = g/q

$$\text{Res}_{z=1} f = \frac{g(1)}{q'(1)} = \frac{2}{1} = 2, \quad g = 2z^{n+1}, \; q' = 4z - 3$$

- The initial value is the limit for z → ∞: $x[0] = \lim_{z \to \infty} F(z) = 1$
- We add the residues: $x[n] = 2 - 2^{-n}, \quad n \geq 0$

Graph



Gauss plane – the poles of F(s).

Known limitations

- Singularities are found for functions with a polynomial in the denominator (and the essential singularities of e^h, sin h, cos h); a denominator like sin z (infinitely many poles) is not supported yet.
- The Laurent series in an annulus works for rational functions, at an essential singularity only around the point when the function has no other singularity.
- Integrals over other curves than a circle (a square, an ellipse), the principal value of an integral with a pole on the axis and the general argument principle are not supported yet.
- A sum of a series from 1 by residues needs an even rational function R; poles of R at whole numbers other than zero are not allowed.
- A conformal map shows the image of a grid; the image of a given region (a half-plane, a strip) only for the axes and the unit circle of a Möbius transformation.
- The Fourier transform works for rational functions without poles on the real axis, the inverse Z-transform for rational functions without a pole at zero.

36. Differential equations

The app solves first-order ordinary differential equations: separable, linear, exact, homogeneous ($y' = f(y/x)$), Bernoulli equations and equations $y' = f(ax + by + c)$. It recognizes the type itself and shows the steps as taught at school – separating the variables, variation of the constant or an integrating factor, a substitution, the potential of an exact equation. The integrals in the steps have steps of their own. It checks the solution: the solution and its derivative are substituted into the equation and both sides edited to the same form.

Higher orders: linear equations with constant coefficients of any order by the characteristic equation (multiple and complex roots too), a particular solution by the special right side (resonance and a sum of right sides too) or by variation of constants, an initial value problem at zero by the Laplace transform too (the "Method" chip). Further Euler's equation by $x = e^t$, $y^{(n)} = f(x)$ by repeated integration and the reduction of order – when y is missing ($p = y'$) or x is ($y' = p(y)$).

The equation $y' = y\cdot(a(x)\cdot\ln y + b(x))$ becomes linear by the substitution $z = \ln y$. A linear second-order equation with variable coefficients is solved when one solution can be guessed (x^k, e^{cx}, …): the second one by reducing the order (d'Alembert), the right side by variation of constants. The order is reduced by more than one too ($p = y^{(m)}$ with a second-order equation for p). With parameters the cases are told apart – a damped oscillator by the sign of $\delta^2 - \omega^2$ (overdamped, critical, underdamped), a resonance ($y'' + \omega^2 y = \cos 2t$ for $\omega = 2$) and the sign of a constant the integral depends on ($y'' = 2yy'$). The Laplace transform also solves an initial value problem at a point other than 0 (by the shift $\tau = t - t_0$) and with parameters.

A Riccati equation $y' = P(x)\cdot y^2 + Q(x)\cdot y + R(x)$ is solved when one particular solution y_1 is known: given on the next line (y1 = 1/x), or guessed of a simple form (a constant, ax, a/x, ax^2, ae^{±x}, x + a). The substitution $y = y_1 + 1/z$ then makes it linear. Equations not linear in y′: Clairaut's $y = x\cdot y' + \psi(y')$ – the lines $y = Cx + \psi(C)$ and the singular solution (their envelope) – and Lagrange's $y = x\cdot\varphi(y') + \psi(y')$ with a parametric solution (the parameter $p = y'$).

The result is the general solution with a constant C, for higher orders with constants C_1, C_2, … (explicitly, or implicitly where it can't be explicit); with conditions, the solution that satisfies them – initial and boundary ones, including "no solution" and "infinitely many solutions". Constant solutions lost by dividing are found too. Every solution is checked by substituting it into the equation. For the first order the graph shows the direction field and solutions for different C, for higher orders solutions for different constants; for an autonomous second-order equation (no x) a switch shows the phase plane (y, y′) too. Under the graph you can enter your own values of the constants and add their curve; the “Default curves” switch hides the ones the app chose. Tapping the graph draws the solution through that point – for a linear second-order equation you also set the tangent’s slope at the point with a slider.

A system of two linear equations with constant coefficients (x′ = 2x + y, y′ = x + 2y, dx/dt too) by the matrix's eigenvalues and eigenvectors – real, complex and double – with the initial conditions x(0), y(0). The stationary point [0; 0] is classified (node, saddle, focus, center), the phase plane shown.

The oscillation of a second-order linear equation (every solution has infinitely many zeros) is studied by oscillation(y" + x y = 0), also for an equation in the self-adjoint form (x y')' + y/x = 0 or with y′ (the app brings it to this form). The steps: exactly for a constant q and Euler's equation y″ + (k/x²)y = 0 with the boundary k = 1/4, the Leighton–Wintner and Kneser criteria (in the critical case also the comparison with the Riemann–Weber equation), the Riccati substitution w = r y′/y, the Prüfer transformation and the numerical count of zeros on an interval (give it after a semicolon: ; ⟨1; 50⟩) with Sturm's estimate of their distance. A half-linear equation (|y'| y')' + q |y| y = 0 (in general the exponent |y'|^(p−2)) is judged by the critical constant ((p − 1)/p)^p, the discrete counterpart Δ^2 y_n + q_n y_{n+1} = 0 (also Δ(r_n Δy_n)) by its generalized zeros. The graph shows the solution with its zeros and Prüfer's angle φ.

The asymptotics of solutions of a nonlinear equation with the p-Laplacian (a(x)|y′|^α sgn y′)′ = b(x)|y|^β sgn y with positive a, b is studied by the command asymptotics((e^(-3x) (y')^3)' = x^(-2) e^(-3x) y^3) (also asymptotika(...) in Czech); |y'|^k y' and |y|^k y can also be written with sgn, as an odd power (y')³, y³, and for α = 1 as (a y')' or y". The solutions are divided into the classes M^+ (|y| grows) and M^- (|y| decreases) by the limits of y and of the quasiderivative y^[1] = a|y′|^α sgn y′. For a half-linear equation (β = α) the app computes the integrals J_1, J_2, Y_1, Y_2 (by Symja, otherwise by comparing the power and logarithmic behaviour), determines one of the cases C1–C8 and the nonempty classes by Theorem 3.1 of Z. Došlá and M. Marini, Asymptotic Properties of Solutions to Half-Linear Differential Equation with Negative Potential (QUALITDE 2020), and shows the reciprocal equation. For an Emden–Fowler type equation with β ≠ α it uses the theorems of M. Cecchi, Z. Došlá, M. Marini, On the dynamics of the generalized Emden–Fowler equation (Georgian Mathematical Journal 7, 2000): the superlinear and sublinear case, strongly and weakly increasing and regularly decaying solutions. The graph shows numerical solutions of the nonempty classes (M^- by shooting) with their quasiderivatives below. Give the interval's start after a semicolon (; x ≥ 1). For an equation with negative q, oscillation offers the chip Asymptotics of solutions, and asymptotics the chip Oscillation of the equation.

How to enter it

`y' = x*y`	The derivative with a prime.
`dy/dx = x y^2`	Leibniz's notation – it names the variable too (dm/dt, dT/dt).
`\frac{dy}{dx} = \frac{y}{x} + x`	Notation from a photo.
`(2xy + 6x)dx + (x^2 + 4y^3)dy = 0`	The form M dx + N dy = 0 (an exact equation).

`y' = y^2 + 1, y(0) = 1`	An initial condition after a comma or on the next line.
`dm/dt = m + 2, m(0) = 0`	The function and the variable can be any letters.
`y' = y^2 - 2/x^2; y1 = 1/x`	A known particular solution of a Riccati equation on the next line (or after a semicolon).
`y'' - 3y' + 2y = 3e^(2x)`	A higher-order equation – the derivatives with more primes.
`y^{(4)} + 2y'' + y = 0`	Higher derivatives as y^{(4)} (from a photo too).
`y'' + y = 0, y(0) = 1, y'(0) = 2`	Initial conditions y(0), y′(0) – or boundary ones at different points.
`\ddot{x} + \omega^2 x = 0`	Newton's dots (derivatives by time t) and Greek parameters.
`x' = 2x + y, y' = x + 2y`	A system of two linear equations – after a comma or on lines, with the conditions x(0) = …, y(0) = ….
`oscillation(y'' + q(x) y = 0; ⟨a; b⟩)`	Oscillation of a second-order equation, optionally with an interval for the zeros.
`oscillation((|y'| y')' + q(x) |y| y = 0)`	A half-linear equation (here p = 3).
`oscillation(Δ^2 y_n + q_n y_{n+1} = 0)`	The discrete counterpart – a second-order linear recurrence.
`asymptotics((a(x) |y'|^(α-1) y')' = b(x) |y|^(β-1) y; x ≥ x₀)`	Asymptotics of solutions of an equation with the p-Laplacian, optionally with the interval's start.

Examples

A separable equation – the general solution and the direction field.

Input: `y' = x*y`

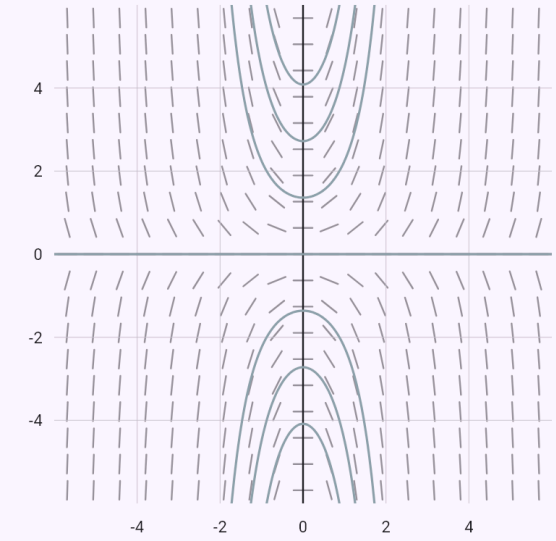
Result

$$y = Ce^{\frac{x^2}{2}}, \quad C \in \mathbb{R}$$

Steps

- A separable equation – the right side is a product of a function of the variable and a function of the unknown: $y' = xy$
- Separating the variables (dividing, assuming $y \neq 0$): $\frac{dy}{y} = x\,dx$
- Integrating both sides: $\int \frac{dy}{y} = \int x\,dx$
- From the table of basic integrals: $\int \frac{1}{y}\,dy = \ln|y|$
- From the table of basic integrals: $\int x\,dx = \frac{x^2}{2}$
- The general solution in implicit form: $\ln|y| = \frac{x^2}{2} + C$
- Exponentiating – the constant ±e^C is called C again: $y = Ce^{\frac{x^2}{2}}$
- The constant solution $y = 0$, excluded by dividing, is in the general solution for C = 0
- Check – the solution's derivative $y' = Cxe^{\frac{x^2}{2}}$ substituted into the equation gives an identity.
- The term differentiated by the rules for a composite function, a product and a quotient $\left(Ce^{\frac{x^2}{2}}\right)' = C \cdot e^{\frac{x^2}{2}} \cdot x = Cxe^{\frac{x^2}{2}}$
- The solution and its derivative substituted into the equation – both sides are equal: $Cxe^{\frac{x^2}{2}} = xCe^{\frac{x^2}{2}}$

Graph



solutions for different C

The short segments show the direction the equation prescribes at each point (the direction field). The grey curves are solutions for different C – they touch the segments everywhere.

Default curves

Your own values of the constants

C

+ Add curve

Tap the graph – the solution through that point is drawn.

The initial condition determines the constant C.

Input: `y' = y^2 + 1, y(0) = 1`

Result

$$y = \tan\left(x + \frac{\pi}{4}\right)$$

Solve by a numerical method:

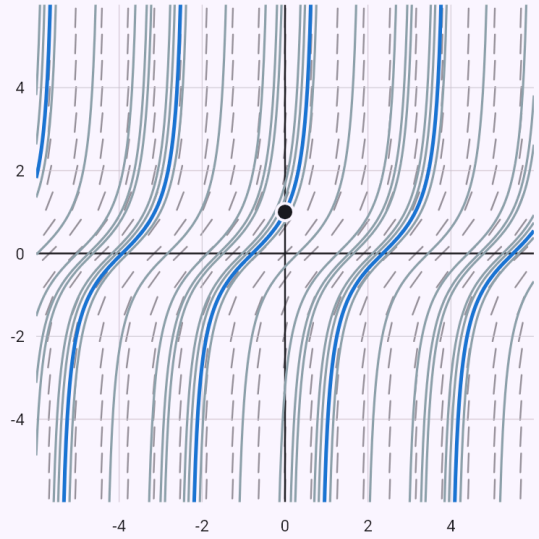
Σ Runge–Kutta 4 | Σ Heun | Σ Euler

Steps

- A separable equation – the right side is a product of a function of the variable and a function of the unknown: $y' = y^2 + 1$
- Separating the variables (dividing, assuming $y^2 + 1 \neq 0$): $\frac{dy}{y^2 + 1} = dx$
- Integrating both sides: $\int \frac{dy}{y^2 + 1} = \int dx$
- From the table of basic integrals: $\int \frac{1}{y^2 + 1}\, dy = \arctan y$
- From the table of basic integrals: $\int 1\, dx = x$
- The general solution in implicit form: $\arctan y = x + C$
- Expressing the solution: $y = \tan(x + C)$
- Substituting the initial condition $y(0) = 1$: $1 = \tan C \Rightarrow C = \frac{\pi}{4}$
- The solution of the initial condition (the particular solution): $y = \tan\left(x + \frac{\pi}{4}\right)$
- Check – the solution's derivative $y' = \frac{1}{\cos^2\left(x + \frac{\pi}{4}\right)}$ substituted into the equation gives an identity.
- The term differentiated by the rules for a composite function, a product and a quotient $\left(\tan\left(x + \frac{\pi}{4}\right)\right)' = \frac{1}{\cos^2\left(x + \frac{\pi}{4}\right)}$
- The solution and its derivative substituted into the equation and simplified with the identity $1 + \tan^2\left(x + \frac{\pi}{4}\right) = \frac{1}{\cos^2\left(x + \frac{\pi}{4}\right)}$ – both sides are equal

$$\frac{1}{\cos^2\left(x + \frac{\pi}{4}\right)} = \tan^2\left(x + \frac{\pi}{4}\right) + 1 = \left(\tan\left(x + \frac{\pi}{4}\right)\right)^2$$

Graph



solutions for different C

$y = \tan\left(x + \frac{\pi}{4}\right)$

The short segments show the direction the equation prescribes at each point (the direction field). The grey curves are solutions for different C – they touch the segments everywhere. Highlighted is the solution of the initial condition; the point is the initial condition.

Default curves

Your own values of the constants

C | + Add curve

Tap the graph – the solution through that point is drawn.

A linear equation by variation of the constant.

Input: `y' + 2y = e^x` Variation of the constant

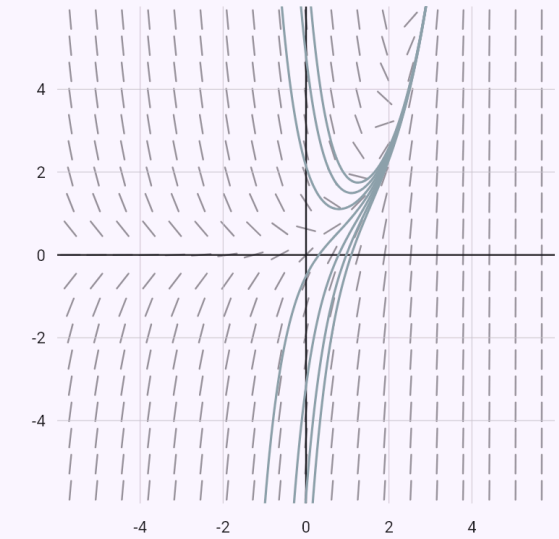
Result

$$y = Ce^{-2x} + \frac{e^x}{3}, \quad C \in \mathbb{R}$$

Steps

- A first-order linear equation y' + p(x)·y = q(x):
  $y' + 2y = e^x$
- First the homogeneous equation (zero right side):
  $y' + 2y = 0$
- From the table of basic integrals: $\int 2\,dx = 2x$
- The homogeneous equation's solution: $y_h = Ce^{-2x}$
- Variation of the constant – we look for the solution as $y = C(x)\,e^{-2x}$; substituted into the equation:
  $C'(x)\,e^{-2x} = e^x$
- Hence $C'(x) = e^{3x}$
- From the table – the inner function is linear, so the result is divided by its coefficient of the variable:
  $\int e^{3x}\,dx = \frac{e^{3x}}{3}$
- Integrating: $C(x) = \int e^{3x}\,dx = \frac{e^{3x}}{3} + C$
- The general solution: $y = Ce^{-2x} + \frac{e^x}{3}$
- Check – the solution's derivative $y' = \frac{e^x}{3} - 2Ce^{-2x}$ substituted into the equation gives an identity.
- The term differentiated by the rules for a composite function, a product and a quotient:
  $\left(Ce^{-2x}\right)' = C \cdot e^{-2x} \cdot (-2) = -2Ce^{-2x}$
- The solution and its derivative substituted into the equation – both sides are equal:
  $\left(\frac{e^x}{3} - 2Ce^{-2x}\right) + 2\left(Ce^{-2x} + \frac{e^x}{3}\right) = e^x$

Graph



solutions for different C

The short segments show the direction the equation prescribes at each point (the direction field). The grey curves are solutions for different C – they touch the segments everywhere.

Default curves

Your own values of the constants

C

+ Add curve

Tap the graph – the solution through that point is drawn.

The same equation by an integrating factor.

Input: `y' + 2y = e^x` Integrating factor

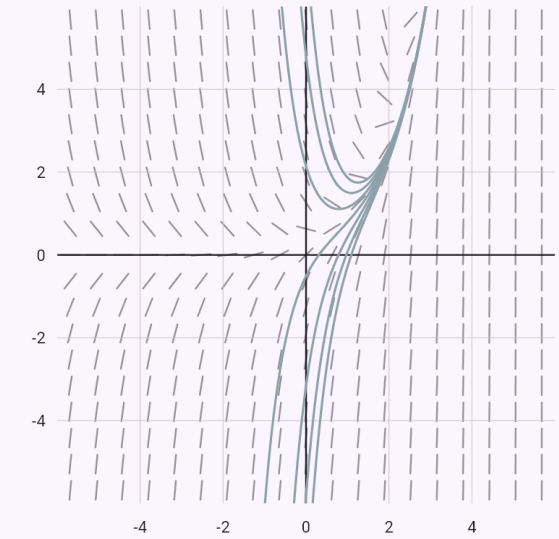
Result

$$y = Ce^{-2x} + \frac{e^x}{3}, \quad C \in \mathbb{R}$$

Steps

- A first-order linear equation y' + p(x)·y = q(x):
  $y' + 2y = e^x$
- From the table of basic integrals: $\int 2\,dx = 2x$
- The integrating factor: $\mu(x) = e^{\int 2\,dx} = e^{2x}$
- Multiplying by the integrating factor – the left side is the derivative of a product: $\left(ye^{2x}\right)' = e^{3x}$
- From the table – the inner function is linear, so the result is divided by its coefficient of the variable:
  $\int e^{3x}\,dx = \frac{e^{3x}}{3}$
- Integrating both sides: $ye^{2x} = \frac{e^{3x}}{3} + C$
- The general solution: $y = Ce^{-2x} + \frac{e^x}{3}$
- Check – the solution's derivative $y' = \frac{e^x}{3} - 2Ce^{-2x}$ substituted into the equation gives an identity.
- The term differentiated by the rules for a composite function, a product and a quotient:
  $\left(Ce^{-2x}\right)' = C \cdot e^{-2x} \cdot (-2) = -2Ce^{-2x}$
- The solution and its derivative substituted into the equation – both sides are equal:
  $\left(\frac{e^x}{3} - 2Ce^{-2x}\right) + 2\left(Ce^{-2x} + \frac{e^x}{3}\right) = e^x$

Graph



solutions for different C

The short segments show the direction the equation prescribes at each point (the direction field). The grey curves are solutions for different C – they touch the segments everywhere.

Default curves

Your own values of the constants

C

+ Add curve

Tap the graph – the solution through that point is drawn.

A homogeneous equation – the substitution u = y/x.

Input: `y' = (x - y)/(x + y)`

Result

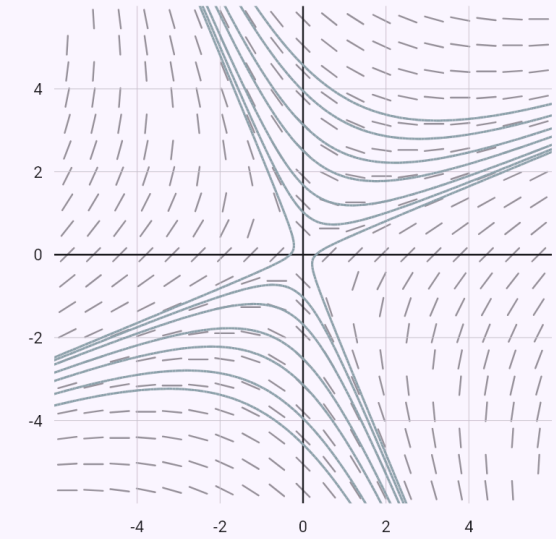
$$x^2 - 2xy - y^2 = C, \quad C \in \mathbb{R}$$

Steps

- A homogeneous equation – the right side depends only on the ratio y/x: $y' = \dfrac{x-y}{x+y}$
- The substitution $u = \dfrac{y}{x},\ y = ux,\ y' = u + xu'$ turns the equation into $xu' = -u - 1 + \dfrac{2}{u+1}$
- Separating the variables (dividing, assuming $u^2 + 2u - 1 \neq 0$): $\dfrac{u+1}{u^2+2u-1}\,du = -\dfrac{1}{x}\,dx$
- Integrating both sides: $\displaystyle\int \frac{u+1}{u^2+2u-1}\,du = \int -\frac{1}{x}\,dx$
- The constant taken out of the integral: $\displaystyle\int -\frac{u+1}{-u^2-2u+1}\,du = -\int \frac{u+1}{-u^2-2u+1}\,du$
- The numerator is the derivative of the denominator (or its multiple), f(x) = $-u^2 - 2u + 1$, by the formula $\displaystyle\int \frac{f'(u)}{f(u)}\,du = \ln|f(u)| + C$

  $\displaystyle\int \frac{u+1}{-u^2-2u+1}\,du = -\frac{1}{2}\int \frac{-2u-2}{-u^2-2u+1}\,du = -\ln$
- The constant taken out of the integral: $\displaystyle\int -\frac{1}{x}\,dx = -\int \frac{1}{x}\,dx$
- From the table of basic integrals: $\displaystyle\int \frac{1}{x}\,dx = \ln|x|$
- The general solution in implicit form: $\dfrac{\ln|-u^2-2u+1|}{2} = C - \ln|x|$
- Exponentiating – the constant ±e^C is called C again: $-u^2 - 2u + 1 = \dfrac{C}{x^2}$
- The constant solution $u = -1 - \sqrt{2}$, excluded by dividing, is in the general solution for C = 0
- The constant solution $u = -1 + \sqrt{2}$, excluded by dividing, is in the general solution for C = 0
- Back to the original unknown: $\dfrac{x^2-2xy-y^2}{x^2} = \dfrac{C}{x^2}$
- The general solution in implicit form: $x^2 - 2xy - y^2 = C$

Graph



solutions for different C

The short segments show the direction the equation prescribes at each point (the direction field). The grey curves are solutions for different C – they touch the segments everywhere.

Default curves

Your own values of the constants

C

+ Add curve

Tap the graph – the solution through that point is drawn.

A Bernoulli equation – the substitution z = y^(1−n) and the constant solution y = 0.

Input: `xy' - y = -xy^2`

Result

$$y = \frac{2x}{x^2 + C}, \quad C \in \mathbb{R}; \quad y = 0$$

Steps

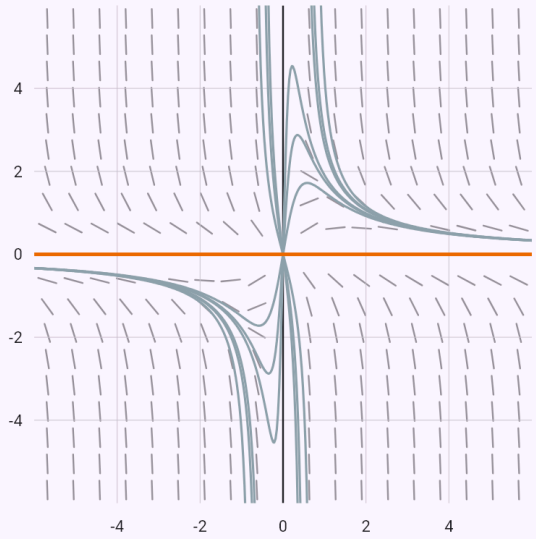
- A Bernoulli equation y' + p(x)·y = q(x)·y^n: $y' - \frac{y}{x} = -y^2$
- The substitution $z = \frac{1}{y}$, $z' = -1\,\frac{1}{y^2}y'$ turns the equation into $z' + \frac{z}{x} = 1$
- First the homogeneous equation (zero right side): $z' + \frac{z}{x} = 0$
- From the table of basic integrals: $\int \frac{1}{x}\,dx = \ln|x|$
- The homogeneous equation's solution: $z_h = \frac{C}{x}$
- Variation of the constant – we look for the solution as $z = C(x) \cdot \frac{1}{x}$; substituted into the equation: $C'(x) \cdot \frac{1}{x} = 1$
- Hence $C'(x) = x$
- From the table of basic integrals: $\int x\,dx = \frac{x^2}{2}$
- Integrating: $C(x) = \int x\,dx = \frac{x^2}{2} + C$
- The general solution: $z = \frac{x}{2} + \frac{C}{x}$
- Back to the original unknown: $\frac{1}{y} = \frac{x}{2} + \frac{C}{x}$
- Expressing the solution: $y = \frac{2x}{x^2 + 2C}$
- Dividing excluded the constant solution $y = 0$ – it is another solution of the equation.
- Renaming the constant (a multiple of an arbitrary constant is an arbitrary constant): $y = \frac{2x}{x^2 + C}$
- Check – the solution's derivative $y' = \frac{2C - 2x^2}{x^4 + 2Cx^2 + C^2}$ substituted into the equation gives an identity.
- The term differentiated by the rules for a composite function, a product and a quotient

$$\left(\frac{2x}{x^2 + C}\right)' = 2 \cdot \frac{x^2 + C - x \cdot 2x}{(x^2 + C)^2} = \frac{2C - 2x^2}{x^4 + 2Cx^2 + C^2}$$

- The solution and its derivative substituted into the equation – both sides are equal

$$x\left(\frac{2C - 2x^2}{x^4 + 2Cx^2 + C^2}\right) - \left(\frac{2x}{x^2 + C}\right) = -\frac{4x^3}{x^4 + 2Cx^2 +}$$

Graph



solutions for different C

$y = 0$

The short segments show the direction the equation prescribes at each point (the direction field). The grey curves are solutions for different C – they touch the segments everywhere.

Default curves

Your own values of the constants

C

+ Add curve

Tap the graph – the solution through that point is drawn.

An exact equation – the potential F(x, y) = C.

Input: `(2xy + 6x)dx + (x^2 + 4y^3)dy = 0`

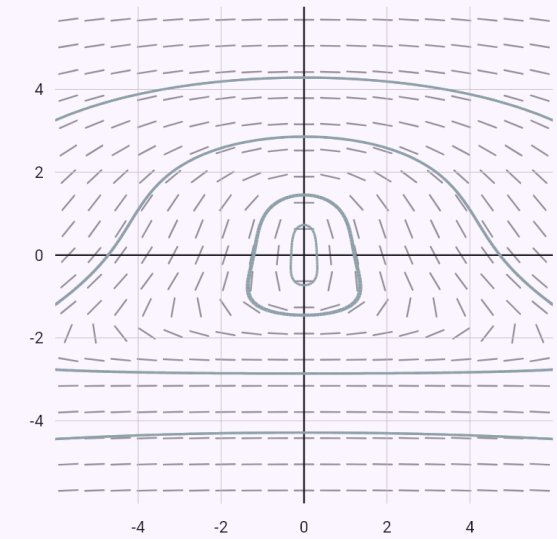
Result

$$y^4 + x^2 y + 3x^2 = C, \quad C \in \mathbb{R}$$

Steps

- An equation M dx + N dy = 0 – we check whether it is exact (the left side a total differential):
  $(2xy + 6x)dx + \left(4y^3 + x^2\right)dy = 0$
- M = $2xy + 6x$, N = $4y^3 + x^2$; the equation is exact, since
  $\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$
- The integral of a sum is the sum of the integrals:
  $\int (6x + 2yx)\,dx = \int 6x\,dx + \int 2yx\,dx$
- The constant taken out of the integral:
  $\int 6x\,dx = 6\int x\,dx$
- From the table of basic integrals: $\int x\,dx = \frac{x^2}{2}$
- The constant taken out of the integral:
  $\int 2yx\,dx = 2y\int x\,dx$
- From the table of basic integrals: $\int x\,dx = \frac{x^2}{2}$
- The potential F by integrating M with respect to x:
  $F(x, y) = \int (2xy + 6x)\,dx = x^2 y + 3x^2 + \varphi(y)$
- The constant taken out of the integral:
  $\int 4y^3\,dy = 4\int y^3\,dy$
- From the table of basic integrals: $\int y^3\,dy = \frac{y^4}{4}$
- From ∂F/∂y = N: $\varphi'(y) = 4y^3 \Rightarrow \varphi(y) = y^4$
- The general solution in implicit form:
  $y^4 + x^2 y + 3x^2 = C$

Graph



solutions for different C

The short segments show the direction the equation prescribes at each point (the direction field). The grey curves are solutions for different C – they touch the segments everywhere.

Default curves

Your own values of the constants

C

+ Add curve

Tap the graph – the solution through that point is drawn.

The substitution z = x − y; Symja would lose a factor 2 here, the app checks the result.

Input: `y' = cos(x - y), y(0) = pi/2`

Result

$$y = x + 2\,\operatorname{arccot}(x+1)$$

Solve by a numerical method:

Σ Runge–Kutta 4 | Σ Heun | Σ Euler

Steps

- The right side is a function of ax + by + c – we take it as a new unknown: $y' = \cos(x-y)$
- The substitution $z = x - y,\ z' = 1 - y'$ turns the equation into $z' = 1 - \cos z$
- Separating the variables (dividing, assuming $1-\cos z \neq 0$): $\dfrac{dz}{1-\cos z} = dx$
- Integrating both sides: $\displaystyle\int \frac{dz}{1-\cos z} = \int dx$
- No simpler substitution fits – the universal substitution $t = \tan\left(\frac{z}{2}\right)$

  $\sin z = \dfrac{2t}{1+t^2},\ \cos z = \dfrac{1-t^2}{1+t^2},\ dz = \dfrac{2\,dt}{(1+t^2)}$

  $$\int \frac{1}{1-\cos z}\,dz = \int \frac{1}{t^2}$$
- From the table of basic integrals: $\displaystyle\int \frac{1}{t^2}\,dt = -\frac{1}{t}$
- Back to the original variable: $t = \tan\left(\frac{z}{2}\right)$: $\quad -\cot\left(\frac{z}{2}\right)$
- From the table of basic integrals: $\displaystyle\int 1\,dx = x$
- The general solution in implicit form: $-\cot\left(\frac{z}{2}\right) = x + C$
- Expressing the solution: $z = 2\,\operatorname{arccot}(C-x)$
- Dividing excluded the constant solution $z = 0$ – it is another solution of the equation.
- Back to the original unknown: $y = x - 2\,\operatorname{arccot}(C-x)$
- Renaming the constant (a multiple of an arbitrary constant is an arbitrary constant): $y = x + 2\,\operatorname{arccot}(x+C)$
- Substituting the initial condition $y(0) = \dfrac{\pi}{2}$: $\dfrac{\pi}{2} = 2\,\operatorname{arccot} C \Rightarrow C = 1$
- The solution of the initial condition (the particular solution): $y = x + 2\,\operatorname{arccot}(x+1)$
- Check – the solution's derivative $y' = \dfrac{x^2+2x}{x^2+2x+2}$ substituted into the equation gives an identity.
- The term differentiated by the rules for a composite function, a product and a quotient

  $$(2\,\operatorname{arccot}(x+1))' = 2\cdot\left(-\frac{1}{(x+1)^2+1}\right) = -\frac{2}{x^2+2}$$
- Added and simplified

  $$y' = 1 - \frac{2}{x^2+2x+2} = \frac{x^2+2x+2-2}{x^2+2x+2} = \frac{x^2+2x}{x^2+2x+}$$
- The solution and its derivative substituted into the equation – both sides are equal

  $$\frac{x^2+2x}{x^2+2x+2} = \cos(x-(x+2\,\operatorname{arccot}(x+1)))$$

A Riccati equation – the particular solution $y_1 = -2/x$ guessed, the substitution $y = y_1 + 1/z$.

Input: `y' = y^2 - 2/x^2`

Result

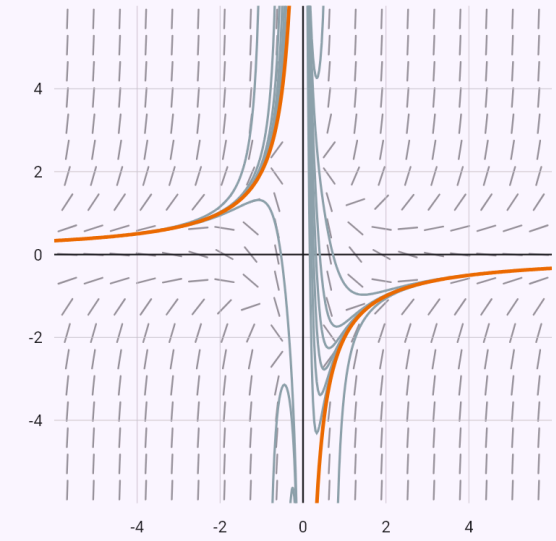
$$y = \frac{1 - 6Cx^3}{3Cx^4 + x}, \quad C \in \mathbb{R}; \quad y = -\frac{2}{x}$$

Steps

- A Riccati equation y' = P(x)·y² + Q(x)·y + R(x): $y' = y^2 - \frac{2}{x^2}$
- Without a particular solution a Riccati equation can't be solved. We try to guess one of a simple form (a constant, ax, a/x, ax², ae^{±x}, x + a) – $y_1 = -\frac{2}{x}$ satisfies the equation: $y_1' = \frac{2}{x^2}$, and substituting it into the right side gives the same.
- The substitution $y = -\frac{2}{x} + \frac{1}{z}$ turns the equation into $z' - \frac{4z}{x} = -1$
- First the homogeneous equation (zero right side): $z' - \frac{4z}{x} = 0$
- The constant taken out of the integral: $\int -\frac{4}{x}\,dx = -4\int \frac{1}{x}\,dx$
- From the table of basic integrals: $\int \frac{1}{x}\,dx = \ln|x|$
- The homogeneous equation's solution: $z_h = Cx^4$
- Variation of the constant – we look for the solution as $z = C(x)\cdot x^4$; substituted into the equation: $C'(x)\cdot x^4 = -1$
- Hence $C'(x) = -\frac{1}{x^4}$
- The constant taken out of the integral: $\int -\frac{1}{x^4}\,dx = -\int \frac{1}{x^4}\,dx$
- From the table of basic integrals: $\int \frac{1}{x^4}\,dx = -\frac{1}{3x^3}$
- Integrating: $C(x) = \int -\frac{1}{x^4}\,dx = C + \frac{1}{3x^3}$
- The general solution: $z = Cx^4 + \frac{x}{3}$
- Back to the original unknown: $y = \frac{1 - 6Cx^3}{3Cx^4 + x}$
- The solution $y = -\frac{2}{x}$ itself is left out by the substitution y = y₁ + 1/z (it corresponds to z → ∞) – we add it to the result.

- Check – the solution's derivative $y' = \frac{18C^2x^6 - 24Cx^3 - 1}{9C^2x^8 + 6Cx^5 + x^2}$ substituted into the equation gives an identity.
- The term differentiated by the rules for a composite function, a product and a quotient $\left(\frac{1 - 6Cx^3}{3Cx^4 + x}\right)' = \frac{(-18Cx^2)\cdot(3Cx^4 + x) - (1 - 6Cx^3}{(3Cx^4 + x)^2}$
- The solution and its derivative substituted into the equation – both sides are equal $\frac{18C^2x^6 - 24Cx^3 - 1}{9C^2x^8 + 6Cx^5 + x^2} = \left(\frac{1 - 6Cx^3}{3Cx^4 + x}\right)^2 - \frac{2}{x^2}$

Graph



solutions for different C

$y = -\frac{2}{x}$

The short segments show the direction the equation prescribes at each point (the direction field). The grey curves are solutions for different C – they touch the segments everywhere.

Default curves

Your own values of the constants

C

+ Add curve

Tap the graph – the solution through that point is drawn.

The same equation with the particular solution $y_1 = 1/x$ given.

Input: `y' = y^2 - 2/x^2`

`y1 = 1/x`

Result

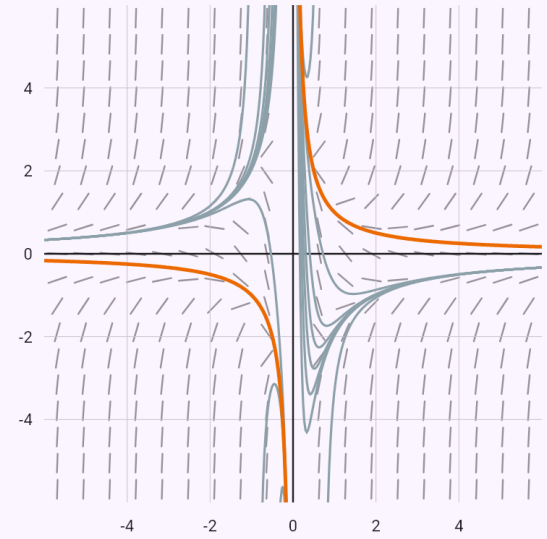
$$y = \frac{-2x^3 - 3C}{x^4 - 3Cx}, \quad C \in \mathbb{R}; \quad y = \frac{1}{x}$$

Steps

- A Riccati equation y' = P(x)·y² + Q(x)·y + R(x): $y' = y^2 - \frac{2}{x^2}$
- The given particular solution $y_1 = \frac{1}{x}$ satisfies the equation: $y_1' = -\frac{1}{x^2}$, and substituting it into the right side gives the same.
- The substitution $y = \frac{1}{x} + \frac{1}{z}$ turns the equation into $z' + \frac{2z}{x} = -1$
- First the homogeneous equation (zero right side): $z' + \frac{2z}{x} = 0$
- The constant taken out of the integral: $\int \frac{2}{x}\,dx = 2\int \frac{1}{x}\,dx$
- From the table of basic integrals: $\int \frac{1}{x}\,dx = \ln|x|$
- The homogeneous equation's solution: $z_h = \frac{C}{x^2}$
- Variation of the constant – we look for the solution as $z = C(x)\cdot\frac{1}{x^2}$; substituted into the equation: $C'(x)\cdot\frac{1}{x^2} = -1$
- Hence $C'(x) = -x^2$
- The constant taken out of the integral: $\int -x^2\,dx = -\int x^2\,dx$
- From the table of basic integrals: $\int x^2\,dx = \frac{x^3}{3}$
- Integrating: $C(x) = \int -x^2\,dx = C - \frac{x^3}{3}$
- The general solution: $z = -\frac{x}{3} + \frac{C}{x^2}$
- Back to the original unknown: $y = \frac{-2x^3 - 3C}{x^4 - 3Cx}$
- The solution $y = \frac{1}{x}$ itself is left out by the substitution y = y_1 + 1/z (it corresponds to z → ∞) – we add it to the result.

- Check – the solution's derivative $y' = \frac{2x^6 + 24Cx^3 - 9C^2}{x^8 - 6Cx^5 + 9C^2x^2}$ substituted into the equation gives an identity.
- The term differentiated by the rules for a composite function, a product and a quotient $\left(\frac{-2x^3 - 3C}{x^4 - 3Cx}\right)' = \frac{(-6x^2)\cdot(x^4 - 3Cx) - (-2x^3 - 3C}{(x^4 - 3Cx)^2}$
- The solution and its derivative substituted into the equation – both sides are equal $\frac{2x^6 + 24Cx^3 - 9C^2}{x^8 - 6Cx^5 + 9C^2x^2} = -\frac{2}{x^2} + \frac{(2x^3 + 3C)^2}{(x^4 - 3Cx)^2} = \left(\frac{-2x^3}{x^4 -}\right.$

Graph



solutions for different C

$y = \frac{1}{x}$

The short segments show the direction the equation prescribes at each point (the direction field). The grey curves are solutions for different C – they touch the segments everywhere.

Default curves

Your own values of the constants

C

+ Add curve

Tap the graph – the solution through that point is drawn.

A Clairaut equation – the lines y = Cx + C² and their envelope y = −x²/4.

Input: `y = xy' + y'^2`

Result

$$y = Cx + C^2, \quad C \in \mathbb{R}; \quad y = -\frac{x^2}{4}$$

Steps

- A Clairaut equation y = x·y' + ψ(y'). Let $p = y'$: $y = xp + p^2$
- We differentiate the equation by x (p = y' depends on x, p' = y''): $p = p + (x + 2p)p' \;\Rightarrow\; (x + 2p)p' = 0$
- If p' = 0, p is a constant and the general solution is the lines: $p' = 0 \;\Rightarrow\; p = C \;\Rightarrow\; y = Cx + C^2$
- The other factor x + ψ'(p) = 0 gives the singular solution – the envelope of all those lines. Parametrically $x = -2p,\ y = -p^2$, with p eliminated: $y = -\frac{x^2}{4}$

Graph



solutions for different C

$y = -\frac{x^2}{4}$

The grey lines are the solutions for various values of C; in colour the singular solution – their envelope, which every line touches.

Through the point [2; −1] on the envelope passes a single line y = 1 − x – and the envelope itself.

Input: `y = xy' + y'^2, y(2) = -1`

Result

$$y = 1 - x; \quad y = -\frac{x^2}{4}$$

Steps

- A Clairaut equation y = x·y' + ψ(y'). Let $p = y'$: $y = xp + p^2$
- We differentiate the equation by x (p = y' depends on x, p' = y''): $p = p + (x + 2p)p' \;\Rightarrow\; (x + 2p)p' = 0$
- If p' = 0, p is a constant and the general solution is the lines: $p' = 0 \;\Rightarrow\; p = C \;\Rightarrow\; y = Cx + C^2$
- The other factor x + ψ'(p) = 0 gives the singular solution – the envelope of all those lines. Parametrically $x = -2p,\ y = -p^2$, with p eliminated: $y = -\frac{x^2}{4}$
- Substituting the initial condition $y(2) = -1$: $-1 = 2C + C^2 \;\Rightarrow\; C = -1$
- The solution of the initial condition (the particular solution): $y = 1 - x; \quad y = -\frac{x^2}{4}$

Graph



$y = 1 - x$

$y = -\frac{x^2}{4}$

In colour the solutions through the given point – lines, possibly the envelope too.

A Lagrange equation – the solution parametrically and the line y = 0.

Input: `y = 2xy' + y'^2`

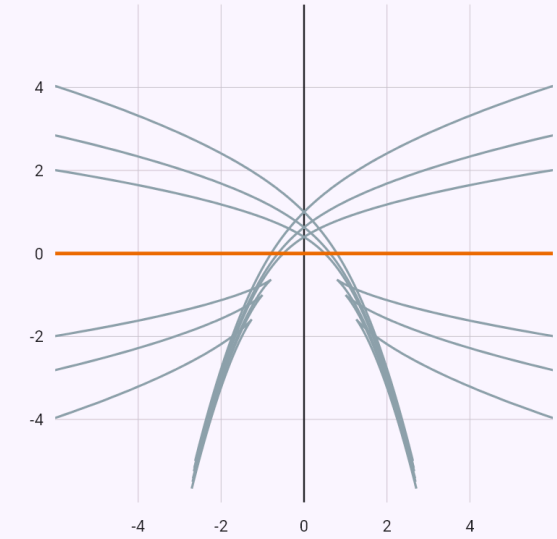
Result

$$x = \frac{C - 2p^3}{3p^2},\ y = \frac{2C - p^3}{3p}, \quad C \in \mathbb{R}$$

Steps

- A Lagrange equation y = x·φ(y') + ψ(y'). Let $p = y'$: $y = 2xp + p^2$
- We differentiate the equation by x (p = y' depends on x): $p = 2p + (2x + 2p)\frac{dp}{dx}$
- We look for x as a function of p: dx/dp from the equation – it is linear in the unknown x: $\frac{dx}{dp} = -\frac{2x}{p} - 2$
- A first-order linear equation y' + p(x)·y = q(x): $x' + \frac{2x}{p} = -2$
- First the homogeneous equation (zero right side): $x' + \frac{2x}{p} = 0$
- The constant taken out of the integral: $\int \frac{2}{p}\,dp = 2\int \frac{1}{p}\,dp$
- From the table of basic integrals: $\int \frac{1}{p}\,dp = \ln|p|$
- The homogeneous equation's solution: $x_h = \frac{C}{p^2}$
- Variation of the constant – we look for the solution as $x = C(p) \cdot \frac{1}{p^2}$; substituted into the equation: $C'(p) \cdot \frac{1}{p^2} = -2$
- Hence $C'(p) = -2p^2$
- The constant taken out of the integral: $\int -2p^2\,dp = -2\int p^2\,dp$
- From the table of basic integrals: $\int p^2\,dp = \frac{p^3}{3}$
- Integrating: $C(p) = \int -2p^2\,dp = C - \frac{2p^3}{3}$
- The general solution: $x = -\frac{2p}{3} + \frac{C}{p^2}$
- Renaming the constant (a multiple of an arbitrary constant is an arbitrary constant): $x = \frac{C - 2p^3}{3p^2}$
- Substituting x into the given equation gives the general solution parametrically (the parameter p): $x = \frac{C - 2p^3}{3p^2},\ y = \frac{2C - p^3}{3p}$
- We divided by p − φ(p), leaving out the values for which $p = 2p$. Each of them gives a line that satisfies the equation too: $y = 0$

Graph



solutions for different C

$y = 0$

The grey curves are solutions for various values of the constants.

The characteristic equation with complex roots −2 ± 3i.

Input: `y'' + 4y' + 13y = 0`

Result

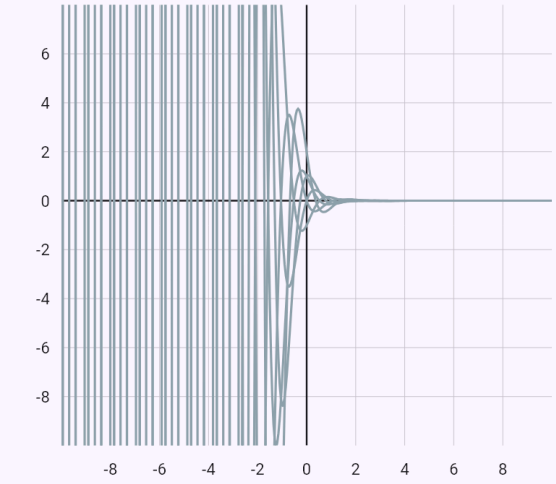
$$y = e^{-2x}(C_1 \cos(3x) + C_2 \sin(3x)),$$

Steps

- A linear equation with constant coefficients:
  $y'' + 4y' + 13y = 0$
- We look for solutions y = e^(λx); substituting and dividing by e^(λx) gives the characteristic equation:
  $\lambda^2 + 4\lambda + 13 = 0$
- The discriminant $D = 4^2 - 4 \cdot 13 = -36$; the roots:
  $\lambda_{1,2} = -2 \pm 3i$
- Each root λ gives the solution e^(λx), a root of multiplicity k also x·e^(λx), …, x^(k−1)·e^(λx), and a pair α ± βi the solutions e^(αx)·cos βx and e^(αx)·sin βx. The fundamental system:
  $y_1 = e^{-2x}\cos(3x),\ y_2 = e^{-2x}\sin(3x)$
- The general solution:
  $y = e^{-2x}(C_1 \cos(3x) + C_2 \sin(3x))$

Graph

Solutions | Phase plane



solutions for different C

The grey curves are solutions for various values of the constants.

Default curves

Your own values of the constants

C_1 C_2 + Add curve

Tap the graph and set the slope with the slider – the solution through that point with that tangent slope is drawn.

A special right side with resonance – 2 is a root of the characteristic equation.

Input: `y'' - 3y' + 2y = 3e^(2x)` Special right side

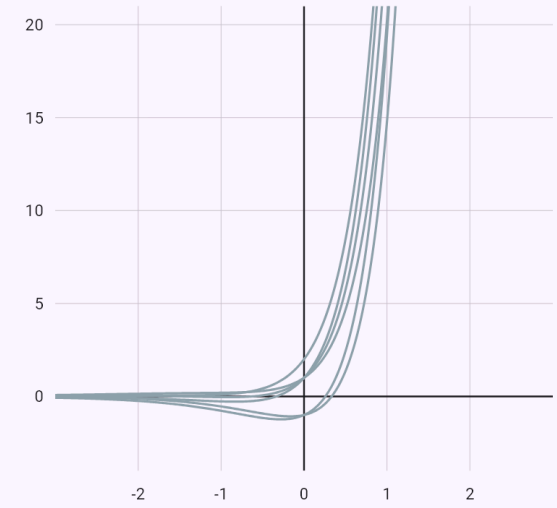
Result

$$y = C_1 e^{x} + C_2 e^{2x} + 3xe^{2x}, \quad C_1, C_2 \in$$

Steps

- A linear equation with constant coefficients:
  $y'' - 3y' + 2y = 3e^{2x}$
- First the homogeneous equation (zero right side):
  $y'' - 3y' + 2y = 0$
- We look for solutions y = e^(λx); substituting and dividing by e^(λx) gives the characteristic equation:
  $\lambda^2 - 3\lambda + 2 = 0$
- The discriminant $D = (-3)^2 - 4 \cdot 2 = 1$; the roots:
  $\lambda_1 = 1,\ \lambda_2 = 2$
- Each root λ gives the solution e^(λx), a root of multiplicity k also x·e^(λx), …, x^(k−1)·e^(λx), and a pair α ± βi the solutions e^(αx)·cos βx and e^(αx)·sin βx. The fundamental system: $y_1 = e^{x},\ y_2 = e^{2x}$
- The homogeneous equation's solution:
  $y_h = C_1 e^{x} + C_2 e^{2x}$
- The right side $3e^{2x}$ has a special form; the number 2 is a root of the characteristic equation (multiplicity 1), so the form is multiplied by x: $y_p = Axe^{2x}$
- The derivatives:
  $y_p' = 2Axe^{2x} + Ae^{2x},\ y_p'' = 4Axe^{2x} + 4Ae^{2x}$
- Substituted into the equation (divided by the exponential), like terms collected: $A = 3$
- A particular solution: $y_p = 3xe^{2x}$
- The general solution:
  $y = y_h + y_p = C_1 e^{x} + C_2 e^{2x} + 3xe^{2x}$

Graph



solutions for different C

The grey curves are solutions for various values of the constants.

Default curves

Your own values of the constants

C₁ C₂ + Add curve

Tap the graph and set the slope with the slider – the solution through that point with that tangent slope is drawn.

The right side tan x isn't special – variation of constants and the Wronskian.

Input: `y'' + y = tan x`

Result

$$y = C_1 \cos x + C_2 \sin x - \cos x \cdot \ln \left| \frac{}{\text{co}} \right.$$

Steps

- A linear equation with constant coefficients: $y'' + y = \tan x$
- First the homogeneous equation (zero right side): $y'' + y = 0$
- We look for solutions y = e^(λx); substituting and dividing by e^(λx) gives the characteristic equation: $\lambda^2 + 1 = 0$
- The roots of the characteristic equation: $\lambda_{1,2} = \pm i$
- Each root λ gives the solution e^(λx), a root of multiplicity k also x·e^(λx), ..., x^(k−1)·e^(λx), and a pair α ± βi the solutions e^(αx)·cos βx and e^(αx)·sin βx. The fundamental system: $y_1 = \cos x,\ y_2 = \sin x$
- The homogeneous equation's solution: $y_h = C_1 \cos x + C_2 \sin x$
- Variation of constants – we look for the solution as $y = C_1(x) \cos x + C_2(x) \sin x$, the derivatives of the functions C_k satisfy the system
$$\begin{cases} C_1'(x) \cos x + C_2'(x) \sin x = 0 \\ -C_1'(x) \sin x + C_2'(x) \cos x = \tan x \end{cases}$$
- The system's determinant (the Wronskian)
$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$
- Hence $C_1'(x) = \dfrac{W_1}{W} = -\sin x \cdot \tan x$
- Hence $C_2'(x) = \dfrac{W_2}{W} = \sin x$
- The constant taken out of the integral:
$$\int -\sin x \cdot \tan x \, dx = -\int \sin x \cdot \tan x \, dx$$
- The function is odd in cosine – the substitution $t = \sin x$
$$dt = \cos x \, dx$$
$$\int \sin x \cdot \tan x \, dx = \int \frac{t^2}{(1-t)(t+1)} \, dt$$
- The rational function split into a polynomial and partial fractions
$$\int \frac{t^2}{(1-t)(t+1)} \, dt = \int \left(-1 - \frac{1}{2(t-1)} + \frac{1}{2(t+1)} \right)$$
- The integral of a sum is the sum of the integrals
$$\int \left(-1 - \frac{1}{2(t-1)} + \frac{1}{2(t+1)} \right) dt = \int -1 \, dt + \int -\frac{}{2}$$
- From the table of basic integrals: $\int -1 \, dt = -t$
- The constant taken out of the integral:
$$\int -\frac{1}{2(t-1)} \, dt = -\frac{1}{2} \int \frac{1}{t-1} \, dt$$
- From the table of basic integrals:
$$\int \frac{1}{t-1} \, dt = \ln |t-1|$$
- The constant taken out of the integral:
$$\int \frac{1}{2(t+1)} \, dt = \frac{1}{2} \int \frac{1}{t+1} \, dt$$
- From the table of basic integrals:
$$\int \frac{1}{t+1} \, dt = \ln |t+1|$$
- Back to the original variable
$$t = \sin x: \quad \frac{\ln |\sin x + 1|}{2} - \frac{\ln |\sin x - 1|}{2} - \sin x = \ln$$
- Integrated (the integration constants are already in the homogeneous solution):
$$C_1(x) = \int -\sin x \cdot \tan x \, dx = \sin x - \ln \left| \frac{1}{\cos x} + \tan x \right|$$
- From the table of basic integrals: $\int \sin x \, dx = -\cos x$
- Integrated (the integration constants are already in the homogeneous solution): $C_2(x) = \int \sin x \, dx = -\cos x$
- A particular solution: $y_p = -\cos x \cdot \ln \left| \frac{1}{\cos x} + \tan x \right|$
- The general solution
$$y = y_h + y_p = C_1 \cos x + C_2 \sin x - \cos x \cdot \ln \left| \frac{1}{\cos x} + \right.$$

Integrals of the variation of constants that can't be expressed by elementary functions – a solution

with an integral.

Input: `y'' + y' + y = tan x`

Result

$$y = e^{-\frac{x}{2}}\left(C_1 \cos\left(\frac{\sqrt{3}x}{2}\right) + C_2 \sin\left(\frac{\sqrt{}}{}\right.\right.$$

Steps

- A linear equation with constant coefficients:
  $y'' + y' + y = \tan x$
- First the homogeneous equation (zero right side):
  $y'' + y' + y = 0$
- We look for solutions y = e^(λx); substituting and dividing by e^(λx) gives the characteristic equation:
  $\lambda^2 + \lambda + 1 = 0$
- The discriminant $D = 1^2 - 4 = -3$; the roots:
  $\lambda_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$
- Each root λ gives the solution e^(λx), a root of multiplicity k also x·e^(λx), ..., x^(k−1)·e^(λx), and a pair α ± βi the solutions e^(αx)·cos βx and e^(αx)·sin βx. The fundamental system
  $y_1 = e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right),\ y_2 = e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}x}{2}\right)$
- The homogeneous equation's solution
  $y_h = e^{-\frac{x}{2}}\left(C_1 \cos\left(\frac{\sqrt{3}x}{2}\right) + C_2 \sin\left(\frac{\sqrt{3}x}{2}\right)\right)$
- Variation of constants – we look for the solution as the derivatives of the functions C_k satisfy the system
  $y = C_1(x)e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right) + C_2(x)e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}x}{2}\right)$
  $$\begin{cases} C_1'(x)e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right) + C_2'(x)e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}x}{2}\right) = 0 \\ C_1'(x)\frac{e^{-\frac{x}{2}}\left(-\cos\left(\frac{\sqrt{3}x}{2}\right) - \sqrt{3}\sin\left(\frac{\sqrt{3}x}{2}\right)\right)}{2} + C_2'(x \end{cases}$$
- The system's determinant (the Wronskian)
  $$W = \begin{vmatrix} e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right) & e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}x}{2}\right) \\ \frac{e^{-\frac{x}{2}}\left(-\cos\left(\frac{\sqrt{3}x}{2}\right) - \sqrt{3}\sin\left(\frac{\sqrt{3}x}{2}\right)\right)}{2} & \frac{e^{-\frac{x}{2}}\left(\sqrt{3}\cos\left(\frac{\sqrt{3}x}{2}\right) - \sin\left(\frac{\sqrt{}}{}\right.\right.\right.}{2} \end{vmatrix}$$
- Hence
  $C_1'(x) = \frac{W_1}{W} = -\frac{2\sqrt{3}e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}x}{2}\right) \tan x}{3}$
- Hence
  $C_2'(x) = \frac{W_2}{W} = \frac{2\sqrt{3}e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right) \tan x}{3}$
- Integrated (the integration constants are already in the homogeneous solution)
  $C_1(x) = \int -\frac{2\sqrt{3}e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}x}{2}\right) \tan x}{3}\, dx$
- Integrated (the integration constants are already in the homogeneous solution)
  $C_2(x) = \int \frac{2\sqrt{3}e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right) \tan x}{3}\, dx$
- These integrals can't be expressed by elementary functions (they lead to higher, non-elementary functions), so the solution keeps them as integrals
  $\int -\frac{2\sqrt{3}e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}x}{2}\right) \tan x}{3}\, dx,\ \int \frac{2\sqrt{3}e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right)}{3}$
- The general solution
  $y = y_h + y_p = e^{-\frac{x}{2}}\left(C_1 \cos\left(\frac{\sqrt{3}x}{2}\right) + C_2 \sin\left(\frac{\sqrt{3}x}{2}\right)\right)$

An initial value problem – a system for the constants C_1, C_2.

Input: `y'' - 4y' + 3y = e^(2x), y(0) = 1, y'(0) = 0`

Result

$$y = 2e^{x} - e^{2x}$$

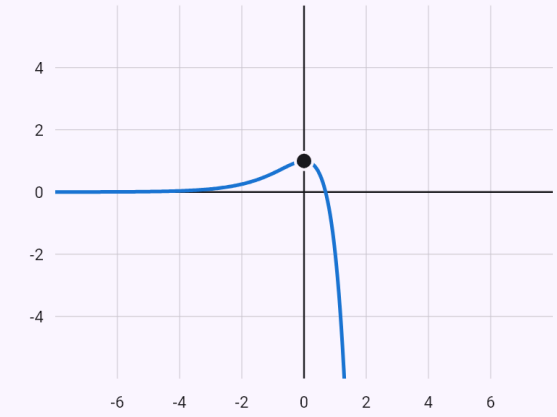
Steps

- A linear equation with constant coefficients: $y'' - 4y' + 3y = e^{2x}$
- First the homogeneous equation (zero right side): $y'' - 4y' + 3y = 0$
- We look for solutions y = e^(λx); substituting and dividing by e^(λx) gives the characteristic equation: $\lambda^2 - 4\lambda + 3 = 0$
- The discriminant $D = (-4)^2 - 4 \cdot 3 = 4$; the roots: $\lambda_1 = 1,\ \lambda_2 = 3$
- Each root λ gives the solution e^(λx), a root of multiplicity k also x·e^(λx), …, x^(k−1)·e^(λx), and a pair α ± βi the solutions e^(αx)·cos βx and e^(αx)·sin βx. The fundamental system: $y_1 = e^{x},\ y_2 = e^{3x}$
- The homogeneous equation's solution: $y_h = C_1 e^{x} + C_2 e^{3x}$
- The right side e^{2x} has a special form; the number 2 is not a root of the characteristic equation, so we look for a particular solution $y_p = Ae^{2x}$
- The derivatives: $y_p' = 2Ae^{2x},\ y_p'' = 4Ae^{2x}$
- Substituted into the equation (divided by the exponential), like terms collected: $-A = 1$
- Hence $A = -1$
- A particular solution: $y_p = -e^{2x}$
- The general solution: $y = y_h + y_p = C_1 e^{x} + C_2 e^{3x} - e^{2x}$
- The general solution's derivatives: $y' = C_1 e^{x} - 2e^{2x} + 3C_2 e^{3x}$. The conditions give the system hence $C_1 = 2,\ C_2 = 0$

$$\begin{cases} -1 + C_1 + C_2 = 1 \\ -2 + C_1 + 3C_2 = 0 \end{cases}$$

- The solution satisfying the given conditions: $y = 2e^{x} - e^{2x}$

Graph



$y = 2e^{x} - e^{2x}$

The highlighted one satisfies the conditions, the points are the given values.

Your own values of the constants

C₁ | C₂ | + Add curve

Tap the graph and set the slope with the slider – the solution through that point with that tangent slope is drawn.

The same problem by the Laplace transform.

Input: `y'' - 4y' + 3y = e^(2x), y(0) = 1, y'(0) = 0` Laplace transform

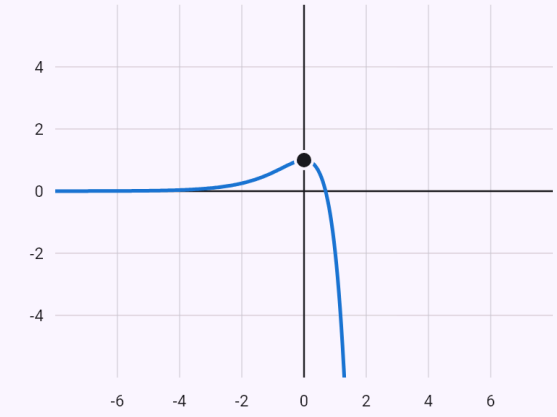
Result

$$y = 2e^x - e^{2x}$$

Steps

- A linear equation with constant coefficients: $y'' - 4y' + 3y = e^{2x}$
- The Laplace transform of both sides: $\mathcal{L}(y) = Y$, $\mathcal{L}(y') = sY - y(0)$, $\mathcal{L}(y'') = s^2Y - s\cdot y(0) - y'(0)$; with the initial conditions: $\left(s^2Y - s\right) - 4(sY - 1) + 3Y = \frac{1}{s-2}$
- We express the image Y: $Y = \frac{s-3}{(s-2)(s-1)}$
- Partial fractions: $Y = -\frac{1}{s-2} + \frac{2}{s-1}$
- The inverse transform (by the table of images) gives the solution: $y = \mathcal{L}^{-1}\{Y\} = 2e^x - e^{2x}$

Graph



$y = 2e^x - e^{2x}$

The highlighted one satisfies the conditions, the points are the given values.

A boundary value problem with infinitely many solutions.

Input: `y'' + y = 0, y(0) = 0, y(pi) = 0`

Result

$$y = C_2 \sin x, \quad C_2 \in \mathbb{R}$$

Steps

- A linear equation with constant coefficients: $y'' + y = 0$
- We look for solutions y = e^(λx); substituting and dividing by e^(λx) gives the characteristic equation: $\lambda^2 + 1 = 0$
- The roots of the characteristic equation: $\lambda_{1,2} = \pm i$
- Each root λ gives the solution e^(λx), a root of multiplicity k also x·e^(λx), ..., x^(k−1)·e^(λx), and a pair α ± βi the solutions e^(αx)·cos βx and e^(αx)·sin βx. The fundamental system: $y_1 = \cos x,\ y_2 = \sin x$
- The general solution: $y = C_1 \cos x + C_2 \sin x$
- We substitute the conditions into the general solution hence $C_1 = 0$

$$\begin{cases} C_1 = 0 \\ -C_1 = 0 \end{cases}$$

- The constant C_2 stays arbitrary – the problem has infinitely many solutions.
- The solution satisfying the given conditions: $y = C_2 \sin x$

Euler's equation – the substitution x = eᵗ.

Input: `x^2y'' - 2xy' + 2y = x^2 + 2`

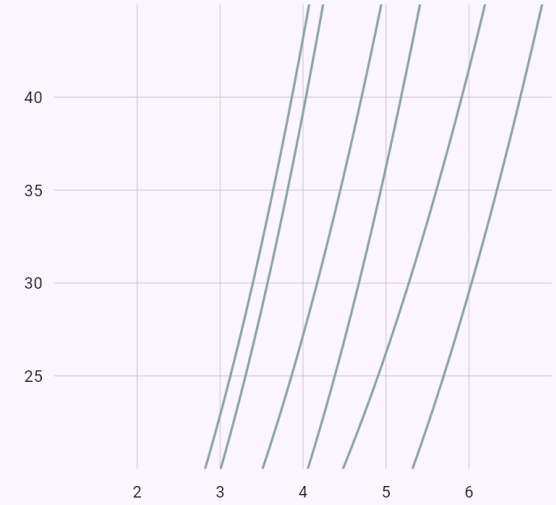
Result

$$y = C_1 x + C_2 x^2 + x^2 \ln x + 1, \quad C_1, C$$

Steps

- Euler's equation – each derivative y^(k) comes with x^k; the substitution x = e^t turns it into one with constant coefficients: $x^2 y'' - 2xy' + 2y = x^2 + 2$
- The substitution $x = e^t,\ xy' = \dot{y},\ x^2 y'' = \ddot{y} - \dot{y}$ turns the equation into $\ddot{y} - 3\dot{y} + 2y = e^{2t} + 2$
- First the homogeneous equation (zero right side): $\ddot{y} - 3\dot{y} + 2y = 0$
- We look for solutions y = e^(λt); substituting and dividing by e^(λt) gives the characteristic equation: $\lambda^2 - 3\lambda + 2 = 0$
- The discriminant $D = (-3)^2 - 4 \cdot 2 = 1$; the roots: $\lambda_1 = 1,\ \lambda_2 = 2$
- Each root λ gives the solution e^(λx), a root of multiplicity k also x·e^(λx), …, x^(k−1)·e^(λx), and a pair α ± βi the solutions e^(αx)·cos βx and e^(αx)·sin βx. The fundamental system: $y_1 = e^t,\ y_2 = e^{2t}$
- The homogeneous equation's solution: $y_h = C_1 e^t + C_2 e^{2t}$
- We split the right side into parts of a special form – a particular solution for each, then we add them up: $f_1 = 2,\ f_2 = e^{2t}$
- The right side 2 has a special form; the number 0 is not a root of the characteristic equation, so we look for a particular solution $y_{p_1} = A$
- The derivatives: $y'_{p_1} = 0,\ y''_{p_1} = 0$
- Substituted into the equation (divided by the exponential), like terms collected: $2A = 2$
- Hence $A = 1$
- A particular solution: $y_{p_1} = 1$
- The right side e^{2t} has a special form; the number 2 is a root of the characteristic equation (multiplicity 1), so the form is multiplied by x: $y_{p_2} = Ate^{2t}$
- The derivatives: $y'_{p_2} = 2Ate^{2t} + Ae^{2t},\ y''_{p_2} = 4Ate^{2t} + 4Ae^{2t}$
- Substituted into the equation (divided by the exponential), like terms collected: $A = 1$
- A particular solution: $y_{p_2} = te^{2t}$
- A particular solution: $y_p = y_{p_1} + y_{p_2} = te^{2t} + 1$
- The general solution: $y = y_h + y_p = C_1 e^t + C_2 e^{2t} + te^{2t} + 1$
- Back to the original variable (t = ln x, for $x > 0$): $y = C_1 x + C_2 x^2 + x^2 \ln x + 1$

Graph



solutions for different C

The grey curves are solutions for various values of the constants.

Default curves

Your own values of the constants

C₁ | C₂ | + Add curve

Tap the graph and set the slope with the slider – the solution through that point with that tangent slope is drawn.

The reduction y' = p(y), the constants from the initial conditions step by step.

Input: `y'' = e^(2y), y(0) = 0, y'(0) = 1`

Result

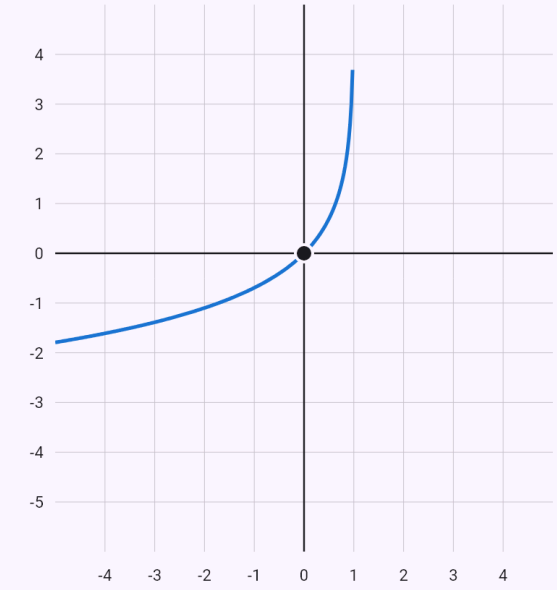
$$y = -\ln(1 - x)$$

Steps

- The independent variable is missing – we take y' as a function of y: $y'' = e^{2y}$
- The substitution $y' = p(y),\ y'' = p\dfrac{dp}{dy}$ turns the equation into $p\dfrac{dp}{dy} = e^{2y}$; the initial conditions give $p(0) = 1$
- A separable equation – the right side is a product of a function of the variable and a function of the unknown: $\dfrac{dp}{dy} = \dfrac{e^{2y}}{p}$
- Separating the variables: $p\,dp = e^{2y}\,dy$
- Integrating both sides: $\int p\,dp = \int e^{2y}\,dy$
- From the table of basic integrals: $\int p\,dp = \dfrac{p^2}{2}$
- From the table – the inner function is linear, so the result is divided by its coefficient of the variable: $\int e^{2y}\,dy = \dfrac{e^{2y}}{2}$
- The general solution in implicit form: $\dfrac{p^2}{2} = \dfrac{e^{2y}}{2} + C_1$
- Expressing the solution: $p^2 = e^{2y} + C_1$
- Substituting the initial condition $p(0) = 1$: $1 = C_1 + 1 \Rightarrow C_1 = 0$
- The solution of the initial condition (the particular solution): $p = e^y$
- The solution and its derivative substituted into the equation – both sides are equal: $e^y e^y - e^{2y} = 0$
- Back to the original unknown: $y' = e^y$
- A separable equation – the right side is a product of a function of the variable and a function of the unknown: $y' = e^y$
- Separating the variables (dividing, assuming $e^y \neq 0$): $\dfrac{dy}{e^y} = dx$
- Integrating both sides: $\int \dfrac{dy}{e^y} = \int dx$
- From the table – the inner function is linear, so the result is divided by its coefficient of the variable: $\int e^{-y}\,dy = -e^{-y}$
- From the table of basic integrals: $\int 1\,dx = x$
- The general solution in implicit form: $-e^{-y} = x + C_2$
- Expressing the solution: $y = -\ln(C_2 - x)$
- Substituting the initial condition $y(0) = 0$: $0 = -\ln C_2 \Rightarrow C_2 = 1$
- The solution of the initial condition (the particular solution): $y = -\ln(1 - x)$
- The term differentiated by the rules for a composite function, a product and a quotient: $(-\ln(1-x))' = -\dfrac{1}{1-x} \cdot (-1) = \dfrac{1}{1-x}$
- The solution and its derivative substituted into the equation – both sides are equal: $\dfrac{1}{1-x} = e^{(-\ln(1-x))}$

Graph

Solutions | Phase plane



$y = -\ln(1 - x)$

The highlighted one satisfies the conditions, the points are the given values.

A damped oscillator – three cases by δ and ω.

Input: `y'' + 2δy' + ω^2y = 0`

Result

$$\delta > \omega: \quad y = C_1 e^{(-\delta-\sqrt{\delta^2-\omega^2})x} + C_2 e^{(}$$

$$\delta = \omega: \quad y = (C_1 + C_2 x)e^{-\delta x}$$

$$\delta < \omega: \quad y = e^{-\delta x}\left(C_1 \cos\left(\sqrt{\omega^2 - \delta^2}\right.\right.$$

Steps

- A linear equation with constant coefficients:
  $y'' + 2\delta y' + \omega^2 y = 0$
- We look for solutions y = e^(λx); substituting and dividing by e^(λx) gives the characteristic equation:
  $\lambda^2 + 2\delta\lambda + \omega^2 = 0$
- The roots of the characteristic equation:
  $\lambda_{1,2} = -\delta \pm \sqrt{\delta^2 - \omega^2}$
- For $\delta > \omega$ so
  $\lambda_1 = -\delta - \sqrt{\delta^2 - \omega^2},\ \lambda_2 = \sqrt{\delta^2 - \omega^2} - \delta$
  $y = C_1 e^{(-\delta-\sqrt{\delta^2-\omega^2})x} + C_2 e^{(\sqrt{\delta^2-\omega^2}-\delta)x}$
- For $\delta = \omega$: $\lambda_{1,2} = -\delta$, so $y = (C_1 + C_2 x)e^{-\delta x}$
- For $\delta < \omega$: $\lambda_{1,2} = -\delta \pm \sqrt{\omega^2 - \delta^2}i$, so
  $y = e^{-\delta x}\left(C_1 \cos\left(\sqrt{\omega^2 - \delta^2}x\right) + C_2 \sin\left(\sqrt{\omega^2 - \delta^2}x\right)\right.$

Resonance – for ω = 2 the solution has a different form.

Input: `y'' + ω^2y = cos(2t)`

Result

$$\omega \neq 2: \quad y = C_1 \cos(\omega t) + C_2 \sin(\omega t)$$

$$\omega = 2: \quad y = C_1 \cos(2t) + C_2 \sin(2t)$$

Steps

- A linear equation with constant coefficients: $y'' + \omega^2 y = \cos(2t)$
- First the homogeneous equation (zero right side): $y'' + \omega^2 y = 0$
- We look for solutions y = e^(λt); substituting and dividing by e^(λt) gives the characteristic equation: $\lambda^2 + \omega^2 = 0$
- The roots of the characteristic equation: $\lambda_{1,2} = \pm \omega i$
- Each root λ gives the solution e^(λx), a root of multiplicity k also x·e^(λx), ..., x^(k−1)·e^(λx), and a pair α ± βi the solutions e^(αx)·cos βx and e^(αx)·sin βx. The fundamental system: $y_1 = \cos(\omega t),\ y_2 = \sin(\omega t)$
- The homogeneous equation's solution: $y_h = C_1 \cos(\omega t) + C_2 \sin(\omega t)$
- The right side $\cos(2t)$ has a special form; the number $2i$ is not a root of the characteristic equation, so we look for a particular solution $y_p = A\cos(2t) + B\sin(2t)$
- The derivatives

$y_p' = 2B\cos(2t) - 2A\sin(2t),\ y_p'' = -4A\cos(2t) - 4I$

- Substituted into the equation (divided by the exponential), like terms collected $(A\omega^2 - 4A)\cos(2t) + (B\omega^2 - 4B)\sin(2t) = \cos(2t)$
- We compare the coefficients of like terms on both sides hence $A = \frac{1}{\omega^2 - 4},\ B = 0$

$$\begin{cases} A\omega^2 - 4A = 1 \\ B\omega^2 - 4B = 0 \end{cases}$$

- A particular solution: $y_p = \frac{\cos(2t)}{\omega^2 - 4}$
- The general solution

$$y = y_h + y_p = C_1 \cos(\omega t) + C_2 \sin(\omega t) + \frac{\cos(2t)}{\omega^2 - 4}$$

- For $\omega = 2$ the number from the right side is a root of the characteristic equation (resonance) – the solution has a different form: $y = C_1 \cos(2t) + C_2 \sin(2t) + \frac{t \sin(2t)}{4}$

A guessed solution $y_1 = x$ and reduction of order (d'Alembert).

Input: `(1-x^2)y'' - 2xy' + 2y = 0`

Result

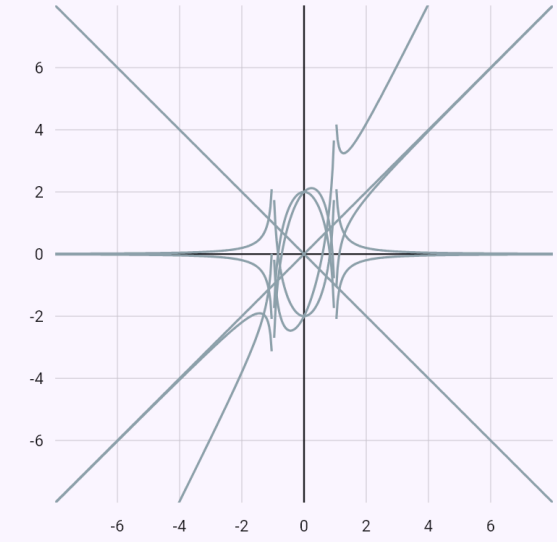
$$y = C_1 x + C_2(x \ln|x-1| - x \ln|x+$$

Steps

- A linear second-order equation with variable coefficients:
  $y'' + \frac{2x}{x^2-1} y' + \frac{2}{1-x^2} y = 0$
- We guess one solution and check it by substitution: $y_1 = x$. The second one by reducing the order (d'Alembert, y = $y_1 \cdot u$)
  $y_2 = y_1 \int \frac{e^{-\int p\,dx}}{y_1^2}\,dx = x \int \frac{1}{x^4 - x^2}\,dx = x \ln|x-1|$
- The rational function split into a polynomial and partial fractions
  $\int \frac{1}{x^4-x^2}\,dx = \int \left(\frac{1}{2(x-1)} - \frac{1}{2(x+1)} - \frac{1}{x^2} \right) dx$
- The integral of a sum is the sum of the integrals
  $\int \left(\frac{1}{2(x-1)} - \frac{1}{2(x+1)} - \frac{1}{x^2} \right) dx = \int \frac{1}{2(x-1)}\,dx$
- The constant taken out of the integral:
  $\int \frac{1}{2(x-1)}\,dx = \frac{1}{2} \int \frac{1}{x-1}\,dx$
- From the table of basic integrals:
  $\int \frac{1}{x-1}\,dx = \ln|x-1|$
- The constant taken out of the integral:
  $\int -\frac{1}{2(x+1)}\,dx = -\frac{1}{2} \int \frac{1}{x+1}\,dx$
- From the table of basic integrals:
  $\int \frac{1}{x+1}\,dx = \ln|x+1|$
- The constant taken out of the integral:
  $\int -\frac{1}{x^2}\,dx = -\int \frac{1}{x^2}\,dx$
- From the table of basic integrals: $\int \frac{1}{x^2}\,dx = -\frac{1}{x}$

- The general solution:
  $y = C_1 x + C_2(x \ln|x-1| - x \ln|x+1| + 2)$

Graph



solutions for different C

The grey curves are solutions for various values of the constants.

Default curves

Your own values of the constants

C_1 C_2 + Add curve

Tap the graph and set the slope with the slider – the solution through that point with that tangent slope is drawn.

A system – eigenvalues of opposite signs, a saddle.

Input: `x' = x + 2y, y' = 3x + 2y`

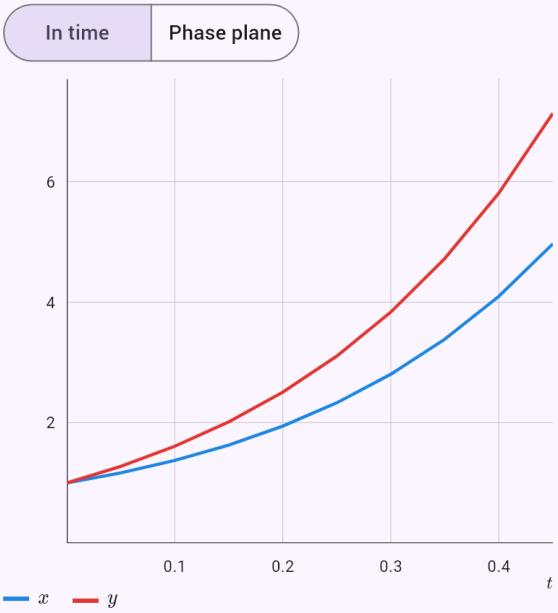
Result

$$x(t) = C_1 e^{-t} + 2C_2 e^{4t}, \quad y(t) = 3C_2 e$$

Steps

- The system in matrix form
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$
- The matrix's eigenvalues from the characteristic equation $\det(A - \lambda E) = \lambda^2 - 3\lambda - 4 = 0$:
$\lambda_1 = -1,\ \lambda_2 = 4$ – two distinct real ones.
- Eigenvectors, $(A - \lambda E)v = 0$
$$v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix},\ v_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$
- The general solution
$x(t) = C_1 e^{-t} + 2C_2 e^{4t}$
$y(t) = 3C_2 e^{4t} - C_1 e^{-t}$
- The stationary point $[0;\ 0]$ is a saddle – the eigenvalues have opposite signs, it is unstable.

Graph



The system's solution in time for the given initial conditions (without them for x(0) = y(0) = 1).

Complex eigenvalues – an unstable focus.

Input: `dx/dt = x - y, dy/dt = x + y`

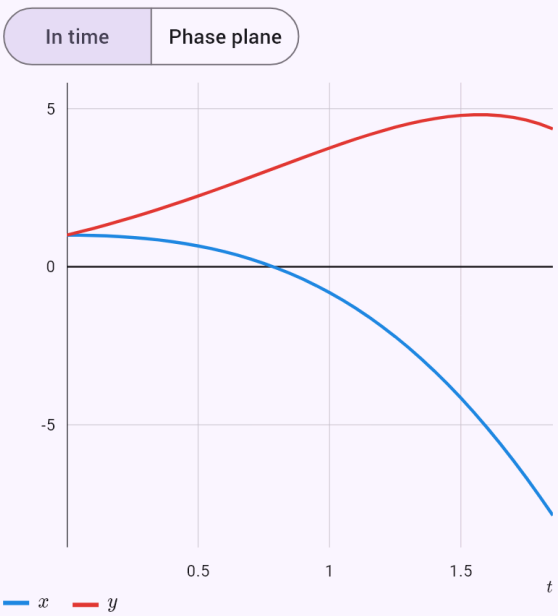
Result

$$x(t) = e^t(-C_1 \cos t - C_2 \sin t), \quad y(t)$$

Steps

- The system in matrix form
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$
- The matrix's eigenvalues from the characteristic equation $\det(A - \lambda E) = \lambda^2 - 2\lambda + 2 = 0$:
$\lambda_{1,2} = 1 \pm i$ – complex conjugate.
- Eigenvectors, $(A - \lambda E)v = 0$
$$v = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
- The general solution
$x(t) = e^t(-C_1 \cos t - C_2 \sin t)$
$y(t) = e^t(C_2 \cos t - C_1 \sin t)$
- The stationary point $[0;\ 0]$ is an unstable focus – the eigenvalues' real part is positive, the trajectories spiral away.

Graph



The system's solution in time for the given initial conditions (without them for x(0) = y(0) = 1).

A system with initial conditions – a center.

Input: `x' = y`
`y' = -x`
`x(0) = 1`
`y(0) = 0`

Result

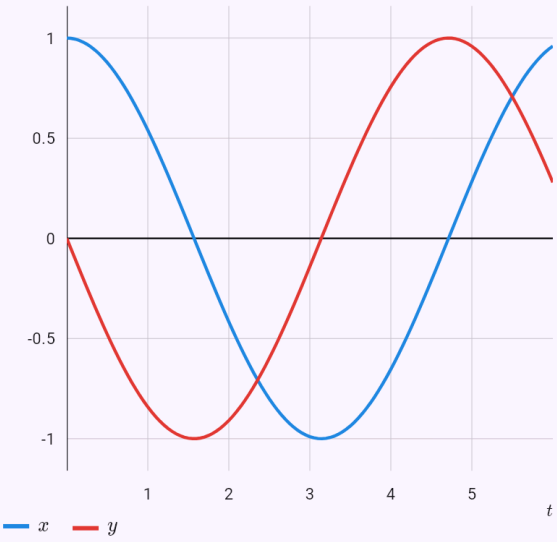
$$x(t) = \cos t, \quad y(t) = -\sin t$$

Steps

- The system in matrix form
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$
- The matrix's eigenvalues from the characteristic equation $\det(A - \lambda E) = \lambda^2 + 1 = 0$: $\lambda_{1,2} = \pm i$ – complex conjugate.
- Eigenvectors, $(A - \lambda E)v = 0$
$$v = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
- The general solution
$$x(t) = C_1 \cos t + C_2 \sin t$$
$$y(t) = C_2 \cos t - C_1 \sin t$$
- The constants from the initial conditions so
$$C_1 = 1,\ C_2 = 0$$
$$C_1 = 1$$
$$C_2 = 0$$
- The stationary point $[0;\ 0]$ is a center – the eigenvalues are purely imaginary, the trajectories are closed curves around it.

Graph

In time | Phase plane



The system's solution in time for the given initial conditions (without them for x(0) = y(0) = 1).

Airy's equation – the Leighton–Wintner criterion and 12 zeros.

Input: `oscillation(y'' + x y = 0)`

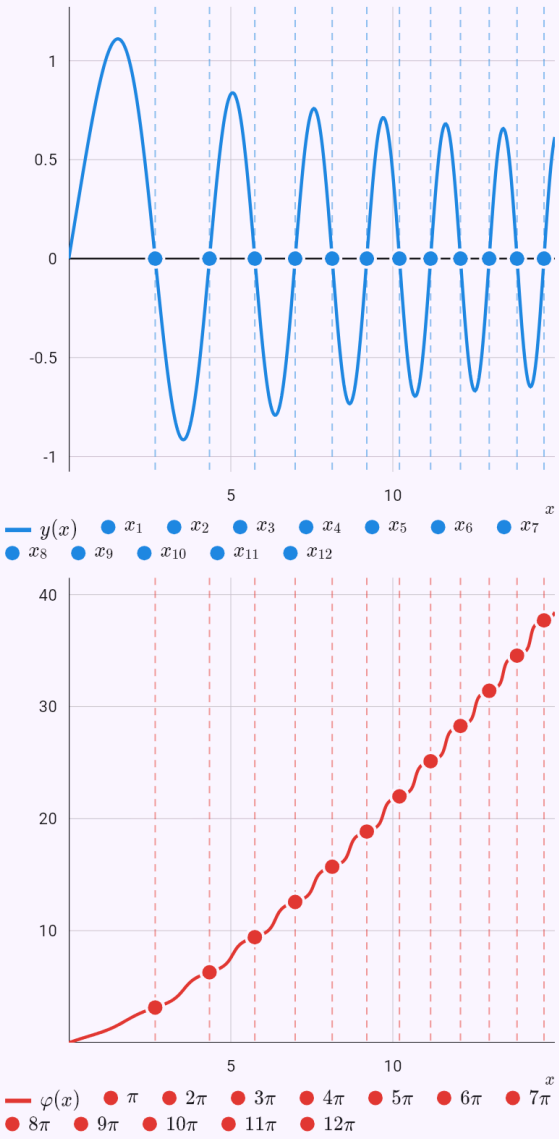
Result

The equation is oscillatory.

Steps

- We study the equation $y'' + xy = 0$ on the interval $[0,\, \infty)$, i.e. y" + q(x)y = 0 with q = x
- The equation is oscillatory if every nontrivial solution has infinitely many zeros on the interval. By Sturm's separation theorem the zeros of two independent solutions alternate, so one solution is enough: either all solutions oscillate or none does.
- The Leighton–Wintner criterion: if ∫ 1/r = ∞ and ∫ q = ∞, the equation is oscillatory. Here $\int_0^\infty 1\,\mathrm{d}x = \infty$ and $\int_0^\infty x\,\mathrm{d}x = \infty$, so the equation is oscillatory.
- Kneser's criterion with the critical constant $\frac{1}{4}$: $\lim_{x\to\infty} x^2 \cdot x = \infty$ > 1/4, the equation is oscillatory.
- The Riccati technique: the substitution $w = \frac{y'}{y}$ turns the equation into the Riccati equation $w' + w^2 + x = 0$. The equation is nonoscillatory exactly when the Riccati equation has a solution on a whole interval ⟨T; ∞) for some T. For an oscillatory equation there is none: w "escapes" to −∞ at every zero of y.
- The Prüfer transformation $y = \rho \sin\varphi,\ y' = \rho\cos\varphi$ turns the equation into an equation for the angle φ: $\varphi' = \cos^2\varphi + x\sin^2\varphi$. At a zero of y the angle φ is a multiple of π and φ' > 0, so φ only grows across multiples of π. The number of zeros on (a; b⟩ is therefore ⌊φ(b)/π⌋. We compute numerically by the Runge–Kutta method (RK4) from the initial values $y(0) = 0,\ y'(0) = 1,\ \varphi(0) = 0$
- On the interval $(0,\, 15]$ the solution has this many zeros: 12; $\varphi(15) \doteq 38.4387 \doteq 12.2354\pi$. Neighbouring zeros are 0.8322 to 1.6761 apart. The zeros $x_1 \doteq 2.6664,\ x_2 \doteq 4.3425,\ x_3 \doteq 5.741,\ x_4 \doteq 6.9861,\ $ ⋯
- Sturm's comparison theorem: on the interval $[0,\, 15]$ we have $q \in [0,\, 15]$. q is not positive everywhere there, so we only get a lower bound of the distance of neighbouring zeros: $0.8112 \le \Delta$
- Conclusion: the equation is oscillatory – every nontrivial solution has infinitely many zeros.

Graph



The solution with its zeros marked; below it Prüfer's angle φ, passing multiples of π at the zeros.

Euler's equation at the boundary k = 1/4 – not oscillatory yet.

Input: `oscillation(y'' + 1/(4x^2) y = 0)`

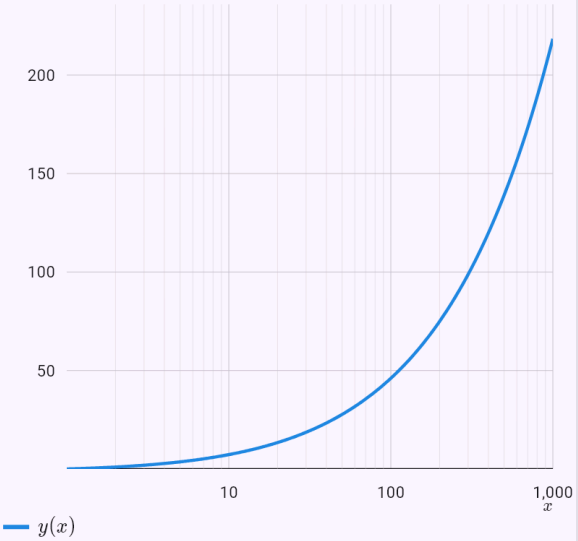
Result

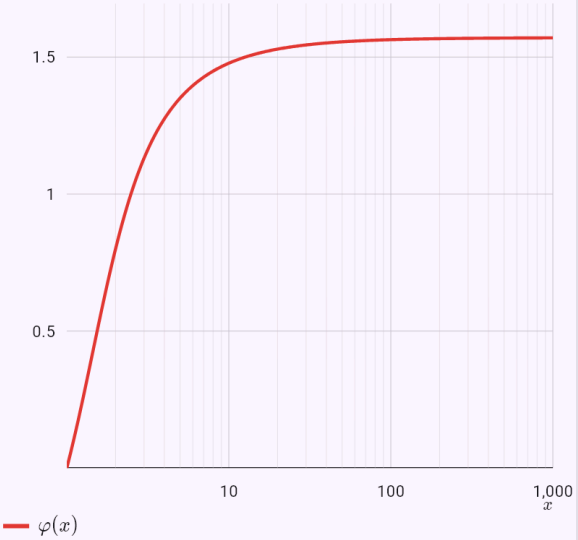
The equation is not oscillatory.

Steps

- We study the equation $y'' + \frac{1}{4x^2} y = 0$ on the interval $[1,\ \infty)$, i.e. y" + q(x)y = 0 with q = $\frac{1}{4x^2}$
- The equation is oscillatory if every nontrivial solution has infinitely many zeros on the interval. By Sturm's separation theorem the zeros of two independent solutions alternate, so one solution is enough: either all solutions oscillate or none does.
- This is Euler's equation y" + (k/x²)y = 0 with the boundary value k = $\frac{1}{4}$. The characteristic equation $\lambda(\lambda - 1) + \frac{1}{4} = 0$ has the double root $\lambda_{1,2} = \frac{1}{2}$, the solution $y = \sqrt{x}(C_1 + C_2 \ln x)$ has at most one zero. The equation is not oscillatory yet.
- The Leighton–Wintner criterion: if ∫ 1/r = ∞ and ∫ q = ∞, the equation is oscillatory. Here $\int_1^\infty 1\,\mathrm{d}x = \infty$, but $\int_1^\infty \frac{1}{4x^2}\,\mathrm{d}x = \frac{1}{4}$, so the criterion does not decide.
- Kneser's criterion: $\lim_{x\to\infty} x^2 \cdot \frac{1}{4x^2} = \frac{1}{4}$ equals the critical constant $\frac{1}{4}$, so it does not decide by itself. The next term decides.
- In the critical case we compare with the Riemann–Weber equation y" + (1/(4x²) + μ/(x² ln² x))y = 0, which oscillates exactly for μ > 1/4: $\lim_{x\to\infty} \left(x^2 \cdot \frac{1}{4x^2} - \frac{1}{4}\right) \ln^2 x = 0 < \frac{1}{4}$, the equation is not oscillatory.
- The Riccati technique: the substitution $w = \frac{y'}{y}$ turns the equation into the Riccati equation $w' + w^2 + \frac{1}{4x^2} = 0$. It has the solution $w = \frac{1}{2x}$ on the whole interval – it belongs to a solution y without zeros, so the equation is not oscillatory.
- The Prüfer transformation $y = \rho \sin\varphi,\ y' = \rho\cos\varphi$ turns the equation into an equation for the angle φ: $\varphi' = \cos^2\varphi + \frac{1}{4x^2}\sin^2\varphi$. At a zero of y the angle φ is a multiple of π and φ' > 0, so φ only grows across multiples of π. The number of zeros on (a; b⟩ is therefore ⌊φ(b)/π⌋. We compute numerically by the Runge–Kutta method (RK4) from the initial values $y(1) = 0,\ y'(1) = 1,\ \varphi(1) = 0$
- On the interval $(1,\ 1000]$ the solution has no zero: $\varphi(1000) \doteq 1.5702 \doteq 0.4998\pi$, the angle did not reach π.
- Conclusion: the equation is not oscillatory – every nontrivial solution has only finitely many zeros.

Graph





The solution has no zero on the chosen interval; Prüfer's angle φ below it does not reach π.

Kneser's criterion: x²q → 1 > 1/4, the zeros grow geometrically.

Input: `oscillation(y'' + 1/(x^2 + 1) y = 0)`

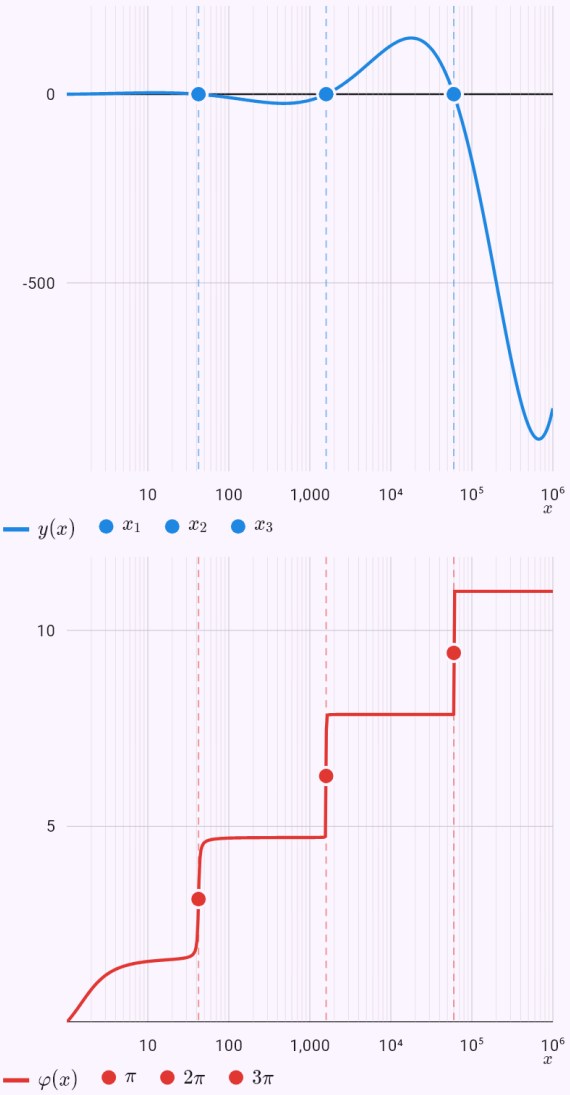
Result

The equation is oscillatory.

Steps

- We study the equation $y'' + \frac{1}{x^2+1} y = 0$ on the interval $[0,\ \infty)$, i.e. y" + q(x)y = 0 with q = $\frac{1}{x^2+1}$
- The equation is oscillatory if every nontrivial solution has infinitely many zeros on the interval. By Sturm's separation theorem the zeros of two independent solutions alternate, so one solution is enough: either all solutions oscillate or none does.
- The Leighton–Wintner criterion: if ∫ 1/r = ∞ and ∫ q = ∞, the equation is oscillatory. Here $\int_0^\infty 1\,dx = \infty$, but $\int_0^\infty \frac{1}{x^2+1}\,dx = \frac{\pi}{2} \doteq 1.5708$, so the criterion does not decide.
- Kneser's criterion with the critical constant $\frac{1}{4}$: $\lim_{x\to\infty} x^2 \cdot \frac{1}{x^2+1} = 1$ > 1/4, the equation is oscillatory.
- The Riccati technique: the substitution $w = \frac{y'}{y}$ turns the equation into the Riccati equation $w' + w^2 + \frac{1}{x^2+1} = 0$. The equation is nonoscillatory exactly when the Riccati equation has a solution on a whole interval ⟨T; ∞) for some T. For an oscillatory equation there is none: w "escapes" to −∞ at every zero of y.
- The Prüfer transformation $y = \rho \sin\varphi,\ y' = \rho\cos\varphi$ turns the equation into an equation for the angle φ: $\varphi' = \cos^2\varphi + \frac{1}{x^2+1}\sin^2\varphi$. At a zero of y the angle φ is a multiple of π and φ' > 0, so φ only grows across multiples of π. The number of zeros on (a; b⟩ is therefore ⌊φ(b)/π⌋. We compute numerically by the Runge–Kutta method (RK4) from the initial values $y(1) = 0,\ y'(1) = 1,\ \varphi(1) = 0$
- On the interval $(1,\ 1\,000\,000]$ the solution has this many zeros: 3; $\varphi(1\,000\,000) \doteq 10.9956 \doteq 3.5\pi$. Neighbouring zeros are 1548.7974 to 58269.3046 apart. The zeros: $x_1 \doteq 42.2876,\ x_2 \doteq 1591.085,\ x_3 \doteq 59860.3896$
- Conclusion: the equation is oscillatory – every nontrivial solution has infinitely many zeros.

Graph



The solution with its zeros marked; below it Prüfer's angle φ, passing multiples of π at the zeros.

The half-linear Euler equation, k = 1 > (2/3)³.

Input: `oscillation((|y'| y')' + 1/x^3 |y| y = 0)`

Result

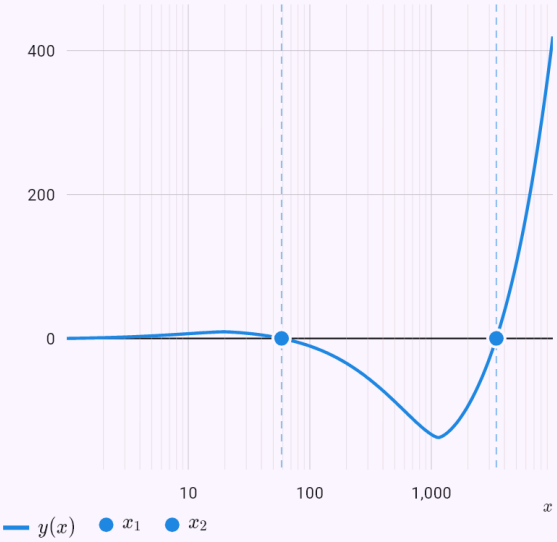
The equation is oscillatory.

Steps

- We study the half-linear equation $(\Phi(y'))' + \frac{1}{x^3}\Phi(y) = 0$ with p = 3 on the interval $[1,\ \infty)$, where $\Phi(s) = |s|^{1}s$. A multiple of a solution is again a solution, but the sum of two solutions in general is not. For p = 2 it is the linear equation y" + q(x)y = 0.
- The equation is oscillatory if every nontrivial solution has infinitely many zeros on the interval. Sturm's separation theorem holds for half-linear equations too: either all solutions oscillate or none does.
- This is the half-linear Euler equation with k = 1. Substituting y = x^λ with 0 < λ < 1 gives k = $G(\lambda) = (p-1)\lambda^{p-1}(1-\lambda)$ for p = 3. The function G attains its maximum, the critical constant $\gamma_p = \left(\frac{p-1}{p}\right)^p = \frac{8}{27} \doteq 0.2963$, at λ = (p − 1)/p. Here k is larger, there is no solution x^λ and the equation is oscillatory.
- The Leighton–Wintner criterion (if ∫ q = ∞, the equation is oscillatory) does not decide here: $\int_1^{\infty} \frac{1}{x^3}\,\mathrm{d}x = \frac{1}{2}$
- Kneser's criterion for half-linear equations with the critical constant $\gamma_p = \frac{8}{27}$: $\lim_{x\to\infty} x^3 \cdot \frac{1}{x^3} = 1$ is larger, the equation is oscillatory.
- The Riccati technique for a half-linear equation: the substitution $w = \frac{\Phi(y')}{\Phi(y)}$ leads to the Riccati equation $w' + \frac{1}{x^3} + 2|w|^{\frac{3}{2}} = 0$. The equation is nonoscillatory exactly when it has a solution on a whole interval ⟨T; ∞).

- Numerically (RK4, y(a) = 0, y'(a) = 1) the solution has this many zeros on the interval $(1,\ 10\,000]$: 2, 3369.2413 apart. The zeros: $x_1 \doteq 58.5473,\ x_2 \doteq 3427.7887$
- Conclusion: the equation is oscillatory – every nontrivial solution has infinitely many zeros.

Graph



The solution with its zeros marked.

The discrete Kneser criterion: n²q_n = 1/8 < 1/4.

Input: `oscillation(Δ^2 y_n + 1/(8n^2) y_{n+1} = 0)`

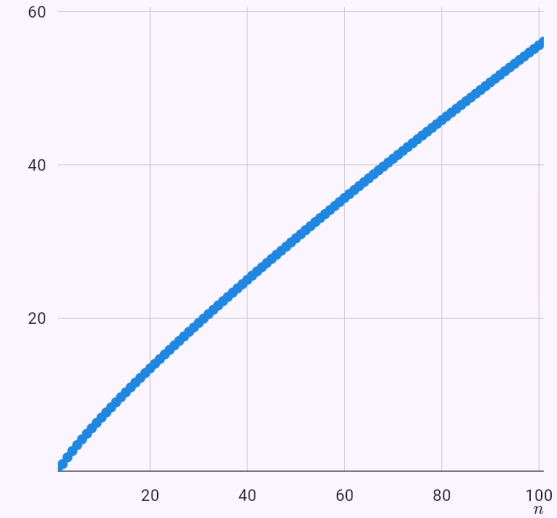
Result

The equation is not oscillatory.

Steps

- We study the discrete counterpart of (r y')' + q y = 0, the linear recurrence $\Delta^2 y_n + \frac{1}{8n^2} y_{n+1} = 0$ for $n \geq 1$, where $\Delta y_n = y_{n+1} - y_n$. As a recurrence: $y_{n+2} = \left(2 - \frac{1}{8n^2}\right) y_{n+1} - y_n$
- For a sequence we count generalized zeros: $y_n = 0$, or a change of sign between y_n and y_{n+1}. The equation is oscillatory if every nontrivial solution has infinitely many generalized zeros. Sturm's separation theorem holds here too.
- The discrete Leighton–Wintner criterion: if $\Sigma\, 1/r_n = \infty$ and $\Sigma\, q_n = \infty$, the equation is oscillatory. Here $\sum_{n=1}^{\infty} 1 = \infty$, but $\sum_{n=1}^{\infty} \frac{1}{8n^2} < \infty$, so the criterion does not decide.
- The discrete Kneser criterion with the critical constant $\frac{1}{4}$: $\lim_{n\to\infty} n^2 \cdot \frac{1}{8n^2} = \frac{1}{8}$ < 1/4, the equation is not oscillatory.
- The discrete Riccati technique: the substitution $w_n = \frac{\Delta y_n}{y_n}$ leads to $\Delta w_n + \frac{1}{8n^2} + \frac{w_n^2}{w_n + 1} = 0$. The equation is nonoscillatory exactly when this equation eventually has a solution with $w_n + r_n > 0$.
- The solution with $y_{n0} = 0$, $y_{n0+1} = 1$ has no generalized zero for n in $(1,\ 101]$
- Conclusion: the equation is not oscillatory – every nontrivial solution has only finitely many zeros.

Graph



The solution has no zero on the chosen interval.

Example 4.2 of Došlá and Marini – the case C5: regularly increasing and strongly decaying solutions.

Input: `asymptotics((e^(-3x) (y')^3)' = x^(-2) e^(-3x) y^3)`

Result

$$M^+ = M^+_{\infty,\ell} \neq \emptyset,\ M^- = M^-_{0,0} \neq \emptyset$$

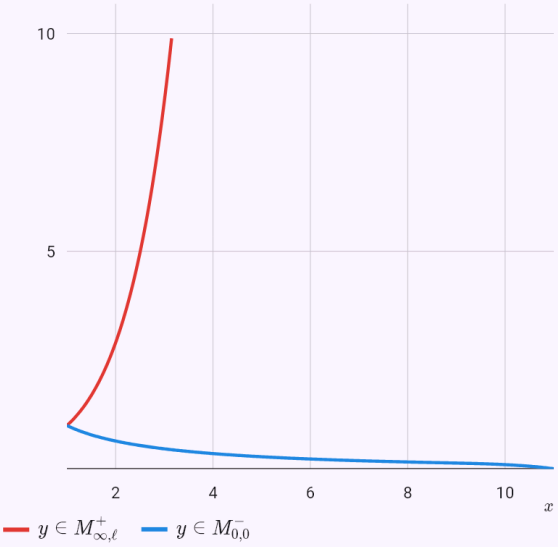
Steps

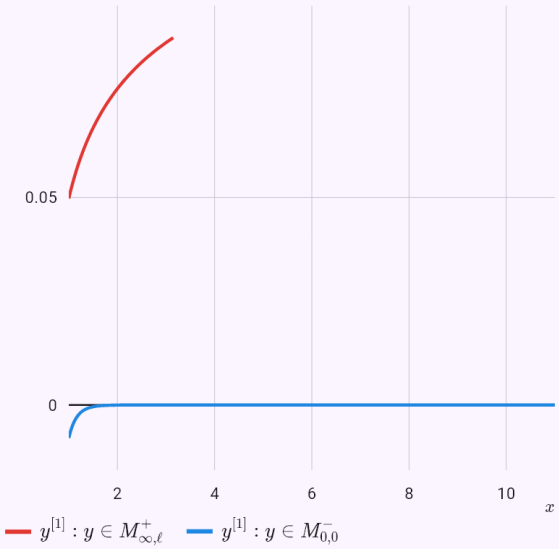
- We study the half-linear equation $\left(e^{-3x}|y'|^2y'\right)' = \frac{e^{-3x}}{x^2}|y|^2y$ on the interval $[1,\ \infty)$ with α = 3, a = e^{-3x} > 0 and b = $\frac{e^{-3x}}{x^2}$ > 0. It is an equation with the p-Laplacian for p = α + 1 and a "negative potential" – it does not oscillate. The general form of the equation is $(a(x)|y'|^\alpha \operatorname{sgn} y')' = b(x)|y|^\alpha \operatorname{sgn} y$
- A solution's behaviour is described together with y by its quasiderivative $y^{[1]} = e^{-3x}|y'|^3 \operatorname{sgn} y'$. Every nontrivial solution belongs to one of two classes: M^+ – y·y' > 0 from some x on and |y| grows – or M^- – y·y' < 0 on the whole interval and |y| decreases. The classes split by the limits of |y| and $|y^{[1]}|$ as x → ∞: the index ∞ (infinite), ℓ (finite nonzero) or 0 (zero), for example $M^+_{\infty,\ell}$
- By Theorem 2.1 (Došlá–Marini) every solution is defined on the whole interval and both classes M^+ and M^- are nonempty. For example, the solutions with $y(x_0) > 0$ and $y'(x_0) \geq 0$ belong to M^+
- The integral A: as x → ∞ the integrand behaves like e^x (up to a positive constant). The exponential decides, hence $\int_1^\infty e^x\,\mathrm{d}x = \infty$
- The integral B $\int_1^\infty \frac{e^{-3x}}{x^2}\,\mathrm{d}x = \frac{1}{e^3} + 3\,\mathrm{Ei}\,(-3) \doteq 0.0106$
- The integral J_1: as x → ∞ the integrand behaves like e^x (up to a positive constant, with the inner integral replaced by its asymptotic behaviour). The exponential decides, hence $\int_1^\infty e^x \left(\int_1^x \frac{e^{-3s}}{s^2}\,\mathrm{d}s\right)^{\frac{1}{3}} \mathrm{d}x = \infty$
- The integral J_2: as x → ∞ the integrand behaves like $x^{-\frac{2}{3}}$ (up to a positive constant, with the inner integral replaced by its asymptotic behaviour). By the comparison test with $\int x^\mu\,dx$, which converges exactly for μ < −1 $\int_1^\infty e^x \left(\int_x^\infty \frac{e^{-3s}}{s^2}\,\mathrm{d}s\right)^{\frac{1}{3}} \mathrm{d}x = \infty$
- The integral Y_1 diverges, since the inner integral already diverges $\int_1^\infty \frac{e^{-3x}}{x^2} \left(\int_x^\infty e^s\,\mathrm{d}s\right)^3 \mathrm{d}x = \infty$
- The integral Y_2: as x → ∞ the integrand behaves like x^{-2} (up to a positive constant, with the inner integral replaced by its asymptotic behaviour). By the comparison test with $\int x^\mu\,dx$, which converges exactly for μ < −1 $\int_1^\infty \frac{e^{-3x}}{x^2} \left(\int_1^x e^s\,\mathrm{d}s\right)^3 \mathrm{d}x < \infty$
- Lemma 2.1 (an extension of Fubini's theorem): for α = 3 ≥ 1, $Y_2 = \infty$ implies $J_2 = \infty$ and $Y_1 = \infty$ implies $J_1 = \infty$.
- The integrals came out as $J_1 = \infty,\ J_2 = \infty,\ Y_1 = \infty,\ Y_2 < \infty$ – this is the case C_5 of the classification (Theorem 3.1).
- The reciprocal equation $\left(\sqrt[3]{x^2}e^x|y'|^{-\frac{2}{3}}y'\right)' = e^x|y|^{-\frac{2}{3}}y$ (a and b swap their roles and α becomes 1/α) has the quasiderivatives of the given equation's solutions as its

solutions. Its integrals J_1, J_2 read as Y_2, Y_1, so the case C_8 occurs. The classes correspond with their indices swapped: $M^+_{\infty,\ell}$ of the given equation corresponds to $M^+_{\ell,\infty}$ of the reciprocal one.

- $M^+_{\infty,\ell} \neq \varnothing$: regularly increasing solutions – $|y| \to \infty$, but the quasiderivative has a finite nonzero limit; y grows like $\int a^{-1/\alpha}\,dx$ (Theorem 3.1).
- $M^-_{0,0} \neq \varnothing$: strongly decaying solutions – $y \to 0$ and $y^{[1]} \to 0$ (Theorem 3.1).
- The solution from M^- in the graph is found numerically by shooting: we look for the quasiderivative's initial value for which the solution with $y(1) = 1$ neither reaches zero nor starts to grow on a long interval – $y^{[1]}(1) \doteq -0.008$. That value is unique, since $\int (a^{-1/\alpha} + b)\,dx = \infty$ (Theorem 4).

Graph





Above, numerical solutions y(x) from the nonempty classes (M^+ from $y(x_0) = 1$, $y'(x_0) = 1$, M^- by shooting); below, their quasiderivatives y^[1](x). A curve ends where |y| exceeds 10.

The case C4 – bounded increasing solutions and a whole interval of decreasing ones.

Input: `asymptotics((x^4 y')' = y/x^4)`

Result

$$M^+ = M^+_{\ell,\ell} \neq \emptyset,\ M^-_{\ell,\ell} \neq \emptyset,\ M^-_{\ell,0} \neq \emptyset,$$

Oscillation of the equation

Steps

- We study the linear equation $\left(x^4 y'\right)' = \frac{1}{x^4} y$ on the interval $[1,\ \infty)$, that is (a(x)y')' = b(x)y with a = x^4 > 0 and b = $\frac{1}{x^4}$ > 0. Written as (a y')' + q y = 0 it has q = −b negative, so the equation is not oscillatory – every solution has at most one zero.
- A solution's behaviour is described together with y by its quasiderivative $y^{[1]} = x^4 y'$. Every nontrivial solution belongs to one of two classes: M^+ – y·y' > 0 from some x on and |y| grows – or M^- – y·y' < 0 on the whole interval and |y| decreases. The classes split by the limits of |y| and $|y^{[1]}|$ as x → ∞: the index ∞ (infinite), ℓ (finite nonzero) or 0 (zero), for example $M^+_{\infty,\ell}$
- By Theorem 2.1 (Došlá–Marini) every solution is defined on the whole interval and both classes M^+ and M^- are nonempty. For example, the solutions with y(x_0) > 0 and y'(x_0) ≥ 0 belong to M^+
- The integral A: $\int_1^\infty \frac{1}{x^4}\,\mathrm{d}x = \frac{1}{3}$
- The integral B: $\int_1^\infty \frac{1}{x^4}\,\mathrm{d}x = \frac{1}{3}$
- The integral J_1: as x → ∞ the integrand behaves like x^{-4} (up to a positive constant, with the inner integral replaced by its asymptotic behaviour). By the comparison test with $\int x^\mu\,dx$, which converges exactly for μ < −1

$$\int_1^\infty \frac{1}{x^4}\left(\int_1^x \frac{1}{s^4}\,\mathrm{d}s\right)^1 \mathrm{d}x < \infty$$

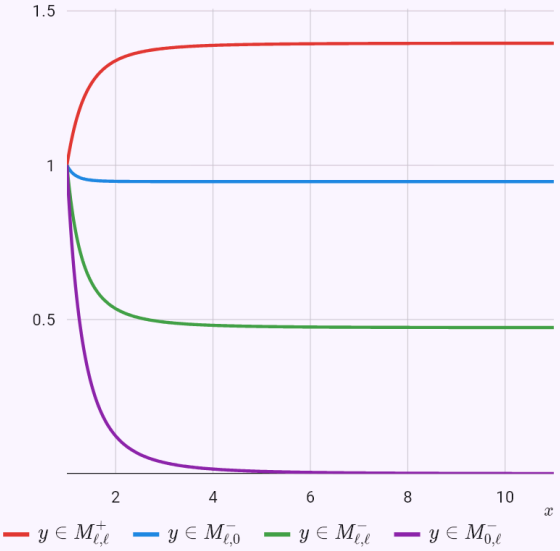
- The integral J_2: as x → ∞ the integrand behaves like x^{-7} (up to a positive constant, with the inner integral replaced by its asymptotic behaviour). By the comparison test with $\int x^\mu\,dx$, which converges exactly for μ < −1

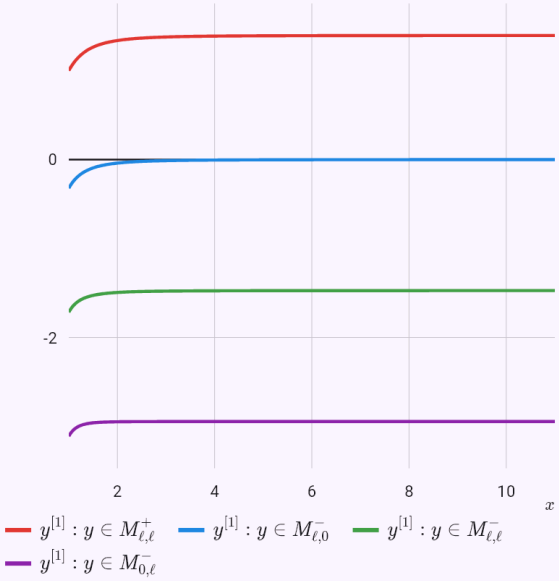
$$\int_1^\infty \frac{1}{x^4}\left(\int_x^\infty \frac{1}{s^4}\,\mathrm{d}s\right)^1 \mathrm{d}x < \infty$$

- The integrals came out as $J_1 < \infty,\ J_2 < \infty$ – this is the case C_4 of the classification (Theorem 3.1).
- $M^+_{\ell,\ell} \neq \emptyset$: increasing solutions with finite limits of y and of the quasiderivative (Theorem 3.1).
- $M^-_{\ell,\ell} \neq \emptyset$: decreasing solutions where y and the quasiderivative have nonzero limits (Theorem 3.1).
- $M^-_{\ell,0} \neq \emptyset$: solutions decreasing to a nonzero limit, the quasiderivative → 0 (Theorem 3.1).
- $M^-_{0,\ell} \neq \emptyset$: regularly decaying solutions – y → 0, but the quasiderivative has a nonzero limit; y decreases like $\int_x^\infty a^{-1/\alpha}\,ds$ (Theorem 3.1).
- $M^-_{0,0} = \emptyset$ – no solution of this kind (Theorem 3.1).
- The solutions from M^- in the graph are found numerically by shooting: with $y(1) = 1$ they stay positive and decreasing for a whole interval of the

quasiderivative's initial values $y^{[1]}(1) \in [-3.1103,\ -0.3215]$, since $\int (a^{-1/\alpha} + b)\,dx < \infty$ (Theorem 4). The interval's ends and its midpoint give solutions of different classes.

Graph





Above, numerical solutions y(x) from the nonempty classes (M^+ from $y(x_0) = 1$, $y'(x_0) = 1$, M^- by shooting); below, their quasiderivatives y^[1](x). A curve ends where |y| exceeds 10.

A superlinear equation – $M^+ = \emptyset$, $M^- = M^-_{0,0}$ (the solution $\sqrt{2}/(x+\sqrt{2})$).

Input: `asymptotics(y'' = y^3)`

Result

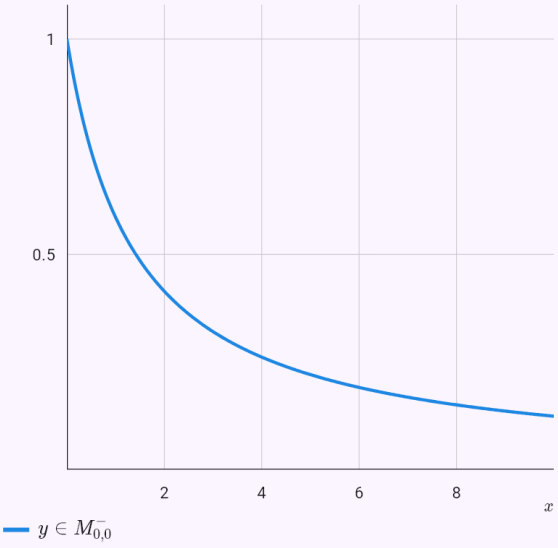
$$M^+ = \emptyset,\ M^- = M^-_{0,0} \neq \emptyset$$

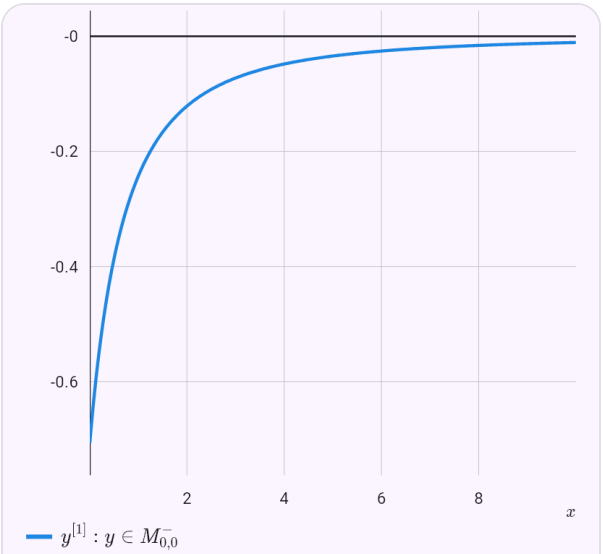
Steps

- We study the Emden–Fowler type equation $y'' = |y|^2 y$ on the interval $[0, \infty)$ with α = 1, β = 3, a = 1 > 0 and b = 1 > 0. Since β > α, the equation is superlinear. The general form of the equation is $(a(x)|y'|^\alpha \operatorname{sgn} y')' = b(x)|y|^\beta \operatorname{sgn} y$
- A solution's behaviour is described together with y by its quasiderivative $y^{[1]} = y'$. Every nontrivial solution defined up to infinity belongs to one of two classes: M^+ – y·y' > 0 from some x on and |y| grows – or M^- – y·y' < 0 and |y| decreases. The classes split by the limits of |y| and $|y^{[1]}|$ as x → ∞: the index ∞ (infinite), ℓ (finite nonzero) or 0 (zero).
- By Theorem 1 (Cecchi–Došlá–Marini) the class M^- is nonempty for β > α. Some solutions, however, "blow up": |y| → ∞ already at a finite point, so they are not defined up to infinity. The class M^+ may be empty.
- The integral A: as x → ∞ the integrand behaves like 1 (up to a positive constant). By the comparison test with $\int x^\mu\,dx$, which converges exactly for μ < −1, $\int_0^\infty 1\,\mathrm{d}x = \infty$
- The integral B: as x → ∞ the integrand behaves like 1 (up to a positive constant). By the comparison test with $\int x^\mu\,dx$, which converges exactly for μ < −1, $\int_0^\infty 1\,\mathrm{d}x = \infty$
- The integral J_1: as x → ∞ the integrand behaves like x (up to a positive constant, with the inner integral replaced by its asymptotic behaviour). By the comparison test with $\int x^\mu\,dx$, which converges exactly for μ < −1, $\int_0^\infty \left(\int_0^x 1\,\mathrm{d}s\right)^1 \mathrm{d}x = \infty$
- The integral $Y_2^{(\beta)}$: as x → ∞ the integrand behaves like x^3 (up to a positive constant, with the inner integral replaced by its asymptotic behaviour). By the comparison test with $\int x^\mu\,dx$, which converges exactly for μ < −1, $\int_0^\infty \left(\int_0^x 1\,\mathrm{d}s\right)^3 \mathrm{d}x = \infty$
- The integral $Y_1^{(\beta)}$ diverges, since the inner integral already diverges: $\int_0^\infty \left(\int_x^\infty 1\,\mathrm{d}s\right)^3 \mathrm{d}x = \infty$
- $M^+ = \varnothing$ – no solution of this kind (Kiguradze, cited in Cecchi–Došlá–Marini).
- $M^-_{\ell,\ell} = \varnothing$ – no solution of this kind (Lemma 1).
- $M^-_{\ell,0} = \varnothing$ – no solution of this kind (Lemma 2).
- $M^-_{0,\ell} = \varnothing$ – no solution of this kind (Lemma 1).
- $M^-_{0,0} \neq \varnothing$: strongly decaying solutions – y → 0 and $y^{[1]} \to 0$ (Theorem 1).
- The solution from M^- in the graph is found numerically

by shooting: we look for the quasiderivative's initial value for which the solution with $y(0) = 1$ neither reaches zero nor starts to grow on a long interval – $y^{[1]}(0) \doteq -0.7071$. That value is unique, since $\int (a^{-1/\alpha} + b)\,dx = \infty$ (Theorem 4).

Graph





Above, numerical solutions y(x) from the nonempty classes (M^{+} from $y(x_0) = 1$, $y'(x_0) = 1$, M^{-} by shooting); below, their quasiderivatives y^[1](x). A curve ends where |y| exceeds 10.

Known limitations

- An equation that needs a hint from the problem (other than z = ln y) can't be solved yet.
- For an equation with a parameter (k, λ, ω) the app draws no graph and takes the parameters as positive.
- Where the solution can't be expressed explicitly, it stays implicit.
- The Laplace transform doesn't know the Heaviside function, shifts and periodic right sides yet.
- When the integrals of the variation of constants can’t be expressed by elementary functions (y″ + y′ + y = tan x), the solution keeps them as integrals; there is no graph then and no initial conditions are applied.
- Systems only of two linear equations with constant coefficients without a right-hand side; nonlinear systems only in the applications (models).
- A Riccati equation without a given particular solution is solved only if one of a simple form is guessed – otherwise the app says one is needed.
- A Lagrange equation only without an initial condition; for a Clairaut equation the condition picks the lines (and the envelope) through the point.
- Oscillation only for equations with numerical coefficients (k must be a number); a half-linear equation only with r = 1. The zeros are counted numerically on a finite interval – only the criteria decide about infinity.

- The asymptotics of an Emden–Fowler type equation ($\beta \neq \alpha$) is decided only where the cited theorems say so – the other classes are marked undecided. The coefficients a, b must be positive; their behaviour at infinity is recognized for products of powers, logarithms and exponentials, otherwise only exact integrals by Symja.

37. Calculus of variations

The app looks for an extremal of a functional J[y] = ∫ F(x, y, y′) dx with the boundary conditions y(a), y(b): it sets up the Euler–Lagrange equation ∂F/∂y − d/dx ∂F/∂y′ = 0 (with the partial derivatives in the steps), shows the first integral of an integrand without y and solves the resulting differential equation by the ODE solver with the boundary conditions and a check. The problem is also a form in the fields Calculus and Physics (Applications).

If the equation has no closed-form solution and F doesn't depend on x, it uses Beltrami's identity and recognizes the known extremals: the brachistochrone (a cycloid, compared with the straight line, with the time of the slide), the catenary and the minimal surface of revolution (two, one or no catenary – Goldschmidt's solution), an isoperimetric problem with a Lagrange multiplier (a circular arc of a given length, a hanging chain). The constants are found numerically. The principle of least action turns a Lagrangian into the equation of motion, for a pendulum with sin θ ≈ θ. An interactive demo – a race of balls along the brachistochrone – is on mathtrace.app under “Try it yourself”.

How to enter it

`functional(sqrt(1 + y'^2); y(0) = 0; y(1) = 2)`	The integrand and the boundary conditions, separated by semicolons.
`functional(y; y(-1) = 0; y(1) = 0; sqrt(1 + y'^2) = pi)`	The constraint ∫ G dx = L as the last item.
`functional(m*x'^2/2 - k*x^2/2; action)`	A Lagrangian L = T − V and the word action – the equation of motion.

Examples

The shortest path between two points is a segment.

Input: `functional(sqrt(1 + y'^2); y(0) = 0; y(1) = 2)`

Result

$$y = 2x$$

Steps

- We look for the function for which the functional has an extremum: $J[y] = \int_0^1 \left(\sqrt{y'^2 + 1}\right) dx$, the boundary conditions $y(0) = 0, \quad y(1) = 2$
- The integrand's partial derivatives by y and y' and the total derivative by x (both y and y' depend on x)

$$\frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{y'^2 + 1}}, \quad \frac{d}{dx}\frac{\partial F}{\partial y'} = \frac{y''}{\sqrt{(y'^2 + 1)^3}}$$

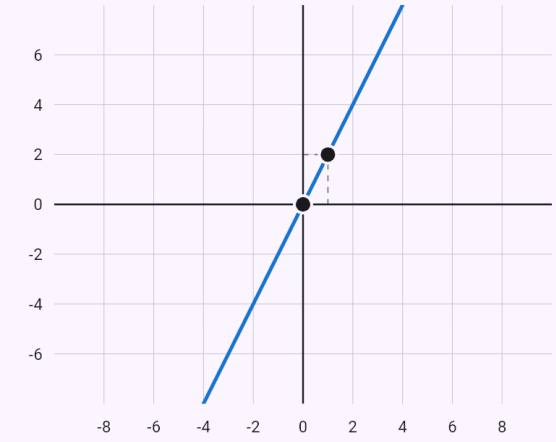
- The Euler–Lagrange equation ∂F/∂y − d/dx ∂F/∂y' = 0: $-y'' = 0$
- The integrand doesn't depend on y – the Euler–Lagrange equation has a first integral: $\frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{y'^2 + 1}} = C_1$
- We solve the differential equation:
- The right side is a function of the independent variable only – we integrate the equation step by step: $y'' = 0$
- Integrating both sides: $y' = \int 0 \, dx = C_1$
- Integrating both sides: $y = \int C_1 \, dx = C_1 x + C_2$
- We substitute the conditions into the general solution hence $C_1 = 2, \; C_2 = 0$

$$\begin{cases} C_2 = 0 \\ C_1 + C_2 = 2 \end{cases}$$

- The solution satisfying the given conditions: $y = 2x$

Graph

Solutions | Phase plane



$y = 2x$

The highlighted one satisfies the conditions, the points are the given values.

Your own values of the constants

C_1 C_2 + Add curve

Tap the graph and set the slope with the slider – the solution through that point with that tangent slope is drawn.

The Euler–Lagrange equation y'' = y with boundary conditions.

Input: `functional(y'^2 + y^2; y(0) = 0; y(1) = 1)`

Result

$$y = \frac{\sinh x}{\sinh 1}$$

Steps

- We look for the function for which the functional has an extremum: $J[y] = \int_0^1 \left(y^2 + y'^2\right) dx$, the boundary conditions $y(0) = 0, \quad y(1) = 1$
- The integrand's partial derivatives by y and y' and the total derivative by x (both y and y' depend on x)
  $\frac{\partial F}{\partial y} = 2y, \quad \frac{\partial F}{\partial y'} = 2y', \quad \frac{d}{dx}\frac{\partial F}{\partial y'} = 2y''$
- The Euler–Lagrange equation ∂F/∂y − d/dx ∂F/∂y' = 0: $-2(-y + y'') = 0$
- We solve the differential equation:
- A linear equation with constant coefficients: $y'' - y = 0$
- We look for solutions y = e^(λx); substituting and dividing by e^(λx) gives the characteristic equation: $\lambda^2 - 1 = 0$
- The roots of the characteristic equation: $\lambda_1 = -1, \ \lambda_2 = 1$
- Each root λ gives the solution e^(λx), a root of multiplicity k also x·e^(λx), …, x^(k−1)·e^(λx), and a pair α ± βi the solutions e^(αx)·cos βx and e^(αx)·sin βx. The fundamental system: $y_1 = e^{-x}, \ y_2 = e^x$
- The general solution: $y = C_1 e^{-x} + C_2 e^x$
- We substitute the conditions into the general solution hence $C_1 = -\frac{e}{e^2 - 1}, \ C_2 = \frac{e}{e^2 - 1}$
  $\begin{cases} C_1 + C_2 = 0 \\ \frac{C_1}{e} + C_2 e = 1 \end{cases}$
- The solution satisfying the given conditions: $y = \frac{e^{x+1}}{e^2 - 1} - \frac{e^{1-x}}{e^2 - 1}$
- The solution satisfying the given conditions: $y = \frac{\sinh x}{\sinh 1}$

The brachistochrone – a cycloid from A [0; 0] to B [2; 1], a slide of 0.806 s against 1.010 s along the

line.

Input: `functional(sqrt(1 + y'^2)/sqrt(2*9.81*y); y(0) = 0; y(2) = 1)`

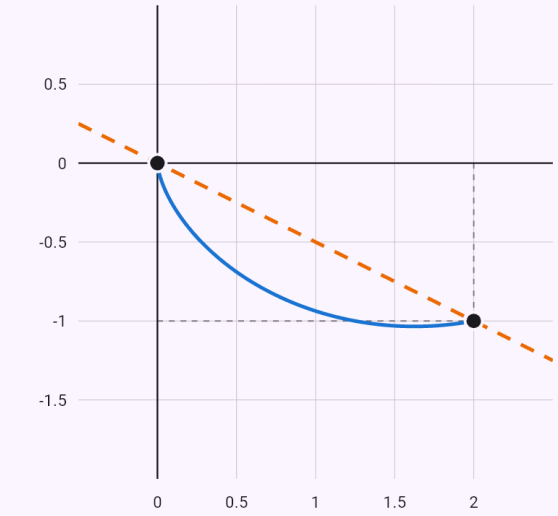
Result

$$x = 0.5172(\theta - \sin\theta),\ y = 0.5172(1 -$$

Steps

- We look for the function for which the functional has an extremum: $J[y] = \int_0^2 \left(\frac{0.225762\sqrt{y'^2+1}}{\sqrt{y}}\right) dx$, the boundary conditions $y(0) = 0, \quad y(2) = 1$
- The integrand's partial derivatives by y and y' and the total derivative by x (both y and y' depend on x)
$$\frac{\partial F}{\partial y} = \frac{-0.112881\sqrt{y'^2+1}}{\sqrt{y^3}}, \quad \frac{\partial F}{\partial y'} = \frac{0.225762y'}{\sqrt{y}\sqrt{y'^2+1}},$$
- The Euler–Lagrange equation ∂F/∂y − d/dx ∂F/∂y' = 0:
$$\sqrt{y'^2+1}\left(-0.225762yy'' - 0.112881y'^2 - 0.112881\right) = 0$$
- The integrand doesn't depend on x − Beltrami's identity F − y'·∂F/∂y' = C_1 holds: $\frac{0.225762}{\sqrt{y}\sqrt{y'^2+1}} = C_1$
- Beltrami's identity gives y·(1 + y'²) = const. The solution is a cycloid – the curve traced by a point on a rolling circle (the y-axis points down, the start at rest at A):
$x = r(\theta - \sin\theta), \quad y = r(1 - \cos\theta)$
- The constants from the end point B (θ_B numerically by bisection)
$$\frac{\theta_B - \sin\theta_B}{1 - \cos\theta_B} = \frac{2}{1} \Rightarrow \theta_B \approx 3.508, \quad r = \frac{1}{1 - \cos\theta_B} \approx$$
- The functional's value: $J_{min} = 0.2258 \cdot \theta_B\sqrt{2r} \approx 0.8056$; along the straight line $J_{y=0.5x} \approx 1.01$. For J = ∫ ds/v with v = √(2gy), J is the time of the slide:
$$T = \frac{J}{\sqrt{2g}} \approx 0.8056 \text{ s}$$
- θ_B > π – the fastest path dips below the level of B and then rises to B.

Graph



The y-axis points down – the path runs below point A. Solid the cycloid (the fastest slide), dashed the straight line to compare.

The minimal surface of revolution between two rings – two catenaries, the shallower is the minimum.

Input: `functional(y*sqrt(1 + y'^2); y(-1) = 2; y(1) = 2)`

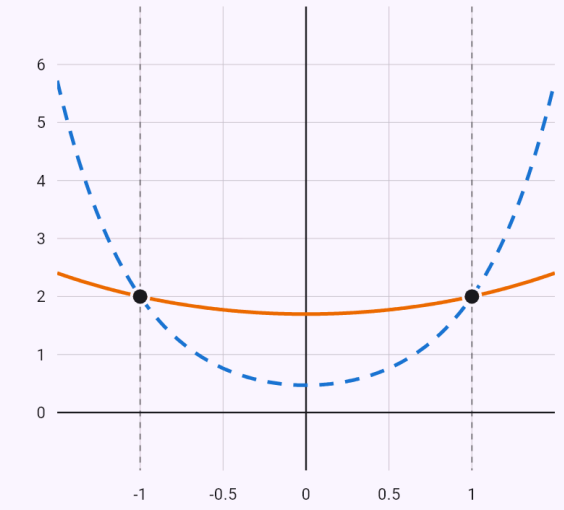
Result

$$y = 1.697 \cosh \frac{x}{1.697}; \quad J_{min} \approx 3.814$$

Steps

- We look for the function for which the functional has an extremum: $J[y] = \int_{-1}^{1} \left(y\sqrt{y'^2 + 1} \right) dx$, the boundary conditions $y(-1) = 2, \quad y(1) = 2$
- The integrand's partial derivatives by y and y' and the total derivative by x (both y and y' depend on x)
  $\frac{\partial F}{\partial y} = \sqrt{y'^2 + 1}, \quad \frac{\partial F}{\partial y'} = \frac{yy'}{\sqrt{y'^2 + 1}}, \quad \frac{d}{dx}\frac{\partial F}{\partial y'} = \frac{}{\sqrt{(y}$
- The Euler–Lagrange equation ∂F/∂y − d/dx ∂F/∂y' = 0: $\sqrt{y'^2 + 1}\left(-yy'' + y'^2 + 1\right) = 0$
- The integrand doesn't depend on x – Beltrami's identity F − y'·∂F/∂y' = C_1 holds: $\frac{y}{\sqrt{y'^2 + 1}} = C_1$
- Beltrami's identity gives a catenary (the hyperbolic cosine): $y = C \cosh \frac{x - x_0}{C}$
- Two catenaries satisfy the boundary conditions (numerically). The minimum is the one with the smaller J – the shallower one; the deeper one is only stationary.
  $C \approx 0.4702, \; x_0 \approx 0, \; J \approx 4.358; \quad C \approx 1.697, \; x_0 \approx 0,$

Graph



$C \approx 0.4702$
$C \approx 1.697, \; J_{min}$

The extremal between the given points; dashed the other solution or a curve to compare.

Rings far apart – no catenary (Goldschmidt's solution).

Input: `functional(y*sqrt(1 + y'^2); y(-2) = 1; y(2) = 1)`

Result

There is no solution.

Steps

- We look for the function for which the functional has an extremum: $J[y] = \int_{-2}^{2} \left(y\sqrt{y'^2+1}\right) dx$, the boundary conditions $y(-2) = 1, \quad y(2) = 1$
- The integrand's partial derivatives by y and y' and the total derivative by x (both y and y' depend on x)
  $\frac{\partial F}{\partial y} = \sqrt{y'^2+1}, \quad \frac{\partial F}{\partial y'} = \frac{yy'}{\sqrt{y'^2+1}}, \quad \frac{d}{dx}\frac{\partial F}{\partial y'} = \frac{}{\sqrt{(y}$
- The Euler–Lagrange equation ∂F/∂y − d/dx ∂F/∂y' = 0: $\sqrt{y'^2+1}\left(-yy'' + y'^2 + 1\right) = 0$
- The integrand doesn't depend on x – Beltrami's identity F − y'·∂F/∂y' = C_1 holds: $\frac{y}{\sqrt{y'^2+1}} = C_1$
- Beltrami's identity gives a catenary (the hyperbolic cosine): $y = C\cosh\frac{x - x_0}{C}$
- No catenary passes through both points – the rings are too small for their distance. The minimal surface is then just the two discs and the axis between them (Goldschmidt's solution); there is no smooth extremal.

The largest area under a curve of length π – a semicircle.

Input: `functional(y; y(-1) = 0; y(1) = 0; sqrt(1 + y'^2) = pi)`

Result

$$x^2 + y^2 = 1, \quad R \approx 1$$

Steps

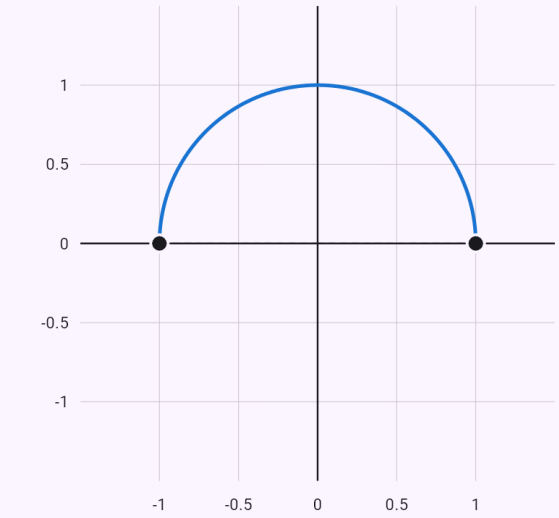
- We look for the function for which the functional has an extremum: $J[y] = \int_{-1}^{1} y\,dx$, the boundary conditions $y(-1) = 0, \quad y(1) = 0$
- An isoperimetric problem with the constraint $\int_{-1}^{1} \left(\sqrt{y'^2 + 1}\right) dx = \pi$: we attach it by a Lagrange multiplier λ and look for an extremum of the functional with the integrand $F^* = F + \lambda G = y + \lambda\sqrt{y'^2 + 1}$
- The integrand's partial derivatives by y and y' and the total derivative by x (both y and y' depend on x)

$$\frac{\partial F}{\partial y} = 1, \quad \frac{\partial F}{\partial y'} = \frac{\lambda y'}{\sqrt{y'^2 + 1}}, \quad \frac{d}{dx}\frac{\partial F}{\partial y'} = \frac{\lambda y''}{\sqrt{(y'^2 + 1)^3}}$$

- The Euler–Lagrange equation ∂F/∂y − d/dx ∂F/∂y' = 0: $y'^2\sqrt{y'^2 + 1} - \lambda y'' + \sqrt{y'^2 + 1} = 0$
- The integrand doesn't depend on x – Beltrami's identity F − y'·∂F/∂y' = C_1 holds: $y + \frac{\lambda}{\sqrt{y'^2 + 1}} = C_1$
- Beltrami's identity for F + λG gives a circle – an arc of radius |λ|: $(x - x_0)^2 + (y - y_0)^2 = R^2$
- The constants from the arc's length and the chord, the central angle 2α (numerically)

$$\frac{\sin\alpha}{\alpha} = \frac{2}{3.142} \Rightarrow \alpha \approx 1.571, \quad R = \frac{2}{2\sin\alpha} \approx 1, \quad S$$

Graph



$x^2 + y^2 = 1, \quad R \approx 1$

The extremal between the given points; dashed the other solution or a curve to compare.

A hanging chain of length 3 – a catenary with a Lagrange multiplier.

Input: `functional(y*sqrt(1 + y'^2); y(-1) = 2; y(1) = 2; sqrt(1 + y'^2) = 3)`

Result

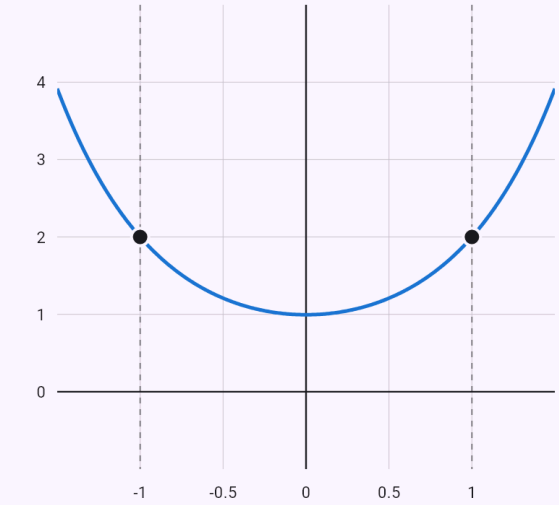
$$y = 0.3783 + 0.6165 \cosh \frac{x}{0.6165}$$

Steps

- We look for the function for which the functional has an extremum: $J[y] = \int_{-1}^{1} \left(y\sqrt{y'^2+1}\right) dx$, the boundary conditions $y(-1) = 2, \quad y(1) = 2$
- An isoperimetric problem with the constraint $\int_{-1}^{1} \left(\sqrt{y'^2+1}\right) dx = 3$: we attach it by a Lagrange multiplier λ and look for an extremum of the functional with the integrand $F^* = F + \lambda G = y\sqrt{y'^2+1} + \lambda\sqrt{y'^2+1}$
- The integrand's partial derivatives by y and y' and the total derivative by x (both y and y' depend on x) $\frac{\partial F}{\partial y} = \sqrt{y'^2+1}, \quad \frac{\partial F}{\partial y'} = \frac{y'(y+\lambda)}{\sqrt{y'^2+1}}, \quad \frac{d}{dx}\frac{\partial F}{\partial y'} = \frac{y'^4 +}$
- The Euler–Lagrange equation ∂F/∂y − d/dx ∂F/∂y' = 0: $-yy'' + y'^2 - \lambda y'' + 1 = 0$
- The integrand doesn't depend on x – Beltrami's identity F − y'·∂F/∂y' = C_1 holds: $\frac{y+\lambda}{\sqrt{y'^2+1}} = C_1$
- Beltrami's identity for F + λG gives a catenary shifted by λ: $y = c + C\cosh\frac{x-x_0}{C}$
- The constants from the chain's length and the boundary conditions (numerically)

$$2C\sinh\frac{2}{2C} = \sqrt{L^2 - \Delta y^2} \;\Rightarrow\; C \approx 0.6165, \quad x_0 \approx 0,$$

Graph



$y = 0.3783 + 0.6165 \cosh \frac{x}{0.6165}$

The extremal between the given points; dashed the other solution or a curve to compare.

The principle of least action – a spring, the equation of motion m x'' + k x = 0.

Input: `functional(m*x'^2/2 - k*x^2/2; action)`

Result

$$x = C_1 \cos\left(\frac{\sqrt{k}t}{\sqrt{m}}\right) + C_2 \sin\left(\frac{\sqrt{k}t}{\sqrt{m}}\right),$$

Steps

- The principle of least action: the motion makes the action stationary: $S[x] = \int \left(-\frac{kx^2}{2} + \frac{mx'^2}{2}\right) dt$
- The integrand's partial derivatives by y and y' and the total derivative by x (both y and y' depend on x) $\frac{\partial F}{\partial x} = -kx, \quad \frac{\partial F}{\partial x'} = mx', \quad \frac{d}{dt}\frac{\partial F}{\partial x'} = mx''$
- The Lagrangian's Euler–Lagrange equation is the equation of motion: $-kx - mx'' = 0$
- We solve the differential equation:
- A linear equation with constant coefficients: $x'' + \frac{k}{m}x = 0$
- We look for solutions x = e^(λt); substituting and dividing by e^(λt) gives the characteristic equation: $\lambda^2 + \frac{k}{m} = 0$
- The roots of the characteristic equation: $\lambda_{1,2} = \pm\frac{\sqrt{k}}{\sqrt{m}}i$
- Each root λ gives the solution e^(λx), a root of multiplicity k also x·e^(λx), ..., x^(k−1)·e^(λx), and a pair α ± βi the solutions e^(αx)·cos βx and e^(αx)·sin βx. The fundamental system $x_1 = \cos\left(\frac{\sqrt{k}t}{\sqrt{m}}\right), \; x_2 = \sin\left(\frac{\sqrt{k}t}{\sqrt{m}}\right)$
- The general solution $x = C_1 \cos\left(\frac{\sqrt{k}t}{\sqrt{m}}\right) + C_2 \sin\left(\frac{\sqrt{k}t}{\sqrt{m}}\right)$

A pendulum – sin θ ≈ θ for small swings.

Input: `functional(m*l^2*θ'^2/2 + m*g*l*cos(θ); action)`

Result

$$\theta = C_1 \cos\left(\frac{\sqrt{g}t}{\sqrt{l}}\right) + C_2 \sin\left(\frac{\sqrt{g}t}{\sqrt{l}}\right),$$

Steps

- The principle of least action: the motion makes the action stationary: $S[\theta] = \int \left(\frac{l^2 m\theta'^2}{2} + glm\cos\theta\right) dt$
- The integrand's partial derivatives by y and y' and the total derivative by x (both y and y' depend on x) $\frac{\partial F}{\partial \theta} = -glm\sin\theta, \quad \frac{\partial F}{\partial \theta'} = l^2 m\theta', \quad \frac{d}{dt}\frac{\partial F}{\partial \theta'} = l^2 m\theta''$
- The Lagrangian's Euler–Lagrange equation is the equation of motion: $lm(-l\theta'' - g\sin\theta) = 0$
- The integrand doesn't depend on x − Beltrami's identity F − y'·∂F/∂y' = C₁ holds: $lm\left(-\frac{l\theta'^2}{2} + g\cos\theta\right) = C_1$
- The equation with sin θ has no elementary solution. For small swings sin θ ≈ θ gives the equation of harmonic oscillation: $lm(-g\theta - l\theta'') = 0$
- We solve the differential equation:
- A linear equation with constant coefficients: $\theta'' + \frac{g}{l}\theta = 0$
- We look for solutions \theta = e^(λt); substituting and dividing by e^(λt) gives the characteristic equation: $\lambda^2 + \frac{g}{l} = 0$
- The roots of the characteristic equation: $\lambda_{1,2} = \pm\frac{\sqrt{g}}{\sqrt{l}}i$
- Each root λ gives the solution e^(λx), a root of multiplicity k also x·e^(λx), ..., x^(k−1)·e^(λx), and a pair α ± βi the solutions e^(αx)·cos βx and e^(αx)·sin βx. The fundamental system $\theta_1 = \cos\left(\frac{\sqrt{g}t}{\sqrt{l}}\right), \; \theta_2 = \sin\left(\frac{\sqrt{g}t}{\sqrt{l}}\right)$
- The general solution $\theta = C_1 \cos\left(\frac{\sqrt{g}t}{\sqrt{l}}\right) + C_2 \sin\left(\frac{\sqrt{g}t}{\sqrt{l}}\right)$

Known limitations

- One unknown function and at most its first derivative in the integrand; functionals of several functions or with y″ not yet.
- The known extremals (cycloid, catenary, circular arc) are recognized by the integrand's form; other nonlinear equations stay as the Euler–Lagrange equation.
- The brachistochrone starts at rest – the start point must have y(a) = 0 and the y-axis points down.

38. Partial differential equations

Type the equation with subscripts u_t, u_xx, u_xy (from a photo the app also reads ∂u/∂t and ∂²u/∂x²), the Laplacian as Δu; the conditions and the domain a semicolon apart. Without conditions the app finds a linear second-order equation's type (elliptic, parabolic, hyperbolic – by region for variable coefficients), the characteristic equation, the canonical form and, without lower-order terms, the general solution. A first-order equation a·u_x + b·u_y + c·u = f is solved by characteristics with a check.

With boundary and initial conditions it solves by separation of variables the heat equation u_t = k·u_xx and the wave equation u_tt = c²·u_xx on an interval (ends held at a temperature or insulated, nonzero values via the steady state) and Laplace's equation on a rectangle and in a disk (or outside it). The Fourier coefficients come as a formula in n; an initial function that is a sum of eigenfunctions gives a finite sum. On the whole line it uses d'Alembert's formula and the Fourier transform (the heat kernel), on a half-line x > 0 the Laplace transform in time. It also handles a source f(x, t) (Duhamel's principle), a heat flux at an end, boundary values changing in time and, for Laplace's equation, derivatives on the sides. Of the nonlinear equations it solves Burgers' – by characteristics until a shock forms, with viscosity by the Cole-Hopf transformation; the wave equation on the line is also shown by the Fourier transform. Poisson's equation Δu = f on a rectangle is solved by a double Fourier series (the source as a sum of products p(x)·q(y)); with all four sides given by the derivative the solvability condition is checked and the solution comes up to a constant. Equations with the variable coefficients αx²u_xx + βxu_x turn into a Sturm–Liouville problem with a weight.

Numerically on a grid with the scheme as the head, as with the other numerical methods: the heat equation explicitly (ftcs), implicitly (btcs), by Crank-Nicolson (cranknicolson) and the θ-method (theta) – the implicit systems by the Thomas algorithm, with von Neumann's stability analysis; the transport equation by upwind, laxwendroff and leapfrog with the CFL condition, the wave equation by leapfrog; Laplace's (Poisson's) equation by the five-point scheme and the iterations jacobi, gaussseidel and sor with their iteration counts compared. Named values: h the space step, k the time step, to the final time, theta, omega, tol, max. The heat equation is also a form in Numerical methods. Heat in 2D on a rectangle is solved by the schemes ftcs and adi (alternating directions), Laplace's equation also by conjugate gradients (cg), multigrid (multigrid) and finite elements with triangles (fem); Burgers' equation by Godunov's scheme (upwind). The transport and Laplace equations are also forms in the field Numerical methods, the applications (a rod, a string, a drum, diffusion) forms in physics and chemistry. For conservation laws there are the schemes laxfriedrichs and muscl (with the minmod limiter) – for the transport and Burgers' equation, the latter also with viscosity (compared with the Cole–Hopf transformation). The finite element method (fem) also works on a general polygon given by its vertices, polygon = (0,0), (2,0), …, with the boundary value u = …; the mesh comes from ear clipping and splitting the triangles. Heat in a plate is a form in physics.

The result is a graph of u(x, t): a curve in time with a slider and a play button, and a 3D surface to turn; a numerical solution has the exact one dashed. Laplace's stationary u(x, y) is a 3D surface.

How to enter it

`u_t = 4u_xx; u(0,t) = 0; u(π,t) = 0; u(x,0) = x(π - x)`	The equation, boundary and initial conditions; a derivative in a condition as u_t(x,0) or u_x(0,t).
`u_t = u_xx; x ∈ ℝ; u(x,0) = e^(-x^2)`	The domain: x ∈ ℝ, x > 0 or 0 < x < π; without boundary conditions the whole line.
`Δu = 0; r < 2; u(2,ϕ) = cos ϕ`	Laplace's equation in a disk: Δu, the radius r and the angle φ (or θ).
`ftcs(u_t = u_xx; u(x,0) = sin(pi x); u(0,t) = 0; u(1,t) = 0; h = 0.1; k = 0.004; to = 0.1)`	Numerically: the scheme as the head, the steps h and k and the final time to.
`adi(u_t = u_xx + u_yy; 0 < x < 1; 0 < y < 1; u(x,y,0) = x(1-x)y(1-y); T = 0.1)`	Heat in 2D: u(x,y,t), conditions like u(x,y,0) = … and u(0,y,t) = …; sides without a condition are 0.
`fem(Δu = -4; polygon = (0,0), (2,0), (2,1), (1,1), (1,2), (0,2); u = 0; h = 0.25)`	Finite elements on a polygon: the vertices after polygon =, the boundary value u = …, the longest edge h.

Examples

A hyperbolic equation – characteristics, the canonical form u_ξη = 0 and the general solution.

Input: `u_xx + 4u_xy + u_yy = 0`

Result

$$u = \Phi\left(-\left(2+\sqrt{3}\right)x + y\right) + \Psi\left(\left(-2\right.\right.$$

Steps

- We solve the partial differential equation $u_{xx} + 4u_{xy} + u_{yy} = 0$
- The coefficients of the second derivatives in the form $a_{11}u_xx + 2a_{12}u_xy + a_{22}u_yy$: $a_{11} = 1, \quad a_{12} = 2, \quad a_{22} = 1$
- The discriminant: $D = a_{12}^2 - a_{11}a_{22} = 3 > 0$
- D > 0, the equation is hyperbolic (like the wave equation): $D = a_{12}^2 - a_{11}a_{22} = 3$
- The equation of the characteristics: $1\,(\mathrm{d}y)^2 - 4\,\mathrm{d}x\,\mathrm{d}y + 1\,(\mathrm{d}x)^2 = 0$
- Two real families of characteristics: $-\left(2+\sqrt{3}\right)x + y = C_1, \quad \left(-2+\sqrt{3}\right)x + y = C_2$
- New variables along the characteristics: $\xi = -\left(2+\sqrt{3}\right)x + y, \quad \eta = \left(-2+\sqrt{3}\right)x + y$
- After the change of variables (the chain rule) we get the canonical form: $u_{\xi\eta} = 0$
- The equation u_ξη = 0 is integrated twice – the general solution with two arbitrary functions $u = \Phi\left(-\left(2+\sqrt{3}\right)x + y\right) + \Psi\left(\left(-2+\sqrt{3}\right)x + y\right)$

Tricomi's equation – the type by region.

Input: `y u_xx + u_yy = 0`

Result

$$D = -y$$

Steps

- We solve the partial differential equation $yu_{xx} + u_{yy} = 0$
- The coefficients of the second derivatives in the form $a_{11}u_xx + 2a_{12}u_xy + a_{22}u_yy$: $a_{11} = y, \quad a_{12} = 0, \quad a_{22} = 1$
- The discriminant depends on the position: $D = a_{12}^2 - a_{11}a_{22} = -y$
- The type by region – hyperbolic for $y < 0$, parabolic for $y = 0$, elliptic for $y > 0$
- The equation of the characteristics: $y\,(\mathrm{d}y)^2 - 0\,\mathrm{d}x\,\mathrm{d}y + 1\,(\mathrm{d}x)^2 = 0$

The transport equation by characteristics – the wave moves at the speed 2.

Input: `u_t + 2u_x = 0; u(x,0) = e^(-x^2)`

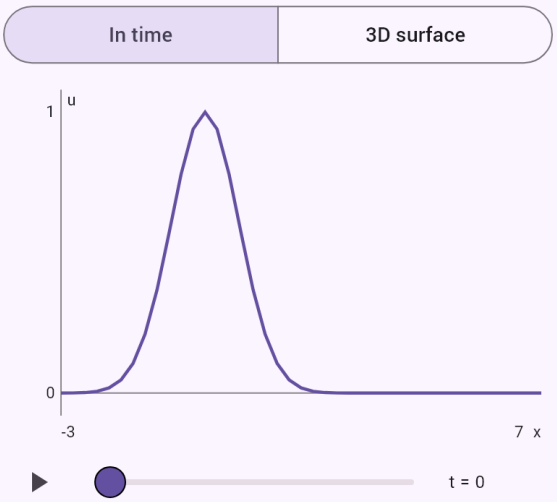
Result

$$u(x,t) = e^{-(x-2t)^2}$$

Steps

- We solve the partial differential equation $u_t + 2u_x = 0$
- Conditions: $u(x,0) = e^{-x^2}$
- The method of characteristics: along the curves solving the system the equation becomes an ordinary differential equation for u.
  $\frac{\mathrm{d}x}{\mathrm{d}s} = 2, \quad \frac{\mathrm{d}t}{\mathrm{d}s} = 1, \quad \frac{\mathrm{d}u}{\mathrm{d}s} = 0$
- The characteristics are the curves on which $x - 2t = \xi$ is constant.
- Along a characteristic we solve an ordinary equation, its initial value from the condition: $\frac{\mathrm{d}u}{\mathrm{d}t} = 0 \;\Rightarrow\; u = e^{-\xi^2}$
- Check: substituted into the equation, the left side equals the right one.
- The solution: $u(x,t) = e^{-(x-2t)^2}$

Graph



The solution u at the time t chosen by the slider; the button plays it.

Heat in a rod with cooled ends – a Fourier series.

Input: `u_t = 4u_xx; u(0,t) = 0; u(π,t) = 0; u(x,0) = x(π - x)`

Result

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4-4(-1)^n}{n^3\pi}\, e^{-4n^2 t}\, \sin(n$$

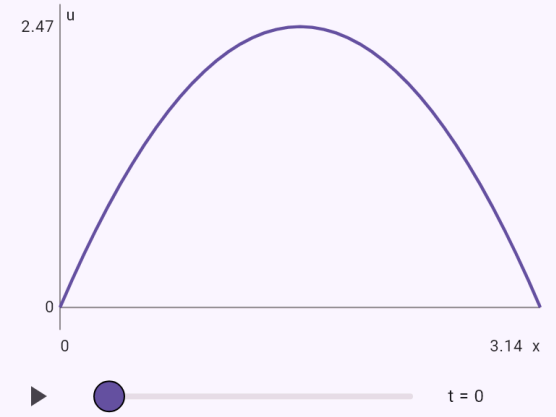
Steps

- We solve the partial differential equation $u_t = 4u_{xx}$
- Conditions:
  $u(0,t) = 0, \quad u(\pi,t) = 0, \quad u(x,0) = x(\pi - x)$
- D = 0, the equation is parabolic (like the heat equation):
  $D = a_{12}^2 - a_{11}a_{22} = 0$
- Separation of variables: we look for a product; substituted and divided, the variables separate – both sides must equal the same constant:
  $u = X(x)\,T(t): \quad \frac{T'}{4T} = \frac{X''}{X} = -\lambda$
- The eigenvalue problem $X'' + \lambda X = 0$ with X = 0 at both ends has nonzero solutions only for
  $\lambda_n = n^2, \quad X_n = \sin(nx), \quad n = 1, 2, \ldots$
- The time part for each n – an exponential decay, higher frequencies damped faster:
  $T_n' = -4n^2\, T_n \;\Rightarrow\; T_n = e^{-4n^2 t}$
- The equation is linear – a sum of solutions is a solution (superposition): $u = \sum_{n=1}^{\infty} c_n\, e^{-4n^2 t}\, \sin(nx)$
- The coefficients from the initial (boundary) function by the eigenfunctions' orthogonality (integration by parts)
  $c_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\, \sin(nx)\, \mathrm{d}x = \frac{4-4(-1)^n}{n^3\pi}$
- The solution

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4-4(-1)^n}{n^3\pi}\, e^{-4n^2 t}\, \sin(nx)$$

Graph

In time | 3D surface



The solution u at the time t chosen by the slider; the button plays it.

Ends at different temperatures – the steady state and the deviation from it.

Input: `u_t = 2u_xx; u(0,t) = 1; u(2,t) = 3; u(x,0) = 0`

Result

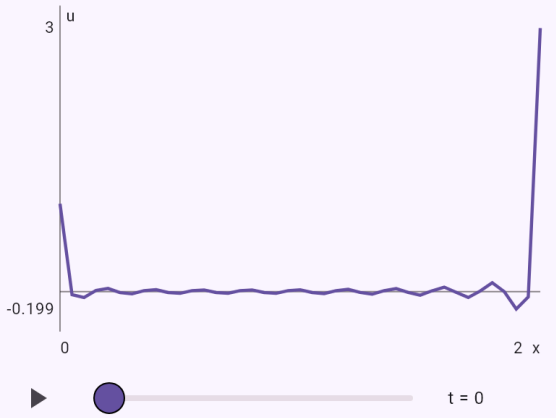
$$u(x,t) = x + 1 + \sum_{n=1}^{\infty} \frac{6(-1)^n - 2}{n\pi} e^{-\frac{n^2\pi^2 t}{2}} \sin\left(\frac{n\pi x}{2}\right)$$

Steps

- We solve the partial differential equation $u_t = 2u_{xx}$
- Conditions: $u(0,t) = 1, \quad u(2,t) = 3, \quad u(x,0) = 0$
- D = 0, the equation is parabolic (like the heat equation): $D = a_{12}^2 - a_{11}a_{22} = 0$
- The ends have nonzero values. First the steady state $w(x) = x + 1$ (the equation without time, w" = 0), then we look for $u = w(x) + v(x,t)$; the deviation v has zero values at the ends.
- Separation of variables: we look for a product; substituted and divided, the variables separate – both sides must equal the same constant:
  $u = X(x)\,T(t): \quad \frac{T'}{2T} = \frac{X''}{X} = -\lambda$
- The eigenvalue problem $X'' + \lambda X = 0$ with X = 0 at both ends has nonzero solutions only for
  $\lambda_n = \frac{n^2\pi^2}{4}, \quad X_n = \sin\left(\frac{n\pi x}{2}\right), \quad n = 1, 2, \ldots$
- The time part for each n – an exponential decay, higher frequencies damped faster:
  $T_n' = -\frac{n^2\pi^2}{2} T_n \;\Rightarrow\; T_n = e^{-\frac{n^2\pi^2 t}{2}}$
- The equation is linear – a sum of solutions is a solution (superposition)
  $u = w(x) + \sum_{n=1}^{\infty} c_n\, e^{-\frac{n^2\pi^2 t}{2}} \sin\left(\frac{n\pi x}{2}\right)$
- The coefficients from the initial (boundary) function by the eigenfunctions' orthogonality (integration by parts)
  $c_n = \int_0^2 (-x-1)\sin\left(\frac{n\pi x}{2}\right) dx = \frac{6(-1)^n - 2}{n\pi}$
- The solution
  $u(x,t) = x + 1 + \sum_{n=1}^{\infty} \frac{6(-1)^n - 2}{n\pi} e^{-\frac{n^2\pi^2 t}{2}} \sin\left(\frac{n\pi x}{2}\right)$

Graph

In time | 3D surface



t = 0

The solution u at the time t chosen by the slider; the button plays it.

A plucked string – standing waves.

Input: `u_tt = 4u_xx; u(0,t) = 0; u(1,t) = 0; u(x,0) = x(1 - x); u_t(x,0) = 0`

Result

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4 - 4(-1)^n}{n^3\pi^3} \cos(2n\pi t)\ s$$

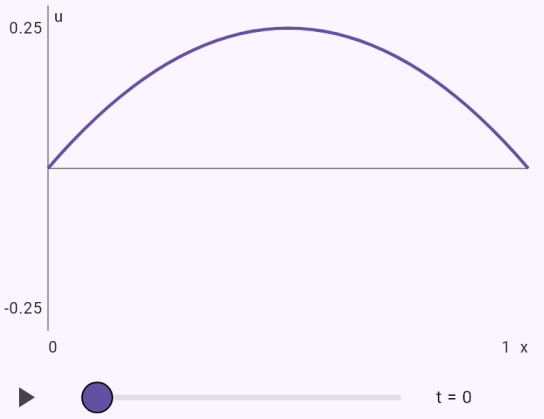
Steps

- We solve the partial differential equation $u_{tt} = 4u_{xx}$
- Conditions:
  $u(0,t) = 0, \quad u(1,t) = 0, \quad u(x,0) = x(1-x), \quad u_t(x,0) = 0$
- D > 0, the equation is hyperbolic (like the wave equation):
  $D = a_{12}^2 - a_{11}a_{22} = 4$
- Separation of variables: we look for a product – standing waves; both sides equal the same constant:
  $u = X(x)\,T(t): \quad \frac{T''}{4T} = \frac{X''}{X} = -\lambda$
- The eigenvalue problem $X'' + \lambda X = 0$ with X = 0 at both ends has nonzero solutions only for
  $\lambda_n = n^2\pi^2, \quad X_n = \sin(n\pi x), \quad n = 1, 2, \ldots$
- The time part for each n – oscillations at its own frequency:
  $T_n'' + 4n^2\pi^2\,T_n = 0 \;\Rightarrow\; T_n = a_n \cos 2n\pi t + b_n \sin 2n\pi t$
- The equation is linear – a sum of solutions is a solution (superposition)
  $$u = \sum_{n=1}^{\infty} (a_n \cos 2n\pi t + b_n \sin 2n\pi t)\ \sin(n\pi x)$$
- The coefficients from the initial (boundary) function by the eigenfunctions' orthogonality (integration by parts)
  $$a_n = 2\int_0^1 x(1-x)\ \sin(n\pi x)\,dx = \frac{4 - 4(-1)^n}{n^3\pi^3}$$

- The solution
  $$u(x,t) = \sum_{n=1}^{\infty} \frac{4 - 4(-1)^n}{n^3\pi^3} \cos(2n\pi t)\ \sin(n\pi x)$$

Graph

In time | 3D surface



The solution u at the time t chosen by the slider; the button plays it.

Laplace's equation on a rectangle with one heated side.

Input: `u_xx + u_yy = 0; 0 < x < π; 0 < y < 1; u(x,1) = sin x`

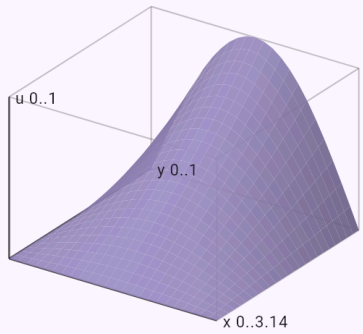
Result

$$u(x,y) = \frac{\sin x \cdot \sinh y}{\sinh 1}$$

Steps

- We solve the partial differential equation $u_{xx} + u_{yy} = 0$
- Conditions: $u(x,1) = \sin x$
- D < 0, the equation is elliptic (like Laplace's equation): $D = a_{12}^2 - a_{11}a_{22} = -1$
- The side with nonzero values (the others have u = 0): $u(x,1) = \sin x$
- Separation of variables: we look for a product: $u = X(x)\,Y(y): \quad \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$
- The eigenvalue problem $X'' + \lambda X = 0$ with X = 0 at both ends has nonzero solutions only for $\lambda_n = n^2, \quad X_n = \sin(nx), \quad n = 1, 2, \ldots$
- The other equation Y" = λY, zero on the opposite side, scaled to 1 on the given one: $Y_n = \frac{\sinh(ny)}{\sinh n}$
- The initial (boundary) function already is a sum of eigenfunctions, so only a few coefficients $c_n = \frac{2}{\pi}\int_0^{\pi} \sin x \, \sin(nx)\,dx$ aren't zero: $c_1 = 1$, the others are zero.
- The solution: $u(x,y) = \frac{\sin x \cdot \sinh y}{\sinh 1}$

Graph



The solution u as a surface over the domain. Turn it by dragging a finger.

Laplace's equation in a disk – a finite sum of harmonics.

Input: `Δu = 0; r < 1; u(1,ϕ) = sin(ϕ)^2`

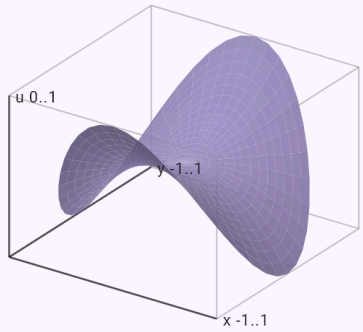
Result

$$u(r,\varphi) = \frac{1 - r^2\cos(2\varphi)}{2}$$

Steps

- We solve the partial differential equation
  $\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\varphi\varphi} = 0$
- Conditions: $u(1,\varphi) = \sin^2\varphi$
- D < 0, the equation is elliptic (like Laplace's equation):
  $\Delta u = 0$
- Separation of variables in polar coordinates:
  $u = R(r)\,\Phi(\varphi):\quad \frac{r^2R'' + rR'}{R} = -\frac{\Phi''}{\Phi} = \lambda$
- The angle returns after a turn, Φ must be periodic:
  $\Phi'' + \lambda\Phi = 0,\quad \Phi(\varphi + 2\pi) = \Phi(\varphi)$ – the solutions
  $\lambda_n = n^2,\quad \Phi_n = a_n\cos n\varphi + b_n\sin n\varphi,\quad n = 0,1,2,\ldots$
- The radial part is Euler's equation; inside the disk the solution must be bounded (r^{-n} and ln r drop out):
  $r^2R'' + rR' - n^2R = 0 \;\Rightarrow\; R_n = r^n$
- The equation is linear – a sum of solutions is a solution (superposition)
  $$u = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n(a_n\cos n\varphi + b_n\sin n\varphi)$$
- The Fourier coefficients of the boundary function:
  $a_0 = \frac{1}{\pi}\int_0^{2\pi}\sin^2\varphi\,\mathrm{d}\varphi = 1$
- The initial (boundary) function already is a sum of eigenfunctions, so only a few coefficients $a_n = \frac{1}{\pi}\int_0^{2\pi}\sin^2\varphi\cos n\varphi\,\mathrm{d}\varphi$ aren't zero: $a_2 = -\frac{1}{2}$, the others are zero.
- All coefficients are zero: $b_n = \frac{1}{\pi}\int_0^{2\pi}\sin^2\varphi\sin n\varphi\,\mathrm{d}\varphi$
- The solution: $u(r,\varphi) = \frac{1 - r^2\cos(2\varphi)}{2}$

Graph



The solution u as a surface over the domain. Turn it by dragging a finger.

d’Alembert’s formula – an initial pulse splits into two waves.

Input: `u_tt = u_xx; x ∈ ℝ; u(x,0) = e^(-x^2); u_t(x,0) = 0`

Result

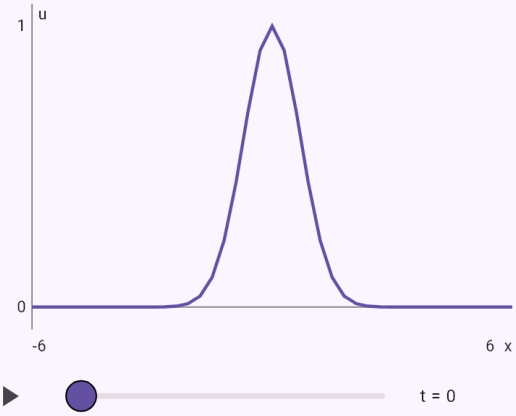
$$u(x,t) = \frac{e^{-(x-t)^2} + e^{-(x+t)^2}}{2}$$

Steps

- We solve the partial differential equation $u_{tt} = u_{xx}$
- Conditions: $u(x,0) = e^{-x^2}, \quad u_t(x,0) = 0$
- D > 0, the equation is hyperbolic (like the wave equation): $D = a_{12}^2 - a_{11}a_{22} = 1$
- The wave equation’s general solution is two waves moving at the speed c to the left and to the right: $u = \Phi(x-t) + \Psi(x+t)$
- The same by the Fourier transform in x: each frequency ω oscillates on its own. Multiplying by e^(±icωt) means a shift by ∓ct, so the inverse transform gives – d’Alembert’s formula.
- The initial conditions u = f and u_t = g give d’Alembert’s formula: $u = \frac{f(x-t) + f(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,\mathrm{d}s$
- Check: substituted into the equation, the left side equals the right one.
- The solution: $u(x,t) = \frac{e^{-(x-t)^2} + e^{-(x+t)^2}}{2}$

Graph

In time | 3D surface



The solution u at the time t chosen by the slider; the button plays it.

The Fourier transform on the whole line – a Gaussian profile widens.

Input: `u_t = u_xx; x ∈ ℝ; u(x,0) = e^(-x^2)`

Result

$$u(x,t) = \frac{e^{-\frac{x^2}{4t+1}}}{\sqrt{4t+1}}$$

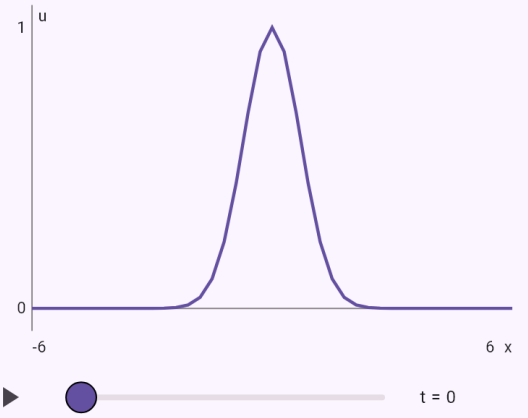
Steps

- We solve the partial differential equation $u_t = u_{xx}$
- Conditions: $u(x,0) = e^{-x^2}$
- D = 0, the equation is parabolic (like the heat equation): $D = a_{12}^2 - a_{11}a_{22} = 0$
- On the whole line we use the Fourier transform in x – the derivative u_xx becomes multiplication by −ω²: $\hat{u}(\omega,t) = \int_{-\infty}^{\infty} u(x,t)\, e^{-i\omega x}\, \mathrm{d}x$. For each ω an ordinary equation in time is left
  $$\frac{\partial \hat{u}}{\partial t} = -\omega^2 \hat{u} \;\Rightarrow\; \hat{u}(\omega,t) = \hat{f}(\omega)\, e^{-\omega^2 t}$$
- The inverse transform of a product is the convolution with the heat kernel (a Gaussian widening in time)
  $$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(s)\, e^{-\frac{(x-s)^2}{4t}}\, \mathrm{d}s$$
- The convolution of two Gaussians is a Gaussian again, just wider and lower
  $$\hat{f}(\omega)\, e^{-\omega^2 t} \;\Rightarrow\; u = \frac{1}{\sqrt{4t+1}}\, e^{-\frac{x^2}{4t+1}}$$
- Check: substituted into the equation, the left side equals the right one.

- The solution: $u(x,t) = \frac{e^{-\frac{x^2}{4t+1}}}{\sqrt{4t+1}}$

Graph

In time | 3D surface



The solution u at the time t chosen by the slider; the button plays it.

The Laplace transform – a half-space heated, the function erfc.

Input: `u_t = 4u_xx; x > 0; u(0,t) = 100; u(x,0) = 20`

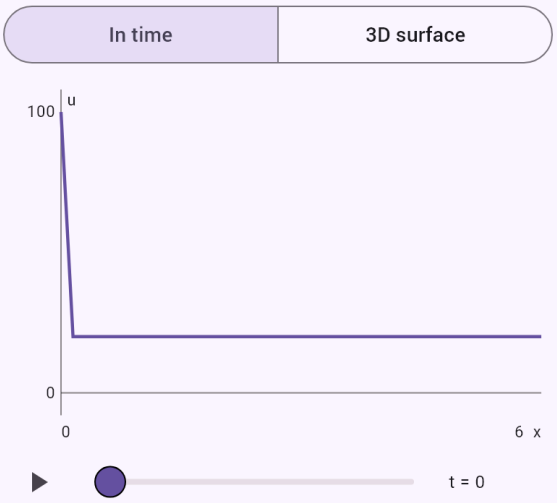
Result

$$u(x,t) = 20 + 80\ \operatorname{erfc}\frac{x}{4\sqrt{t}}$$

Steps

- We solve the partial differential equation $u_t = 4u_{xx}$
- Conditions: $u(0,t) = 100, \quad u(x,0) = 20$
- D = 0, the equation is parabolic (like the heat equation): $D = a_{12}^2 - a_{11}a_{22} = 0$
- On a half-line we use the Laplace transform in time $U(x,s) = \int_0^\infty u(x,t)\, e^{-st}\, \mathrm{d}t$ – the derivative u_t becomes sU − u(x, 0) and an ordinary equation in x is left: $sU - 20 = 4U_{xx}$
- As x → ∞ U must stay bounded, the growing exponential drops out: $U = \frac{20}{s} + A(s)\, e^{-x\sqrt{\frac{s}{4}}}$; the constant from the boundary condition: $U(0,s) = \frac{100}{s} \Rightarrow A(s) = \frac{80}{s}$
- The inverse transform from the table $\mathcal{L}^{-1}\left\{\frac{e^{-a\sqrt{s}}}{s}\right\} = \operatorname{erfc}\frac{a}{2\sqrt{t}}$
- The solution: $u(x,t) = 20 + 80\ \operatorname{erfc}\frac{x}{4\sqrt{t}}$

Graph



The solution u at the time t chosen by the slider; the button plays it.

The explicit scheme with r = 0.7 > 1/2 – unstable by von Neumann.

Input: `ftcs(u_t = u_xx; u(x,0) = x(1-x); u(0,t) = 0; u(1,t) = 0; h = 0.1; k = 0.007; to =`

1)

Result

$$u(0.5, 0.35) \approx 1.35312 \cdot 10^{8}$$

Steps

- Solved numerically on a grid: $u_t = u_{xx}$, $u(x,0) = x(1-x), \quad u(0,t) = 0, \quad u(1,t) = 0$
- The grid: 11 nodes in x, 143 time steps up to t = 1.001; $h = 0.1, \quad k = 0.007, \quad r = \dfrac{k}{h^2} = 0.7000$ (r is the time step over the space step squared).
- The explicit scheme (forward in time, central in space) – each new value straight from three old ones: $u_i^{n+1} = u_i^n + r\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$
- Von Neumann's analysis: the error is split into waves e^(iβx), each multiplied by the amplification G in a step: $G = 1 - 4r\sin^2\dfrac{\beta h}{2}$. Stability needs $r = 0.7000 > \dfrac{1}{2}$ – not here, the errors will grow exponentially.
- The first time level

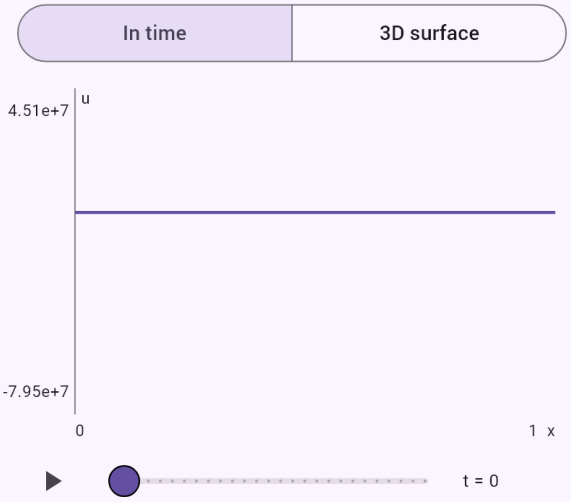
i	x_i	u_i^0	u_i^1
0	0	0.000000	0.000000
1	0.1	0.090000	0.076000
2	0.2	0.160000	0.146000
3	0.3	0.210000	0.196000
4	0.4	0.240000	0.226000
5	0.5	0.250000	0.236000
6	0.6	0.240000	0.226000
7	0.7	0.210000	0.196000
8	0.8	0.160000	0.146000
9	0.9	0.090000	0.076000
10	1	0.000000	0.000000

- The solution at chosen time levels

n	t_n	$x = 0$	$x = 0.1$	$x = 0.2$
0	0	0.0000	0.0900	0.1600
8	0.056	0.0000	0.0518	0.0752
16	0.112	0.0000	0.3660	−0.5894
24	0.168	0.0000	26.5321	−50.3647
32	0.224	0.0000	2137.3470	−4065.2082
40	0.28	0.0000	172643.2102	−328385.7013
48	0.336	0.0000	$1.395 \cdot 10^{7}$	$-2.653 \cdot 10^{7}$
50	0.35	0.0000	$4.181 \cdot 10^{7}$	$-7.953 \cdot 10^{7}$

- At the level level.50 the values grew beyond all bounds – the scheme is unstable, the computation was stopped.
- The result at the middle node: $u(0.5, 0.35) \approx 1.35312 \cdot 10^{8}$

Graph



The solution u at the time t chosen by the slider; the button plays it.

The Crank-Nicolson scheme with the Thomas algorithm.

Input: `cranknicolson(u_t = u_xx; u(x,0) = sin(pi x); u(0,t) = 0; u(1,t) = 0; h = 0.1; k =`

`0.01; to = 0.1)`

Result

$$u(0.5, 0.1) \approx 0.375442$$

Steps

- Solved numerically on a grid: $u_t = u_{xx}$, $u(x,0) = \sin(\pi x), \quad u(0,t) = 0, \quad u(1,t) = 0$
- The grid: 11 nodes in x, 10 time steps up to t = 0.1; $h = 0.1, \quad k = 0.01, \quad r = \dfrac{k}{h^2} = 1.0000$ (r is the time step over the space step squared).
- The Crank-Nicolson scheme – the average of the explicit and the implicit one, second order in time too $-\frac{1}{2}r\,u_{i-1}^{n+1} + (1+r)u_i^{n+1} - \frac{1}{2}r\,u_{i+1}^{n+1} = u_i^{n} + \frac{1}{2}r\left(u_{i+1}^{n} - 2\right.$
- Von Neumann's analysis: the error is split into waves e^(iβx), the amplification in a step is $G = \dfrac{1 - 2r\sin^2\frac{\beta h}{2}}{1 + 2r\sin^2\frac{\beta h}{2}}$; always $|G| \le 1$, the scheme is stable for any step (unconditionally).
- Each time level needs a tridiagonal system a□u□₋₁ + b□u□ + c□u□₊₁ = d□ solved (the boundary values on the right side). The first level

i	a_i	b_i	c_i	d_i
1		2.0000	−0.5000	0.293893
2	−0.5000	2.0000	−0.5000	0.559017
3	−0.5000	2.0000	−0.5000	0.769421
4	−0.5000	2.0000	−0.5000	0.904508
5	−0.5000	2.0000	−0.5000	0.951057
6	−0.5000	2.0000	−0.5000	0.904508
7	−0.5000	2.0000	−0.5000	0.769421
8	−0.5000	2.0000	−0.5000	0.559017
9	−0.5000	2.0000		0.293893

- The system by the Thomas algorithm – the forward sweep and the back substitution $c_i' = \dfrac{c_i}{b_i - a_i c_{i-1}'}, \quad d_i' = \dfrac{d_i - a_i d_{i-1}'}{b_i - a_i c_{i-1}'}, \quad u_i^1 = d_i' - c_i' u_{i+}^1$

i	c_i'	d_i'	u_i^1
1	−0.250000	0.146946	0.280180
2	−0.266667	0.337328	0.532933
3	−0.267857	0.502545	0.733520
4	−0.267943	0.619366	0.862304
5	−0.267949	0.675627	0.906680
6	−0.267949	0.665758	0.862304
7	−0.267949	0.590721	0.733520
8	−0.267949	0.457859	0.532933
9		0.280180	0.280180

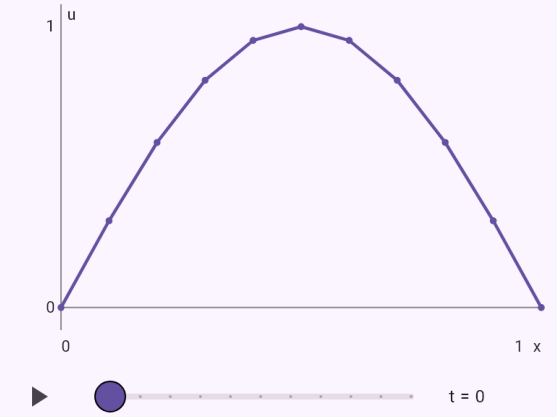
- The solution at chosen time levels

n	t_n	$x = 0$	$x = 0.1$	$x = 0.2$	$x = 0.3$	$x =$
0	0	0.0000	0.3090	0.5878	0.8090	0.9
2	0.02	0.0000	0.2540	0.4832	0.6651	0.7
4	0.04	0.0000	0.2088	0.3972	0.5467	0.6
6	0.06	0.0000	0.1717	0.3265	0.4495	0.5
8	0.08	0.0000	0.1411	0.2684	0.3695	0.4
10	0.1	0.0000	0.1160	0.2207	0.3037	0.3

- The largest error against the exact solution at the last level: $\max_i \left|u(x_i, 0.1) - u_i^{10}\right| \approx 0.00273$ (exactly 0.372708 in the middle).
- The result at the middle node: $u(0.5, 0.1) \approx 0.375442$

Graph

In time | 3D surface



Dots: the numerical solution at the grid nodes; dashed: the exact solution. The slider picks the time level.

The Lax-Wendroff scheme for the transport equation, the CFL condition.

Input: `laxwendroff(u_t + u_x = 0; u(x,0) = e^(-50(x-0.3)^2); 0 < x < 1; h = 0.02; k = 0.016;`

```
to = 0.4)
```

Result

$$u(0.5, 0.4) \approx 0.133866$$

Steps

- Solved numerically on a grid: $u_t + u_x = 0$, $u(x,0) = e^{-50(x-0.3)^2}$
- The grid on $0 \le x \le 1$: 51 nodes, 25 time steps up to t = 0.4; $h = 0.02, \quad k = 0.016, \quad \nu = \dfrac{ak}{h} = 0.8000$ is the Courant number.
- The exact solution is carried along the characteristics $x - t = C$ at the speed a = 1
- The Lax-Wendroff scheme – Taylor's expansion to the second order, u_tt replaced by a²u_xx

$$u_i^{n+1} = u_i^n - \frac{\nu}{2}\left(u_{i+1}^n - u_{i-1}^n\right) + \frac{\nu^2}{2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

- The CFL condition (the numerical domain of dependence contains the characteristic): $|\nu| = 0.8000 \le 1$
- Von Neumann's analysis: the error is split into waves e^(iβx), each multiplied by the amplification G in a step. The scheme is stable when |G| ≤ 1 for all β: $|G|^2 = 1 - 4\nu^2(1-\nu^2)\sin^4\dfrac{\beta h}{2}$; this holds because $|\nu| \le 1$
- The first time level

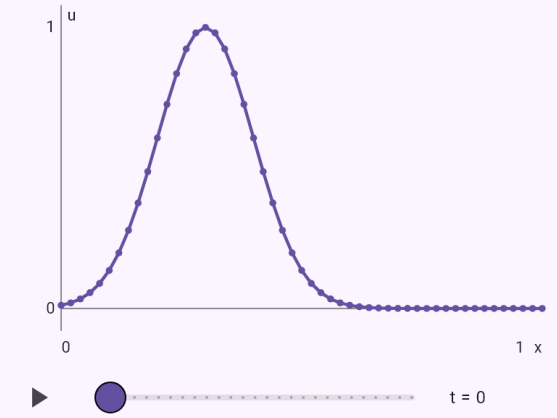
i	x_i	u_i^0	u_i^1
0	0	0.011109	0.006787
1	0.02	0.019841	0.012417
2	0.04	0.034047	0.022052
3	0.06	0.056135	0.037609
4	0.08	0.088922	0.061602
5	0.1	0.135335	0.096912
6	0.12	0.197899	0.146442
7	0.14	0.278037	0.212556
8	0.16	0.375311	0.296359
9	0.18	0.486752	0.396932
10	0.2	0.606531	0.510721
11	0.22	0.726149	0.631294
12	0.24	0.835270	0.749675
13	0.26	0.923116	0.855301
14	0.28	0.980199	0.937515
15	0.3	1.000000	0.987327
16	0.32	0.980199	0.999022
17	0.34	0.923116	0.971243
18	0.36	0.835270	0.907249
19	0.38	0.726149	0.814286
20	0.4	0.606531	0.702238
21	0.42	0.486752	0.581908
22	0.44	0.375311	0.463331
23	0.46	0.278037	0.354486
24	0.48	0.197899	0.260604
⋮	⋮	⋮	⋮
41	0.82	0.000001	0.000003
42	0.84	$4.65572 \cdot 10^{-7}$	0.000001
43	0.86	$1.54975 \cdot 10^{-7}$	$3.87038 \cdot 10^{-7}$
44	0.88	$4.95641 \cdot 10^{-8}$	$1.28207 \cdot 10^{-7}$
45	0.9	$1.523 \cdot 10^{-8}$	$4.08092 \cdot 10^{-8}$
46	0.92	$4.49635 \cdot 10^{-9}$	$1.24822 \cdot 10^{-8}$
47	0.94	$1.27541 \cdot 10^{-9}$	$3.66871 \cdot 10^{-9}$
48	0.96	$3.47589 \cdot 10^{-10}$	$1.03614 \cdot 10^{-9}$
49	0.98	$9.10147 \cdot 10^{-11}$	$2.81198 \cdot 10^{-10}$
50	1	$2.28973 \cdot 10^{-11}$	$7.73912 \cdot 10^{-11}$

- The solution at chosen time levels

n	t_n	$x = 0$	$x = 0.1$	$x = 0.2$
0	0	0.0111	0.1353	0.6065
4	0.064	0.0013	0.0301	0.2611
8	0.128	0.0001	0.0042	0.0733
12	0.192	$5.542 \cdot 10^{-6}$	0.0004	0.0128
16	0.256	$1.937 \cdot 10^{-7}$	$2.235 \cdot 10^{-5}$	0.0014
20	0.32	$4.496 \cdot 10^{-9}$	$8.568 \cdot 10^{-7}$	$9.388 \cdot 10^{-5}$
24	0.384	$6.928 \cdot 10^{-11}$	$1.815 \cdot 10^{-8}$	$3.337 \cdot 10^{-6}$
25	0.4	$2.29 \cdot 10^{-11}$	$7.929 \cdot 10^{-9}$	$1.511 \cdot 10^{-6}$

- The largest error against the exact solution at the last level: $\max\limits_i \left|u(x_i, 0.4) - u_i^{25}\right| \approx 0.0129$ (exactly 0.135335 in the middle).
- The result at the middle node: $u(0.5, 0.4) \approx 0.133866$

Graph



Dots: the numerical solution at the grid nodes; dashed: the exact solution. The slider picks the time level.

The five-point scheme and SOR with the optimal ω – compared with Jacobi and Gauss-Seidel.

Input: `sor(u_xx + u_yy = 0; u(x,1) = sin(pi x); u(x,0) = 0; u(0,y) = 0; u(1,y) = 0; h =`

0.1)

Result

$$u(0.5, 0.5) \approx 0.201565$$

Steps

- Solved numerically on a grid: $u_{xx} + u_{yy} = 0$, $u(x,1) = \sin(\pi x), \quad u(x,0) = 0, \quad u(0,y) = 0, \quad u(1,y) = 0$
- A square grid: $h = 0.1$, 121 nodes, 81 of them unknowns inside.
- The five-point scheme – u_xx + u_yy replaced by differences, one equation at each node: $u_{i,j} = \frac{1}{4}(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1})$
- Successive over-relaxation (SOR) – the Gauss-Seidel step "overshot" by the factor ω: $u_{i,j}^{(k+1)} = (1-\omega)\, u_{i,j}^{(k)} + \omega\, \tilde{u}_{i,j}^{GS}$
- The optimal relaxation factor from the Jacobi iteration's spectral radius $\rho_J = \frac{1}{2}\left(\cos\frac{\pi}{10} + \cos\frac{\pi}{10}\right) = 0.9511, \quad \omega_{opt} = \frac{2}{1+\sqrt{1}}$
- The iterations from a zero guess inside (chosen nodes and the largest change)

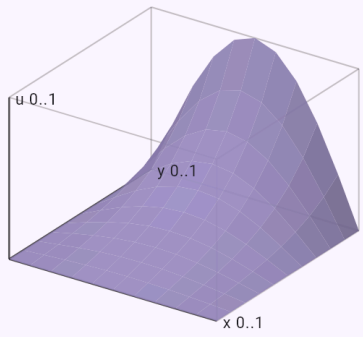
k	$u_{5,5}$	$u_{4,5}$	$u_{5,6}$	$\max\lvert\Delta u\rvert$
0	0.000000	0.000000	0.000000	
1	0.000000	0.000000	0.000000	0.585
2	0.000000	0.000000	0.000000	0.349
3	0.000000	0.000000	0.000000	0.206
4	0.000000	0.000000	0.096188	0.122
5	0.050359	0.035655	0.170299	0.0741
6	0.104395	0.084053	0.213423	0.0541
7	0.140768	0.119960	0.241149	0.0389
8	0.165712	0.145515	0.259286	0.0289
9	0.182811	0.163713	0.265151	0.0219
10	0.187653	0.176637	0.270685	0.0171
11	0.193498	0.181824	0.275244	0.0127
12	0.197266	0.185954	0.277934	0.00885
13	0.199257	0.188778	0.279539	0.00584
14	0.200171	0.190116	0.280249	0.00352
15	0.200753	0.190684	0.280659	0.002
16	0.201149	0.191113	0.280933	0.00109
17	0.201350	0.191408	0.281068	$7.75 \cdot 10^{-4}$
18	0.201464	0.191552	0.281137	$6.1 \cdot 10^{-4}$
19	0.201535	0.191649	0.281190	$4.2 \cdot 10^{-4}$
20	0.201565	0.191686	0.281214	$7.43 \cdot 10^{-5}$

- The iteration stopped after 20 steps, when the largest change fell below 10^{-4}
- The methods on the same grid with the same tolerance: Jacobi 102 iterations, Gauss-Seidel 61, SOR with ω = 1.528 only 20
- Faster methods for the same system (81 unknowns): conjugate gradients 10 iterations, multigrid 5 V-cycles. Multigrid's number of V-cycles hardly depends on how fine the grid is.
- The values at the grid's nodes

$y\backslash x$	0	0.2	0.4	0.6	0.8	
1	0.0000	0.5878	0.9511	0.9511	0.5878	0.
0.8	0.0000	0.3136	0.5075	0.5075	0.3136	0.
0.6	0.0000	0.1653	0.2674	0.2675	0.1653	0.
0.4	0.0000	0.0832	0.1347	0.1347	0.0833	0.
0.2	0.0000	0.0346	0.0561	0.0561	0.0347	0.
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.

- The largest error against the exact solution at the last level: $\max_{i,j} |u(x_i, y_j) - u_{i,j}| \approx 0.0028$ (exactly 0.199268 in the middle).
- The result at the middle node: $u(0.5, 0.5) \approx 0.201565$

Graph



The solution u as a surface over the domain. Turn it by dragging a finger.

A heat source fading with time – Duhamel's principle.

Input: `u_t = u_xx + sin(x)e^(-t); u(0,t) = 0; u(π,t) = 0; u(x,0) = 0`

Result

$$u(x,t) = te^{-t}\sin x$$

Steps

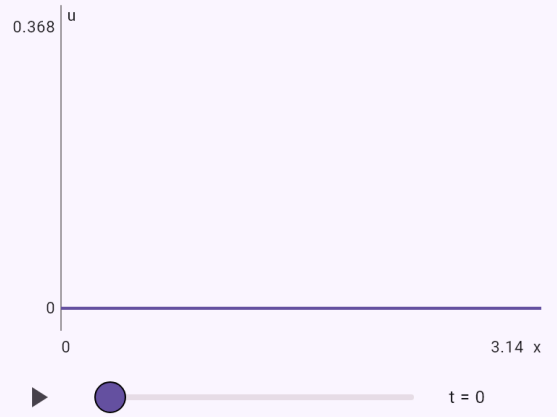
- We solve the partial differential equation $u_t = u_{xx} + e^{-t}\sin x$
- Conditions: $u(0,t) = 0, \quad u(\pi,t) = 0, \quad u(x,0) = 0$
- D = 0, the equation is parabolic (like the heat equation): $D = a_{12}^2 - a_{11}a_{22} = 0$
- Separation of variables: we look for a product; substituted and divided, the variables separate – both sides must equal the same constant: $u = X(x)\,T(t): \quad \frac{T'}{T} = \frac{X''}{X} = -\lambda$
- The eigenvalue problem $X'' + \lambda X = 0$ with X = 0 at both ends has nonzero solutions only for $\lambda_n = n^2, \quad X_n = \sin(nx), \quad n = 1, 2, \ldots$
- The time part for each n – an exponential decay, higher frequencies damped faster: $T_n' = -n^2 T_n \;\Rightarrow\; T_n = e^{-n^2 t}$
- The equation is linear – a sum of solutions is a solution (superposition): $u = \sum_{n=1}^{\infty} T_n(t)\,\sin(nx)$
- All coefficients are zero: $c_n = \frac{2}{\pi}\int_0^{\pi} 0\,\sin(nx)\,\mathrm{d}x$
- We expand the source $F(x,t) = e^{-t}\sin x$ into the eigenfunctions just like the initial condition: $F_n(t) = f_n\,e^{-t}$, the coefficients by an integral.
- The initial (boundary) function already is a sum of eigenfunctions, so only a few coefficients $f_n = \frac{2}{\pi}\int_0^{\pi} \sin x\,\sin(nx)\,\mathrm{d}x$ aren't zero: $f_1 = 1$, the others are zero.
- Duhamel's principle: each coefficient of the series satisfies a linear ordinary equation with a source; the solution is the free decay plus the convolution of the source with the decay

$$T_n' + n^2 T_n = F_n(t) \;\Rightarrow\; T_n(t) = c_n\,e^{-n^2 t} + \int_0^t e^{-n^2(t-s)}$$

- The source has only finitely many nonzero coefficients, their contributions come out exactly: $T_1^F(t) = te^{-t}$
- The solution: $u(x,t) = te^{-t}\sin x$

Graph

In time | 3D surface



The solution u at the time t chosen by the slider; the button plays it.

A constant heat source in a rod – the steady state is a parabola.

Input: `u_t = u_xx + 1; u(0,t) = 0; u(1,t) = 0; u(x,0) = 0`

Result

$$u(x,t) = \frac{x(1-x)}{2} + \sum_{n=1}^{\infty} \frac{2((-1)^n - 1}{n^3\pi^3}$$

Steps

- We solve the partial differential equation $u_t = u_{xx} + 1$
- Conditions: $u(0,t) = 0, \quad u(1,t) = 0, \quad u(x,0) = 0$
- D = 0, the equation is parabolic (like the heat equation): $D = a_{12}^2 - a_{11}a_{22} = 0$
- Neither the source nor the boundary values depend on time. First we find the steady state – the solution of the equation without the time derivative $w'' + 1 = 0$ with the boundary conditions: $w(x) = \frac{x(1-x)}{2}$. We look for $u = w(x) + v(x,t)$; the deviation v has zero boundary conditions and no source.
- Separation of variables: we look for a product; substituted and divided, the variables separate – both sides must equal the same constant: $u = X(x)\,T(t): \quad \frac{T'}{T} = \frac{X''}{X} = -\lambda$
- The eigenvalue problem $X'' + \lambda X = 0$ with X = 0 at both ends has nonzero solutions only for $\lambda_n = n^2\pi^2, \quad X_n = \sin(n\pi x), \quad n = 1, 2, \ldots$
- The time part for each n – an exponential decay, higher frequencies damped faster: $T_n' = -n^2\pi^2\,T_n \;\Rightarrow\; T_n = e^{-n^2\pi^2 t}$
- The equation is linear – a sum of solutions is a solution (superposition): $u = w(x) + \sum_{n=1}^{\infty} c_n\, e^{-n^2\pi^2 t} \sin(n\pi x)$
- The coefficients from the initial (boundary) function by the eigenfunctions' orthogonality (integration by parts)

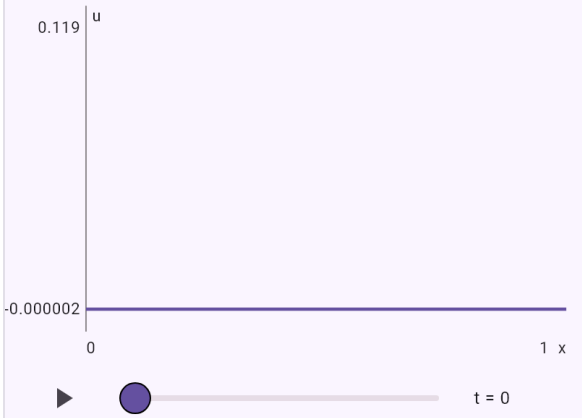
$$c_n = 2\int_0^1 \frac{x(x-1)}{2} \sin(n\pi x)\,dx = \frac{2((-1)^n - 1)}{n^3\pi^3}$$

- The solution

$$u(x,t) = \frac{x(1-x)}{2} + \sum_{n=1}^{\infty} \frac{2((-1)^n - 1)}{n^3\pi^3}\, e^{-n^2\pi^2 t} \sin(n\pi$$

Graph

In time | 3D surface



The solution u at the time t chosen by the slider; the button plays it.

An insulated rod with heat flowing in at one end – the mean temperature grows linearly.

Input: `u_t = u_xx; u_x(0,t) = 0; u_x(π,t) = 1; u(x,0) = 0`

Result

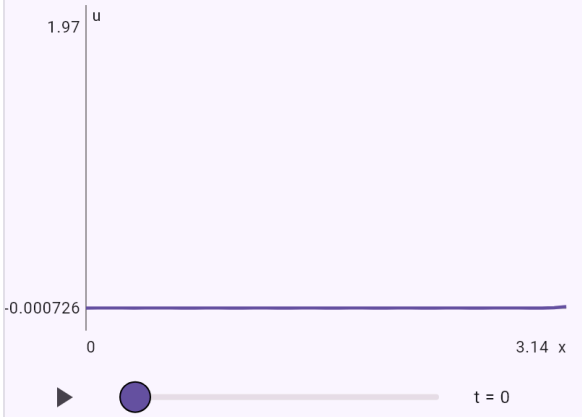
$$u(x,t) = \frac{x^2+2t}{2\pi} - \frac{\pi}{6} + \sum_{n=1}^{\infty} -\frac{2(-1)^n}{n^2\pi}$$

Steps

- We solve the partial differential equation $u_t = u_{xx}$
- Conditions: $u_x(0,t) = 0, \quad u_x(\pi,t) = 1, \quad u(x,0) = 0$
- D = 0, the equation is parabolic (like the heat equation): $D = a_{12}^2 - a_{11}a_{22} = 0$
- Both ends are insulated, but heat flows into the rod (a flux at the ends or a source). There is no steady state – the mean temperature grows at the rate $R = \frac{1}{\pi}$. We look for $u = \frac{t}{\pi} + W(x) + v(x,t)$, where W satisfies the equation without the time derivative: $W(x) = \frac{x^2}{2\pi}$
- Separation of variables: we look for a product; substituted and divided, the variables separate – both sides must equal the same constant: $u = X(x)\,T(t): \quad \frac{T'}{T} = \frac{X''}{X} = -\lambda$
- The eigenvalue problem $X'' + \lambda X = 0$ with X' = 0 at both ends (insulated ends) has the solutions $\lambda_n = n^2, \quad X_n = \cos(nx), \quad n = 0, 1, 2, \ldots$
- The time part for each n – an exponential decay, higher frequencies damped faster: $T_n' = -n^2 T_n \;\Rightarrow\; T_n = e^{-n^2 t}$
- The equation is linear – a sum of solutions is a solution (superposition): $u = w(x,t) + c_0 + \sum_{n=1}^{\infty} c_n\, e^{-n^2 t} \cos(nx)$
- The coefficients from the initial (boundary) function by the eigenfunctions' orthogonality (integration by parts): $c_0 = \frac{1}{\pi}\int_0^{\pi} -\frac{x^2}{2\pi}\,\mathrm{d}x = -\frac{\pi}{6}$
- The coefficients from the initial (boundary) function by the eigenfunctions' orthogonality (integration by parts) $c_n = \frac{2}{\pi}\int_0^{\pi} -\frac{x^2}{2\pi}\cos(nx)\,\mathrm{d}x = -\frac{2(-1)^n}{n^2\pi}$
- The solution $u(x,t) = \frac{x^2+2t}{2\pi} - \frac{\pi}{6} + \sum_{n=1}^{\infty} -\frac{2(-1)^n}{n^2\pi}\, e^{-n^2 t}\cos(nx)$

Graph

In time | 3D surface



The solution u at the time t chosen by the slider; the button plays it.

A string shaken at its end at its own frequency – resonance, the amplitude grows with t.

Input: `u_tt = u_xx; u(0,t) = 0; u(π,t) = sin(t); u(x,0) = 0; u_t(x,0) = 0`

Result

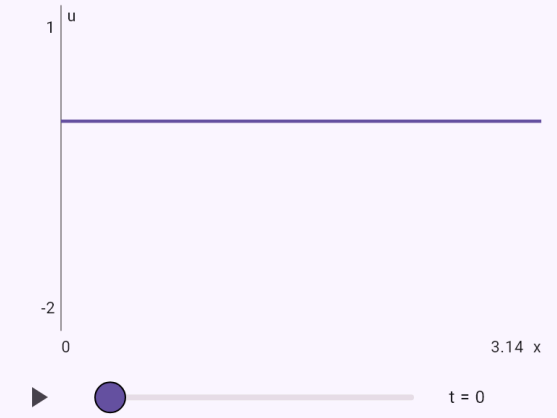
$$u(x,t) = \frac{x \sin t}{\pi} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2 \pi} \sin(nt$$

Steps

- We solve the partial differential equation $u_{tt} = u_{xx}$
- Conditions:
  $u(0,t) = 0, \quad u(\pi,t) = \sin t, \quad u(x,0) = 0, \quad u_t(x,0) = 0$
- D > 0, the equation is hyperbolic (like the wave equation):
  $D = a_{12}^2 - a_{11} a_{22} = 1$
- The boundary values change with time. We choose a function $w(x,t) = \frac{x \sin t}{\pi}$ that satisfies the boundary conditions and look for $u = w + v$. The deviation v has zero boundary conditions, but its equation gains a source: $v_{tt} = v_{xx} + \frac{x \sin t}{\pi}$
- Separation of variables: we look for a product – standing waves; both sides equal the same constant:
  $u = X(x)\,T(t): \quad \frac{T''}{T} = \frac{X''}{X} = -\lambda$
- The eigenvalue problem $X'' + \lambda X = 0$ with X = 0 at both ends has nonzero solutions only for
  $\lambda_n = n^2, \quad X_n = \sin(nx), \quad n = 1, 2, \ldots$
- The time part for each n – oscillations at its own frequency:
  $T_n'' + n^2 T_n = 0 \;\Rightarrow\; T_n = a_n \cos nt + b_n \sin nt$
- The equation is linear – a sum of solutions is a solution (superposition): $u = w(x,t) + \sum_{n=1}^{\infty} T_n(t) \sin(nx)$
- All coefficients are zero: $a_n = \frac{2}{\pi} \int_0^{\pi} 0 \sin(nx)\,\mathrm{d}x$
- The coefficients from the initial (boundary) function by the eigenfunctions' orthogonality (integration by parts)
  $b_n = \frac{2}{n\pi} \int_0^{\pi} -\frac{x}{\pi} \sin(nx)\,\mathrm{d}x = \frac{2(-1)^n}{n^2 \pi}$
- We expand the source $F(x,t) = \frac{x \sin t}{\pi}$ into the eigenfunctions just like the initial condition:
  $F_n(t) = f_n \sin t$, the coefficients by an integral.
- The coefficients from the initial (boundary) function by the eigenfunctions' orthogonality (integration by parts)
  $f_n = \frac{2}{\pi} \int_0^{\pi} \frac{x}{\pi} \sin(nx)\,\mathrm{d}x = -\frac{2(-1)^n}{n\pi}$
- Duhamel's principle: each coefficient of the series is a forced oscillation – an oscillator with its own frequency and an external force F_n(t). The solution is the free oscillation plus the convolution of the force with the impulse response sin(ω_n t)/ω_n
  $T_n'' + n^2 T_n = F_n(t) \;\Rightarrow\; T_n(t) = a_n \cos(nt) + b_n \sin(n$
- The integral is computed generally for every n (K_n is the kernel from the previous step); where the general formula would divide by zero (resonance), separately
  $\int_0^t K_n(t-s) \sin s\,\mathrm{d}s = \frac{n \sin t - \sin(nt)}{n^3 - n}, \quad n = 1: \int$
- The solution
  $u(x,t) = \frac{x \sin t}{\pi} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2 \pi} \sin(nt) \sin(nx) + \frac{(\sin}{}$

Graph

In time | 3D surface



The solution u at the time t chosen by the slider; the button plays it.

Laplace's equation with an insulated bottom side (u_y = 0) – cosh instead of sinh.

Input: `Δu = 0; u_y(x,0) = 0; u(x,1) = sin(pi x); u(0,y) = 0; u(1,y) = 0`

Result

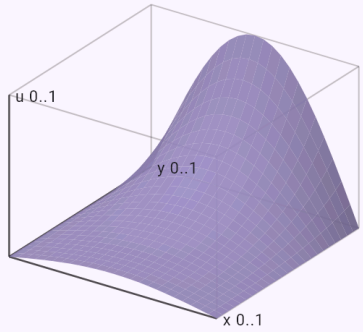
$$u(x,y) = \frac{\cosh(\pi y)\sin(\pi x)}{\cosh \pi}$$

Steps

- We solve the partial differential equation $u_{xx} + u_{yy} = 0$
- Conditions:
  $u_y(x,0) = 0, \quad u(x,1) = \sin(\pi x), \quad u(0,y) = 0, \quad u(1,y) = 0$
- D < 0, the equation is elliptic (like Laplace's equation):
  $D = a_{12}^2 - a_{11}a_{22} = -1$
- The side with nonzero values (the others have u = 0):
  $u(x,1) = \sin(\pi x)$
- Separation of variables: we look for a product:
  $u = X(x)\,Y(y): \quad \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$
- The eigenvalue problem $X'' + \lambda X = 0$ with X = 0 at both ends has nonzero solutions only for
  $\lambda_n = n^2\pi^2, \quad X_n = \sin(n\pi x), \quad n = 1, 2, \ldots$
- The second equation Y" = λY with a zero derivative on the opposite (insulated) side – hence cosh –, normalized to 1 on the given side: $Y_n = \frac{\cosh(n\pi y)}{\cosh(n\pi)}$
- The initial (boundary) function already is a sum of eigenfunctions, so only a few coefficients $c_n = 2\int_0^1 \sin(\pi x)\,\sin(n\pi x)\,\mathrm{d}x$ aren't zero: $c_1 = 1$, the others are zero.

- The solution: $u(x,y) = \frac{\cosh(\pi y)\sin(\pi x)}{\cosh \pi}$

Graph



The solution u as a surface over the domain. Turn it by dragging a finger.

Burgers' equation – the characteristics cross and a shock wave forms.

Input: `u_t + u u_x = 0; u(x,0) = e^(-x^2)`

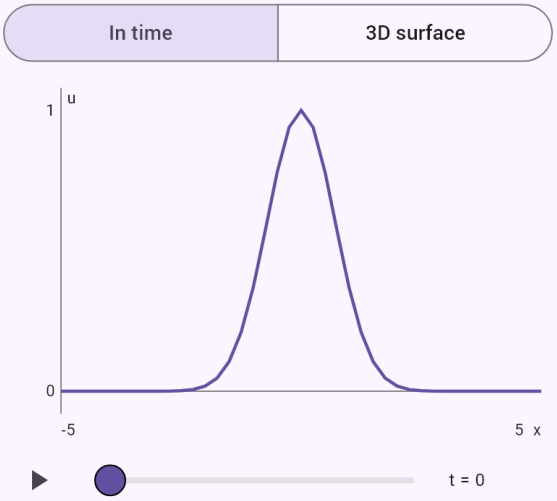
Result

$$u = f(x - ut), \quad t < 1.1658$$

Steps

- We solve the partial differential equation $u_t + uu_x = 0$
- Conditions: $u(x,0) = e^{-x^2}$
- Burgers' equation is nonlinear – the propagation speed is u itself. Along the characteristics u is constant, so the characteristics are straight lines
  $\frac{\mathrm{d}x}{\mathrm{d}t} = u, \quad \frac{\mathrm{d}u}{\mathrm{d}t} = 0 \;\Rightarrow\; u = f(x_0), \quad x = x_0 + f(x_0)\,t$
- The solution is given implicitly by $u = f(x - ut)$ – for each point we look for the x_0 whose characteristic leads to it.
- Where f decreases, the faster part of the wave catches up with the slower one and the characteristics cross. The first crossing comes at the time
  $t_b = -\frac{1}{\min f'} = -\frac{1}{-0.8577} \approx 1.1658$ $(x_0 \approx 0.710)$ – a shock wave forms and the smooth solution ceases to exist.
- The solution (implicitly, until a shock forms):
  $u = f(x - ut), \quad t < 1.1658$

Graph



The solution u at the time t chosen by the slider; the button plays it.

Burgers' equation with viscosity – the Cole-Hopf transformation into the heat equation.

Input: `u_t + u u_x = 0.1u_xx; u(x,0) = -tanh(x)`

Result

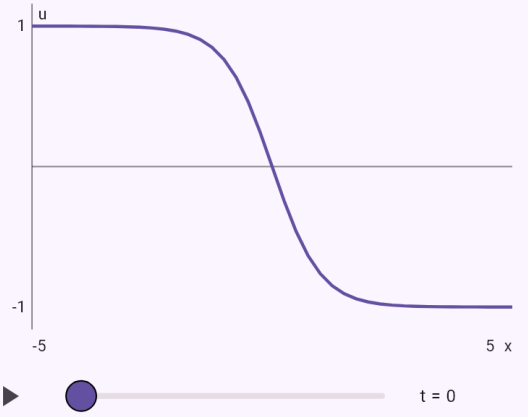
$$u(x,t) = -0.2\,\frac{\varphi_x}{\varphi}$$

Steps

- We solve the partial differential equation $u_t + uu_x = 0.1u_{xx}$
- Conditions: $u(x,0) = -\tanh x$
- With viscosity v > 0 the Cole-Hopf transformation linearizes the equation: $u = -0.2\,\frac{\varphi_x}{\varphi} \;\Rightarrow\; \varphi_t = 0.1\,\varphi_{xx}$ – Burgers' equation turns into the heat equation.
- The initial condition for φ $\varphi(x,0) = e^{-\frac{1}{0.2}\int_0^x f(s)\,\mathrm{d}s} = e^{-\frac{1}{0.2}(-\ln(\cosh x))}$
- The heat equation is solved by the convolution with the heat kernel; after substituting, u is a ratio of two integrals, computed numerically for the graph $\varphi(x,t) = \frac{1}{\sqrt{4\pi\nu t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\nu t}}\,\varphi(y,0)\,\mathrm{d}y, \quad u = \frac{\int \frac{x-y}{t}\,I}{\int K}$
- The integral has no elementary form – the solution stays a convolution, the graph is computed numerically: $u(x,t) = -0.2\,\frac{\varphi_x}{\varphi}$

Graph

In time | 3D surface



The solution u at the time t chosen by the slider; the button plays it.

Godunov's scheme – the shock moves at the speed 1/2.

Input: `upwind(u_t + u u_x = 0; u(x,0) = (1 - tanh(10x))/2; -2 < x < 4; h = 0.1; T = 2)`

Result

$$u(1, 2) \approx 0.513583$$

Steps

- Solved numerically on a grid: $u_t + uu_x = 0$, $u(x, 0) = \dfrac{1 - \tanh(10x)}{2}$
- Grid: 61 nodes, 25 time steps; $h = 0.1, \quad k = 0.08$
- A shock wave is handled by Godunov's conservative scheme – the flux F between the cells from the exact solution of the Riemann problem (a jump)

$$u_i^{n+1} = u_i^n - \frac{k}{h}\left(F_{i+1/2} - F_{i-1/2}\right), \quad F_{i+1/2} = \begin{cases} \min \\ u_i \le u \le u_{i+1} \\ \max \\ u_{i+1} \le u \le u_i \end{cases}$$

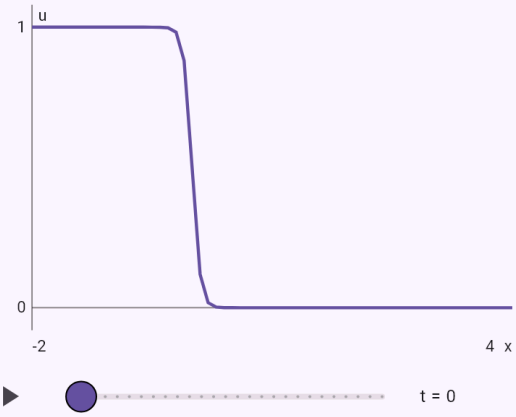
- The CFL condition (the numerical domain of dependence contains the characteristic): $\dfrac{k}{h} \max |u| = 0.8000 \le 1$
- The drop from u_L to u_R < u_L travels as a shock wave at the speed of the Rankine-Hugoniot condition: $s = \dfrac{u_L + u_R}{2} = \dfrac{1.0000 + 0.0000}{2} = 0.5000$
- The solution at chosen time levels

n	t_n	$x = -2$	$x = -1.4$	$x = -0.8$	$x = -0.$
0	0	1.0000	1.0000	1.0000	0.9820
4	0.32	1.0000	1.0000	1.0000	0.9998
8	0.64	1.0000	1.0000	1.0000	1.0000
12	0.96	1.0000	1.0000	1.0000	1.0000
16	1.28	1.0000	1.0000	1.0000	1.0000
20	1.6	1.0000	1.0000	1.0000	1.0000
24	1.92	1.0000	1.0000	1.0000	1.0000
25	2	1.0000	1.0000	1.0000	1.0000

- The result at the middle node: $u(1, 2) \approx 0.513583$

Graph

In time | 3D surface



The solution u at the time t chosen by the slider; the button plays it.

Heat in a square plate by ADI – stable even for a large step.

Input: `adi(u_t = u_xx + u_yy; 0 < x < 1; 0 < y < 1; u(x,y,0) = sin(pi x)sin(pi y); h = 0.1;`

k = 0.01; T = 0.1)

Result

$$u(0.5, 0.5, 0.1) \approx 0.140956$$

Steps

- Solved numerically on a grid: $u_t = u_{xx} + u_{yy}$, $u(x, y, 0) = \sin(\pi y)\sin(\pi x)$
- Grid: 121 nodes, 10 time steps (the ratios of the time step to the squared space steps).

$$h = 0.1, \quad k = 0.01, \quad r_x = \frac{\kappa k}{h_x^2} = 1.0000, \quad r_y = \frac{\kappa k}{h_y^2} =$$

- The alternating direction implicit method (ADI) of Peaceman and Rachford: half a step is implicit in x and explicit in y, the other half the other way round

$$\left(1 - \frac{r_x}{2}\delta_x^2\right)u^{n+1/2} = \left(1 + \frac{r_y}{2}\delta_y^2\right)u^n, \quad \left(1 - \frac{r_y}{2}\delta_y^2\right)u^{n+1}$$

- Von Neumann analysis: both halves of the step together have the amplification for any step – ADI is unconditionally stable.

$$|G| = \left|\frac{1 - 2r_y \sin^2\frac{\beta h_y}{2}}{1 + 2r_x \sin^2\frac{\alpha h_x}{2}} \cdot \frac{1 - 2r_x \sin^2\frac{\alpha h_x}{2}}{1 + 2r_y \sin^2\frac{\beta h_y}{2}}\right| \le 1$$

- In the first half of the step we solve 9 tridiagonal systems along the grid's rows, in the second 9 along its columns – each by the Thomas algorithm. That is far cheaper than one large system for the whole grid.
- The solution at chosen time levels

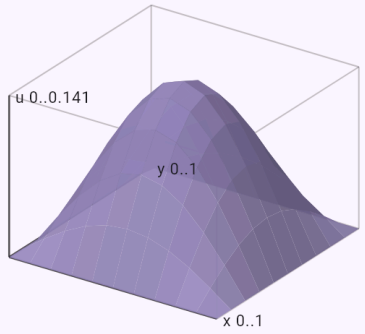
t	$u(0.5, 0.5)$	$\max\lvert u\rvert$
0	1.000000	1.000000
0.01	0.822069	0.822069
0.02	0.675798	0.675798
0.03	0.555553	0.555553
0.04	0.456703	0.456703
0.05	0.375442	0.375442
0.06	0.308639	0.308639
0.07	0.253723	0.253723
0.08	0.208578	0.208578
0.09	0.171465	0.171465
0.1	0.140956	0.140956

- The largest error against the exact solution at the last level (exactly 0.138911 in the middle).

$$u = \sum_{m,n} c_{mn}\, e^{-\kappa\pi^2(m^2/a^2 + n^2/b^2)t} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}, \quad \max$$

- The result at the middle node: $u(0.5, 0.5, 0.1) \approx 0.140956$

Graph



The solution u as a surface over the domain. Turn it by dragging a finger.

Laplace's equation by conjugate gradients – compared with SOR and multigrid.

Input: cg(u_xx + u_yy = 0; u(x,1) = sin(pi x); u(x,0) = 0; u(0,y) = 0; u(1,y) = 0; h =

0.125)

Result

$$u(0.5, 0.5) \approx 0.202915$$

Steps

- Solved numerically on a grid: $u_{xx} + u_{yy} = 0$, $u(x,1) = \sin(\pi x)$, $u(x,0) = 0$, $u(0,y) = 0$, $u(1,y) = 0$
- A square grid: $h = 0.125$, 81 nodes, 49 of them unknowns inside.
- The five-point scheme – u_xx + u_yy replaced by differences, one equation at each node: $u_{i,j} = \frac{1}{4}(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1})$
- The conjugate gradient method – the system $-\Delta_h u = -f$ is symmetric and positive definite; each step reduces the error along a new direction p_k, conjugate to the previous ones

$$\alpha_k = \frac{r_k^T r_k}{p_k^T A p_k}, \quad u_{k+1} = u_k + \alpha_k p_k, \quad r_{k+1} = r_k - \alpha_k A_l$$

- The iterations from a zero guess inside (chosen nodes and the largest change)

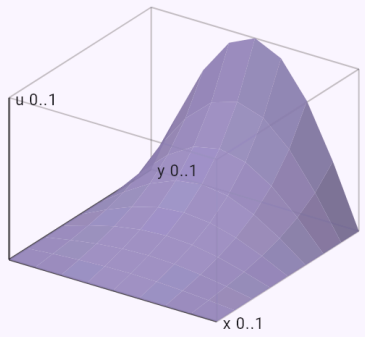
k	$u_{4,4}$	$u_{3,4}$	$u_{4,5}$	$\max \lvert\Delta u\rvert$
0	0.000000	0.000000	0.000000	
1	0.000000	0.000000	0.000000	0.465
2	0.000000	0.000000	0.000000	0.275
3	0.000000	0.000000	0.176522	0.177
4	0.116818	0.107926	0.251421	0.117
5	0.168686	0.155846	0.284676	0.0784
6	0.192179	0.177550	0.299739	0.0529
7	0.202915	0.187469	0.306622	0.0358
8	0.202915	0.187469	0.306622	$1.02 \cdot 10^{-15}$

- The iteration stopped after 8 steps, when the largest change fell below 10^{-4}
- The methods on the same grid with the same tolerance: Jacobi 71 iterations, Gauss-Seidel 42, SOR with ω = 1.446 only 16
- Faster methods for the same system (49 unknowns): conjugate gradients 8 iterations, multigrid 5 V-cycles. Multigrid's number of V-cycles hardly depends on how fine the grid is.
- The values at the grid's nodes

$y \backslash x$	0	0.125	0.25	0.375	0.5
1	0.0000	0.3827	0.7071	0.9239	1.0000
0.875	0.0000	0.2591	0.4787	0.6254	0.6770
0.75	0.0000	0.1749	0.3232	0.4222	0.4570
0.625	0.0000	0.1173	0.2168	0.2833	0.3066
0.5	0.0000	0.0777	0.1435	0.1875	0.2029
0.375	0.0000	0.0498	0.0920	0.1202	0.1301
0.25	0.0000	0.0295	0.0545	0.0712	0.0771
0.125	0.0000	0.0137	0.0253	0.0331	0.0358
0	0.0000	0.0000	0.0000	0.0000	0.0000

- The largest error against the exact solution at the last level: $\max_{i,j} \lvert u(x_i, y_j) - u_{i,j}\rvert \approx 0.00432$ (exactly 0.199268 in the middle).
- The result at the middle node: $u(0.5, 0.5) \approx 0.202915$

Graph



The solution u as a surface over the domain. Turn it by dragging a finger.

Poisson's equation by multigrid – a few V-cycles on a fine grid.

Input: `multigrid(u_xx + u_yy = -2; 0 < x < 1; 0 < y < 1; h = 0.0625)`

Result

$$u(0.5, 0.5) \approx 0.146886$$

Steps

- Solved numerically on a grid: $u_{xx} + u_{yy} = -2$,
- A square grid: $h = 0.0625$, 289 nodes, 225 of them unknowns inside.
- The five-point scheme – u_xx + u_yy replaced by differences, one equation at each node

$$u_{i,j} = \frac{1}{4}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - h^2 f_{i,j}\right)$$

- Multigrid (a V-cycle): the Gauss-Seidel iteration quickly damps only the rapidly varying part of the error (S² – two sweeps). The smooth remainder r goes to the coarser grid 2h, is corrected there (by multigrid again) and the correction is interpolated back

$$u \leftarrow S^2 u, \quad r = f - A_h u, \quad r_{2h} = Rr, \quad A_{2h} e_{2h} = r_{2h},$$

- The iterations from a zero guess inside (chosen nodes and the largest change)

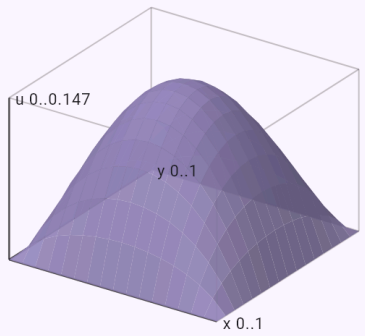
k	$u_{8,8}$	$u_{7,8}$	$u_{8,9}$	$\max \lvert\Delta u\rvert$
0	0.000000	0.000000	0.000000	
1	0.134291	0.132314	0.132794	0.134
2	0.145872	0.143893	0.143981	0.0116
3	0.146813	0.144856	0.144867	$9.83 \cdot 10^{-4}$
4	0.146886	0.144932	0.144933	$8.06 \cdot 10^{-5}$

- The iteration stopped after 4 steps, when the largest change fell below 10^{-4}
- The methods on the same grid with the same tolerance: Jacobi 179 iterations, Gauss-Seidel 108, SOR with ω = 1.674 only 25
- Faster methods for the same system (225 unknowns): conjugate gradients 15 iterations, multigrid 4 V-cycles. Multigrid's number of V-cycles hardly depends on how fine the grid is.
- The values at the grid's nodes

$y \backslash x$	0	0.125	0.25	0.375	0.5
1	0.0000	0.0000	0.0000	0.0000	0.0000
0.875	0.0000	0.0362	0.0561	0.0665	0.0697
0.75	0.0000	0.0561	0.0903	0.1086	0.1143
0.625	0.0000	0.0665	0.1085	0.1317	0.1390
0.5	0.0000	0.0697	0.1143	0.1390	0.1469
0.375	0.0000	0.0665	0.1085	0.1317	0.1390
0.25	0.0000	0.0561	0.0902	0.1085	0.1143
0.125	0.0000	0.0362	0.0561	0.0665	0.0697
0	0.0000	0.0000	0.0000	0.0000	0.0000

- The largest error against the exact solution at the last level: $\max_{i,j} \lvert u(x_i, y_j) - u_{i,j} \rvert \approx 4.56 \cdot 10^{-4}$ (exactly 0.147342 in the middle).
- The result at the middle node: $u(0.5, 0.5) \approx 0.146886$

Graph



The solution u as a surface over the domain. Turn it by dragging a finger.

The finite element method with triangles – the stiffness matrix and the same system as the five-point

scheme.

Input: `fem(u_xx + u_yy = -2; 0 < x < 1; 0 < y < 1; h = 0.25)`

Result

$$u(0.5, 0.5) \approx 0.140625$$

Steps

- Solved numerically on a grid: $u_{xx} + u_{yy} = -2$,
- A square grid: $h = 0.25$, 25 nodes, 9 of them unknowns inside.
- The finite element method: the equation is multiplied by a test function v and integrated by parts (the weak formulation); the solution is sought as a combination of "hat" functions φ_j

$$\iint_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x\,\mathrm{d}y = -\iint_\Omega f\, v \,\mathrm{d}x\,\mathrm{d}y, \quad u = \sum_j u_j \varphi_j$$

- The mesh: each cell is split by a diagonal into two right triangles – 32 triangles, 25 nodes, 9 of them unknown.
- The stiffness matrix of one triangle (the gradients of the linear functions are constant on it)

$$K^e_{ab} = \iint_T \nabla\varphi_a \cdot \nabla\varphi_b = |T|\, \nabla\varphi_a \cdot \nabla\varphi_b, \quad K^e = \begin{pmatrix} 1.0 \\ -0. \\ -0. \end{pmatrix}$$

- Adding the contributions of the six triangles around an inner node gives its equation – on a regular grid it is the five-point scheme multiplied by h_x·h_y

$$4.0000\, u_{i,j} - 1.0000(u_{i+1,j} + u_{i-1,j}) - 1.0000(u_{i,j+1} + $$

- The conjugate gradient method – the system $-\Delta_h u = -f$ is symmetric and positive definite; each step reduces the error along a new direction p_k, conjugate to the previous ones

$$\alpha_k = \frac{r_k^T r_k}{p_k^T A p_k}, \quad u_{k+1} = u_k + \alpha_k p_k, \quad r_{k+1} = r_k - \alpha_k A_$$

- The iterations from a zero guess inside (chosen nodes and the largest change)

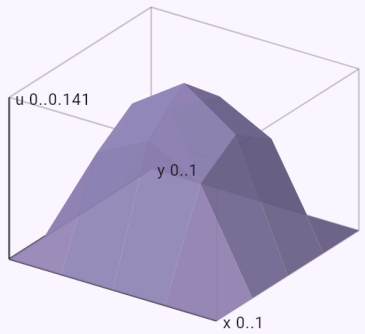
k	$u_{2,2}$	$u_{1,2}$	$u_{2,3}$	$\max \lvert\Delta u\rvert$
0	0.000000	0.000000	0.000000	
1	0.093750	0.093750	0.093750	0.0937
2	0.136364	0.110795	0.110795	0.0426
3	0.140625	0.109375	0.109375	0.00426
4	0.140625	0.109375	0.109375	$1.16 \cdot 10^{-17}$

- The iteration stopped after 4 steps, when the largest change fell below 10^{-4}
- The methods on the same grid with the same tolerance: Jacobi 19 iterations, Gauss-Seidel 11, SOR with ω = 1.172 only 8
- Faster methods for the same system (9 unknowns): conjugate gradients 4 iterations, multigrid 3 V-cycles. Multigrid's number of V-cycles hardly depends on how fine the grid is.
- The values at the grid's nodes

$y \backslash x$	0	0.25	0.5	0.75	1
1	0.0000	0.0000	0.0000	0.0000	0.0000
0.75	0.0000	0.0859	0.1094	0.0859	0.0000
0.5	0.0000	0.1094	0.1406	0.1094	0.0000
0.25	0.0000	0.0859	0.1094	0.0859	0.0000
0	0.0000	0.0000	0.0000	0.0000	0.0000

- The largest error against the exact solution at the last level: $\max_{i,j} |u(x_i, y_j) - u_{i,j}| \approx 0.00672$ (exactly 0.147342 in the middle).
- The result at the middle node: $u(0.5, 0.5) \approx 0.140625$

Graph



The solution u as a surface over the domain. Turn it by dragging a finger.

Poisson's equation on a square – a double Fourier series, compare with the sor scheme.

Input: `Δu = -2; u(0,y) = 0; u(1,y) = 0; u(x,0) = 0; u(x,1) = 0`

Result

$$u(x,y) = -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{4((-1)^m - 1)(2 -$$

$$mn\pi^4\left(m^2 +$$

Steps

- We solve the partial differential equation $u_{xx} + u_{yy} = -2$
- Conditions: $u(0,y) = 0, \quad u(1,y) = 0, \quad u(x,0) = 0, \quad u(x,1) = 0$
- D < 0, the equation is elliptic (like Laplace's equation): $D = a_{12}^2 - a_{11}a_{22} = -1$
- The part with the source is a double series of both directions' eigenfunctions – each product X_m·Y_n meets the conditions on the sides and the Laplacian only multiplies it by a number: $v = \sum_m \sum_n c_{mn}\, X_m(x)\, Y_n(y)$, where

$$X_m = \sin(m\pi x), \quad Y_n = \sin(n\pi y), \quad \Delta(X_m Y_n) = -($$

- Expand the source into the same double series. Substituting into Δv = f and comparing coefficients gives $c_{mn} = -\frac{f_{mn}}{m^2\pi^2 + n^2\pi^2}$, so

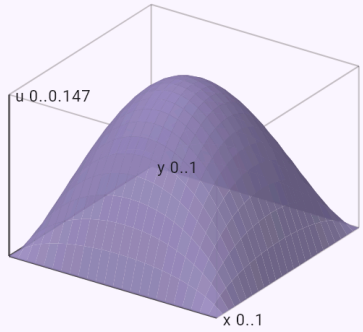
$$f_{mn} = 4\int_0^1\int_0^1 -2\,\sin(m\pi x)\,\sin(n\pi y)\,\mathrm{d}y\,\mathrm{d}x = \frac{4((-}{}$$

$$c_{mn} = -\frac{4((-1)^m - 1)(2 - 2(-1)^n)}{mn\pi^4\left(m^2 + n^2\right)}$$

- The solution

$$u(x,y) = -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{4((-1)^m - 1)(2 - 2(-1)^n)}{mn\pi^4\left(m^2 + n^2\right)} \sin(m$$

Graph



The solution u as a surface over the domain. Turn it by dragging a finger.

Derivatives on all four sides – the solvability condition and a solution up to a constant.

Input: `u_xx + u_yy = 0; u_x(0,y) = 0; u_x(1,y) = 1; u_y(x,0) = 0; u_y(x,1) = -1 + cos(πx)`

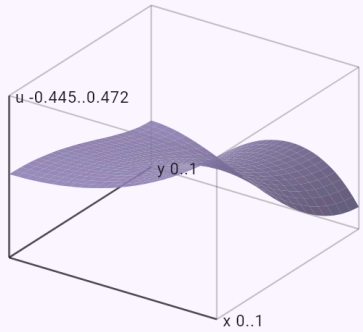
Result

$$u(x,y) = \frac{x^2 - y^2}{2} + \frac{\cos(\pi x)\cosh(\pi y)}{\pi \sinh \pi}$$

Steps

- We solve the partial differential equation $u_{xx} + u_{yy} = 0$
- Conditions
  $u_x(0,y) = 0, \quad u_x(1,y) = 1, \quad u_y(x,0) = 0, \quad u_y(x,1)$
- D < 0, the equation is elliptic (like Laplace's equation):
  $D = a_{12}^2 - a_{11}a_{22} = -1$
- All four sides have the derivative (a flux) given. By Gauss's theorem the total flux through the boundary must match the source inside ($\iint_\Omega f \, \mathrm{d}S = 0$) – the solvability condition. It holds; a solution exists, but only up to an added constant. The flux through the boundary
  $\oint_{\partial\Omega} \frac{\partial u}{\partial n} \, \mathrm{d}s = \int_0^1 (u_x(1,y) - u_x(0,y))\mathrm{d}y + \int_0^1 (u_y(x,1$
- A quadratic polynomial P takes over the sides' mean fluxes; the rest v has a zero-mean flux on every side, so it is summed side by side as before:
  $u = P + v, \quad P = \frac{x^2 - y^2}{2}$; the mean fluxes
  $\overline{u_x(0,y)} = 0, \quad \overline{u_x(1,y)} = 1, \quad \overline{u_y(x,0)} = 0, \quad \overline{u_y(x,1)}$
- The side with nonzero values (the others have u = 0):
  $v_y(x,1) = \cos(\pi x)$
- Separation of variables: we look for a product:
  $u = X(x)\,Y(y): \quad \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$
- The eigenvalue problem $X'' + \lambda X = 0$ with X' = 0 at both ends (insulated ends) has the solutions
  $\lambda_n = n^2\pi^2, \quad X_n = \cos(n\pi x), \quad n = 0, 1, 2, \ldots$
- A derivative is prescribed on the given side and the opposite side is insulated: cosh, normalized so that its derivative on the given side is 1: $Y_n = \frac{\cosh(n\pi y)}{n\pi \sinh(n\pi)}$
- The initial (boundary) function already is a sum of eigenfunctions, so only a few coefficients
  $c_n = 2\int_0^1 \cos(\pi x)\,\cos(n\pi x)\,\mathrm{d}x$ aren't zero: $c_1 = 1$, the others are zero.
- The solution (C is an arbitrary constant – with derivatives on all sides u is fixed only up to a shift)
  $u(x,y) = \frac{x^2 - y^2}{2} + \frac{\cos(\pi x)\cosh(\pi y)}{\pi \sinh \pi} + C$

Graph



The solution u as a surface over the domain. Turn it by dragging a finger.

Variable coefficients – a Sturm–Liouville problem with the weight 1/x, eigenfunctions sin(n ln x).

Input: `u_t = x^2u_xx + x u_x; u(1,t) = 0; u(e^π,t) = 0; u(x,0) = 1`

Result

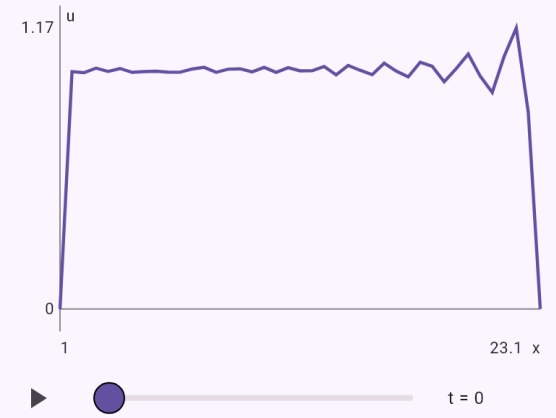
$$u(x,t) = \sum_{n=1}^{\infty} \frac{2-2(-1)^n}{n\pi}\, e^{-n^2 t}\, \sin(n$$

Steps

- We solve the partial differential equation $u_t = x^2 u_{xx} + x u_x$
- Conditions: $u(1,t) = 0, \quad u(e^\pi, t) = 0, \quad u(x,0) = 1$
- Separation of variables for an equation with variable coefficients – both sides must equal the same constant: $u = X(x)\,T(t): \quad \frac{T'}{T} = \frac{x^2 X'' + x X'}{X} = -\lambda$
- Multiplying X's equation by a suitable function gives a Sturm–Liouville problem $(xX')' + \lambda \frac{1}{x} X = 0, \quad X(1) = X(e^\pi) = 0$ with p = x and the weight w = $\frac{1}{x}$. Its eigenfunctions are orthogonal with the weight w.
- X's equation is Euler's, try X = x^r: $x^2 X'' + x X' + \lambda X = 0, \quad X = x^r: \quad r^2 + \lambda = 0$. A nonzero solution with X = 0 at both ends exists only for complex roots: $r = \pm i\omega, \quad \lambda = \omega^2$. The ends' conditions give $\omega_n = n, \quad \lambda_n = n^2, \quad X_n = \sin(n \ln x), \quad n = 1, 2, \ldots$
- The eigenfunctions' orthogonality with the weight w: $\int_1^{e^\pi} X_m X_n \frac{1}{x}\,\mathrm{d}x = 0 \quad (m \neq n)$; their norm $\int_1^{e^\pi} X_n^2 \frac{1}{x}\,\mathrm{d}x = \frac{\pi}{2}$
- The time part for each n – an exponential decay, higher frequencies damped faster: $T_n' = -\lambda_n T_n \;\Rightarrow\; T_n = e^{-\lambda_n t}$
- The initial function f's coefficients are its projections with the weight w: $c_n = \frac{1}{\|X_n\|^2} \int_1^{e^\pi} f(x)\, X_n \frac{1}{x}\,\mathrm{d}x$. After a substitution they become an ordinary sine series' coefficients of $\xi = \ln x: \quad 1$
- The coefficients from the initial (boundary) function by the eigenfunctions' orthogonality (integration by parts) $c_n = \frac{2}{\pi} \int_0^{\pi} 1 \cdot \sin(n\xi)\,\mathrm{d}\xi = \frac{2-2(-1)^n}{n\pi}$
- The solution $u(x,t) = \sum_{n=1}^{\infty} \frac{2-2(-1)^n}{n\pi}\, e^{-n^2 t}\, \sin(n \ln x)$

Graph

In time | 3D surface



The solution u at the time t chosen by the slider; the button plays it.

The MUSCL scheme with a limiter – compare its error with the Lax–Wendroff and Lax–Friedrichs schemes.

Input: `muscl(u_t + u_x = 0; u(x,0) = e^(-50(x-0.3)^2); 0 < x < 1; h = 0.02; k = 0.016; to =`

0.4)

Result

$$u(0.5, 0.4) \approx 0.137131$$

Steps

- Solved numerically on a grid: $u_t + u_x = 0$, $u(x,0) = e^{-50(x-0.3)^2}$
- The grid on $0 \le x \le 1$: 51 nodes, 25 time steps up to t = 0.4; $h = 0.02, \quad k = 0.016, \quad \nu = \dfrac{ak}{h} = 0.8000$ is the Courant number.
- The exact solution is carried along the characteristics $x - t = C$ at the speed a = 1
- The MUSCL scheme – a linear profile in each cell with the slope σ_i bounded by a limiter; the flux comes from the upwind cell moved by half a step. Second order where the solution is smooth, no wiggles at jumps

$$u_i^{n+1} = u_i^n - \nu\big(F_{i+1/2} - F_{i-1/2}\big), \quad F_{i+1/2} = u_i^n + \frac{1-\iota}{2}$$

- The slope in a cell is bounded by minmod – zero at an extremum, otherwise the smaller of the two one-sided differences

$$\sigma_i = \operatorname{minmod}(u_i - u_{i-1},\, u_{i+1} - u_i), \quad \operatorname{minmod}(p,q) =$$

- The CFL condition (the numerical domain of dependence contains the characteristic): $|\nu| = 0.8000 \le 1$
- A scheme with a limiter is nonlinear; instead of von Neumann's analysis, it doesn't increase the total variation (TVD – no new wiggles): $TV(u^{n+1}) \le TV(u^n), \quad TV(u) = \sum_i |u_{i+1} - u_i|$, provided $|\nu| \le 1$
- The first time level

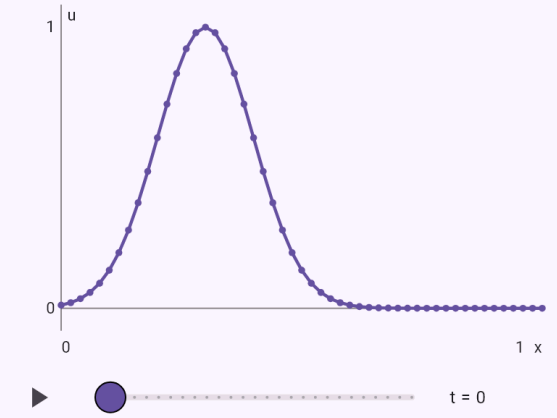
i	x_i	u_i^0	u_i^1
0	0	0.011109	0.006787
1	0.02	0.019841	0.012157
2	0.04	0.034047	0.022244
3	0.06	0.056135	0.037834
4	0.08	0.088922	0.061836
5	0.1	0.135335	0.097114
6	0.12	0.197899	0.146556
7	0.14	0.278037	0.212520
8	0.16	0.375311	0.296121
9	0.18	0.486752	0.396466
10	0.2	0.606531	0.510054
11	0.22	0.726149	0.631294
12	0.24	0.835270	0.749675
13	0.26	0.923116	0.855301
14	0.28	0.980199	0.937515
15	0.3	1.000000	0.985743
16	0.32	0.980199	0.997624
17	0.34	0.923116	0.971765
18	0.36	0.835270	0.908008
19	0.38	0.726149	0.815148
20	0.4	0.606531	0.703065
21	0.42	0.486752	0.581921
22	0.44	0.375311	0.463331
23	0.46	0.278037	0.354486
24	0.48	0.197899	0.260604
⋮	⋮	⋮	⋮
41	0.82	0.000001	0.000003
42	0.84	$4.65572 \cdot 10^{-7}$	0.000001
43	0.86	$1.54975 \cdot 10^{-7}$	$3.87038 \cdot 10^{-7}$
44	0.88	$4.95641 \cdot 10^{-8}$	$1.28207 \cdot 10^{-7}$
45	0.9	$1.523 \cdot 10^{-8}$	$4.08092 \cdot 10^{-8}$
46	0.92	$4.49635 \cdot 10^{-9}$	$1.24822 \cdot 10^{-8}$
47	0.94	$1.27541 \cdot 10^{-9}$	$3.66871 \cdot 10^{-9}$
48	0.96	$3.47589 \cdot 10^{-10}$	$1.03614 \cdot 10^{-9}$
49	0.98	$9.10147 \cdot 10^{-11}$	$2.81198 \cdot 10^{-10}$
50	1	$2.28973 \cdot 10^{-11}$	$7.73912 \cdot 10^{-11}$

- The solution at chosen time levels

n	t_n	$x = 0$	$x = 0.1$	$x = 0.2$
0	0	0.0111	0.1353	0.6065
4	0.064	0.0013	0.0311	0.2605
8	0.128	0.0001	0.0046	0.0750
12	0.192	$5.542 \cdot 10^{-6}$	0.0005	0.0144
16	0.256	$1.937 \cdot 10^{-7}$	$3.261 \cdot 10^{-5}$	0.0019
20	0.32	$4.496 \cdot 10^{-9}$	$1.529 \cdot 10^{-6}$	0.0002
24	0.384	$6.928 \cdot 10^{-11}$	$4.869 \cdot 10^{-8}$	$1.032 \cdot 10^{-5}$
25	0.4	$2.29 \cdot 10^{-11}$	$1.937 \cdot 10^{-8}$	$4.85 \cdot 10^{-6}$

- The largest error against the exact solution at the last level: $\max_i \left|u(x_i, 0.4) - u_i^{25}\right| \approx 0.0326$ (exactly 0.135335 in the middle).
- The result at the middle node: $u(0.5, 0.4) \approx 0.137131$

Graph



Dots: the numerical solution at the grid nodes; dashed: the exact solution. The slider picks the time level.

Burgers’ equation with viscosity by the MUSCL scheme – compared with the Cole–Hopf

transformation.

Input: `muscl(u_t + u u_x = 0.05u_xx; u(x,0) = (1 - tanh(x))/2; -5 < x < 5; T = 1)`

Result

$$u(0,1) \approx 0.805836$$

Steps

- Solved numerically on a grid: $u_t + uu_x = 0.05u_{xx}$, $u(x,0) = \dfrac{1-\tanh x}{2}$
- Grid: 41 nodes, 10 time steps; $h = 0.25, \quad k = 0.1$
- The MUSCL scheme for Burgers' equation: the limited slopes σ_i (limiter) give the values at the cells' edges, the flux between them comes from Godunov's solution of the jump, and Heun's method steps in time (two stages)

$$u^* = u^n + k\,L(u^n), \quad u^{n+1} = \frac{1}{2}(u^n + u^* + k\,L(u^*)),$$

- The slope in a cell is bounded by minmod – zero at an extremum, otherwise the smaller of the two one-sided differences

$$\sigma_i = \operatorname{minmod}(u_i - u_{i-1},\, u_{i+1} - u_i), \quad \operatorname{minmod}(p,q) =$$

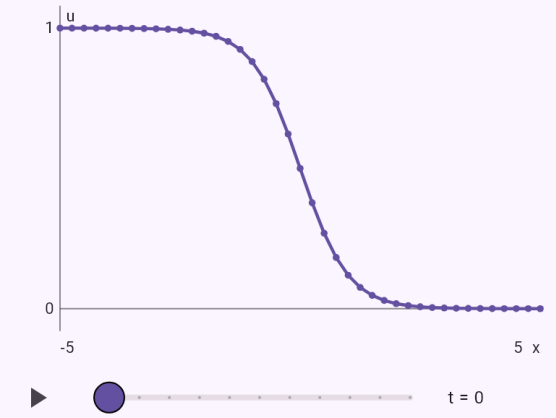
- The CFL condition (the numerical domain of dependence contains the characteristic): $\dfrac{k}{h}\max|u| = 0.4000 \le 0.5$
- The viscous term ν·u_xx is added by a central difference: $+\dfrac{\nu k}{h^2}\left(u^n_{i+1} - 2u^n_i + u^n_{i-1}\right)$. The explicit step is stable since $\dfrac{\nu k}{h^2} = 0.0800 \le \dfrac{1}{2}$
- The drop from u_L to u_R < u_L travels as a shock wave at the speed of the Rankine-Hugoniot condition: $s = \dfrac{u_L + u_R}{2} = \dfrac{1.0000 + 4.54\cdot 10^{-5}}{2} = 0.5000$
- The solution at chosen time levels

n	t_n	$x=-5$	$x=-4$	$x=-3$	$x=-2$	$x=$
0	0	1.0000	0.9997	0.9975	0.9820	0.8
2	0.2	1.0000	0.9998	0.9982	0.9870	0.9
4	0.4	1.0000	0.9998	0.9987	0.9907	0.9
6	0.6	1.0000	0.9999	0.9991	0.9933	0.9
8	0.8	1.0000	0.9999	0.9993	0.9952	0.9
10	1	1.0000	0.9999	0.9995	0.9965	0.9

- The largest error against the exact solution at the last level: $\max_i |u(x_i,1) - u_i^{10}| \approx 0.0136$ (exactly 0.811680 in the middle).
- The result at the middle node: $u(0,1) \approx 0.805836$

Graph

In time | 3D surface



Dots: the numerical solution at the grid nodes; dashed: the exact solution. The slider picks the time level.

Finite elements on an L-shaped domain – triangulation by ear clipping, the 3D surface over the domain

only.

Input: `fem(Δu = -4; polygon = (0,0), (2,0), (2,1), (1,1), (1,2), (0,2); u = 0; h = 0.25)`

Result

$$u(0.8125, 0.8125) \approx 0.506527$$

Steps

- Solved numerically on a grid: $u_{xx} + u_{yy} = -4$, $u = 0 \quad (\partial\Omega)$
- The domain is a polygon with the vertices $(0,0),\ (2,0),\ (2,1),\ (1,1),\ (1,2),\ (0,2)$, its area $|\Omega| = 3.0000$
- The finite element method: the equation is multiplied by a test function v and integrated by parts (the weak formulation); the solution is sought as a combination of "hat" functions φ_j

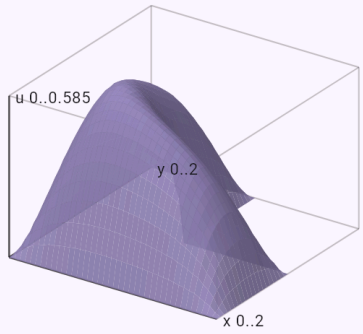
$$\iint_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x\,\mathrm{d}y = -\iint_\Omega f\, v \,\mathrm{d}x\,\mathrm{d}y, \quad u = \sum_j u_j \varphi_j$$

- Triangulation: clipping "ears" (a convex vertex whose triangle holds no other vertex) gives 4 triangles; $4\times$ each is split by its edges' midpoints into four – 1024 triangles, 561 nodes, 465 of them inside (unknown), the longest edge $h = 0.1398$
- The stiffness matrix of one triangle (the gradients of the linear functions are constant on it)

$$K^e_{ab} = \frac{b_a b_b + c_a c_b}{4|T|}, \quad b_a = y_{a+1} - y_{a+2}, \quad c_a = x_{a+2} - \dots$$

- The values at the boundary nodes are given, their contributions move to the right-hand side: $u_j = 0 \quad (j \in \partial\Omega)$
- The sparse system K·U = F for 465 unknowns is solved by the conjugate gradient method in 128 iterations. The largest value: $u(0.6875, 0.625) \approx 0.585873$
- The result at the middle node: $u(0.8125, 0.8125) \approx 0.506527$

Graph



The solution u as a surface over the domain. Turn it by dragging a finger.

Known limitations

- Analytically only linear equations with one unknown function u of two variables, and Burgers' equation; other nonlinear equations and systems not.
- By separation of variables the heat and wave equations on an interval; a source only as a sum of products p(x)·q(t), for Poisson's equation at most three products p(x)·q(y). Variable coefficients only of Euler's type αx²u_xx + βxu_x with zero values at the ends.
- The Fourier transform in closed form for a Gaussian, a polynomial and functions with f″ = λf; otherwise the convolution stays and the graph is computed numerically.
- Numerically in 2D the heat and Laplace's equations on a rectangle with a regular grid; on a polygon only the finite element method with linear triangles and a boundary value (not a derivative), without holes and with at most 40 vertices.

39. Recurrences and generating functions

The app solves linear recurrences with constant coefficients of any order by the characteristic equation – multiple and complex roots too (the complex form of the solution and its conversion to the real one by de Moivre's theorem) – and with a right side of the form $P(n)\cdot q^n$ (also cos nφ, sin nφ, resonance and a sum of right sides). The initial conditions determine the constants; as a check, the app computes further terms from the recurrence and compares them with the formula.

The “Method” chip offers the generating function too ($A(x) = \Sigma\, a_n x^n$, partial fractions, geometric series), matrix powers (diagonalization) or, for the first order, summing. A first-order recurrence with a variable coefficient ($a_{n+1} = (n + 1)a_n$) is solved by a product (factorials), convertible nonlinear recurrences by the substitution $b_n = 1/a_n$ or $b_n = \ln a_n$. A system of two recurrences is solved by eliminating one sequence or by the matrix, a divide-and-conquer recurrence $T(n) = aT(n/b) + f(n)$ by $n = b^k$ – with the master theorem's estimate.

With the “Generating function” operation the app finds the generating function of a sequence given by a formula ($n^2 + 2^n$) or by its first terms (1, 0, 1, 0, …), and conversely the formula for a_n from a function $A(x)$. The number of non-negative whole solutions of $x_1 + x_2 + \ldots = N$ (also with bounds or coefficients – paying with coins) is the coefficient of a generating function.

The graph shows the terms of the sequence as points (both sequences of a system).

A difference $\Delta f(n) = f(n + 1) - f(n)$ (“Δ(n^3)”) and the difference equation Δy_n = n as the recurrence y_{n+1} − y_n = n. An arithmetic or geometric sequence from two given values (“a_3 = 7, a_7 = 19”, “geometric sequence a_2 = 6, a_5 = 162”) – a_1, d or q, the n-th term and the terms and sums asked for (s_{10}, a_{20} = ?).

A sequence given by its n-th term (a_n = (n+1)/n) is examined: the first five terms, monotonicity from the difference of neighbouring terms, boundedness and the limit.

How to enter it

`a(n+2) = 5a(n+1) - 6a(n), a(0) = 1, a(1) = 4`	Brackets, the conditions after a comma.
`a_{n+2} = 5a_{n+1} - 6a_n`	Subscripts (also from a photo). Without conditions: the general solution with constants C_1, C_2.
`F_n = F_{n-1} + F_{n-2}, F_1 = F_2 = 1`	The index may go down (a_{n-1}, a_{n-2}); one value can be given to several terms.

`a_{n+1} = a_n + 2b_n`
`b_{n+1} = 3a_n + b_n`
`a_0 = 1, b_0 = 0`
	A system – the equations on lines, the conditions on the last one.

`T(n) = 2T(n/2) + n, T(1) = 1`	Divide and conquer – a term at n/b.
`n^2 + 2^n`	A sequence's formula with the “Generating function” operation (also a_n = …).
`1, 0, 1, 0, ...`	The first terms separated by commas.
`1/(1 - x - x^2)`	A generating function in x (also A(x) = …) – the formula for a_n.
`number of solutions x_1 + x_2 + x_3 = 10, x_1 <= 3`	The number of non-negative whole solutions with bounds.
`Δ(n^3), \Delta y_n = n`	A difference and a difference equation.
`a_3 = 7, a_7 = 19, s_{10}`	A sequence from given terms – arithmetic without a word, geometric with “geometric”.

Examples

Two distinct real roots, the check by further terms.

Input: `a_{n+2} = 5a_{n+1} - 6a_n, a_0 = 1, a_1 = 4`

Result

$$a_n = 2 \cdot 3^n - 2^n$$

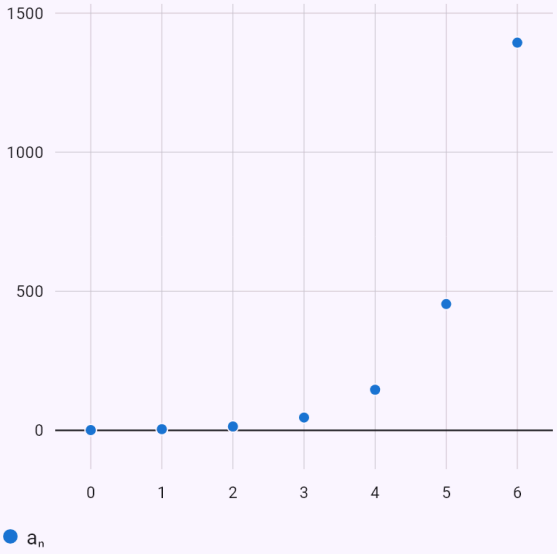
Steps

- A linear recurrence of order 2 with constant coefficients: $a_{n+2} = 5a_{n+1} - 6a_n$
- We look for solutions $a_n = \lambda^n$; substituted and divided by λ^n, they give the characteristic equation: $\lambda^2 - 5\lambda + 6 = 0$
- The discriminant $D = (-5)^2 - 4 \cdot 6 = 1$; the roots: $\lambda_1 = 2,\ \lambda_2 = 3$
- The general solution: $a_n = C_1 \cdot 2^n + C_2 \cdot 3^n$
- We substitute the conditions into the general solution hence $C_1 = -1,\ C_2 = 2$

$$C_1 + C_2 = 1$$
$$2C_1 + 3C_2 = 4$$

- The solution satisfying the given conditions: $a_n = 2 \cdot 3^n - 2^n$
- Check: the recurrence gives $a_2 = 14,\ a_3 = 46,\ a_4 = 146$ – the same as the formula.

Graph



The points are the terms computed from the formula – the recurrence itself gives the same values.

The Fibonacci sequence – Binet's formula.

Input: `F_n = F_{n-1} + F_{n-2}, F_0 = 0, F_1 = 1`

Result

$$F_n = \frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{\sqrt{5}}{5}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

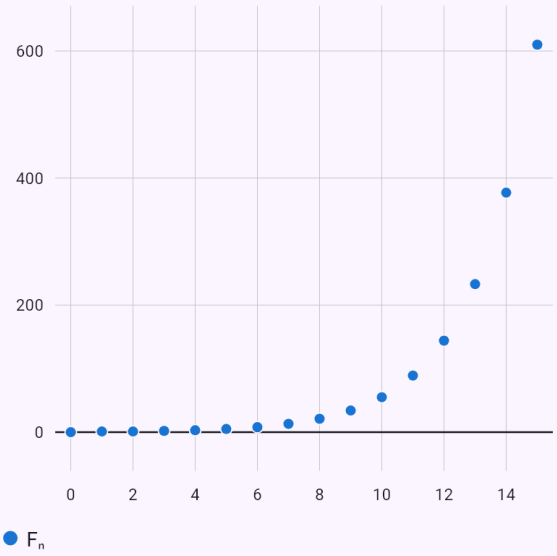
Steps

- A linear recurrence of order 2 with constant coefficients: $F_n = F_{n-1} + F_{n-2}$
- We look for solutions $F_n = \lambda^n$; substituted and divided by λ^{n-2}, they give the characteristic equation: $\lambda^2 - \lambda - 1 = 0$
- The discriminant $D = (-1)^2 - 4 \cdot (-1) = 5$; the roots: $\lambda_1 = \frac{1-\sqrt{5}}{2},\ \lambda_2 = \frac{1+\sqrt{5}}{2}$
- The general solution $F_n = C_1\left(\frac{1-\sqrt{5}}{2}\right)^n + C_2\left(\frac{1+\sqrt{5}}{2}\right)^n$
- We substitute the conditions into the general solution

  hence $C_1 = -\frac{\sqrt{5}}{5},\ C_2 = \frac{\sqrt{5}}{5}$

  $$C_1 + C_2 = 0$$
  $$\frac{1-\sqrt{5}}{2}C_1 + \frac{1+\sqrt{5}}{2}C_2 = 1$$
- The solution satisfying the given conditions $F_n = \frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{\sqrt{5}}{5}\left(\frac{1-\sqrt{5}}{2}\right)^n$
- Check: the recurrence gives $F_2 = 1,\ F_3 = 2,\ F_4 = 3$ – the same as the formula.

Graph



The points are the terms computed from the formula – the recurrence itself gives the same values.

Complex roots 1 ± i – the complex form and de Moivre's theorem.

Input: `a_{n+2} = 2a_{n+1} - 2a_n, a_0 = 1, a_1 = 2`

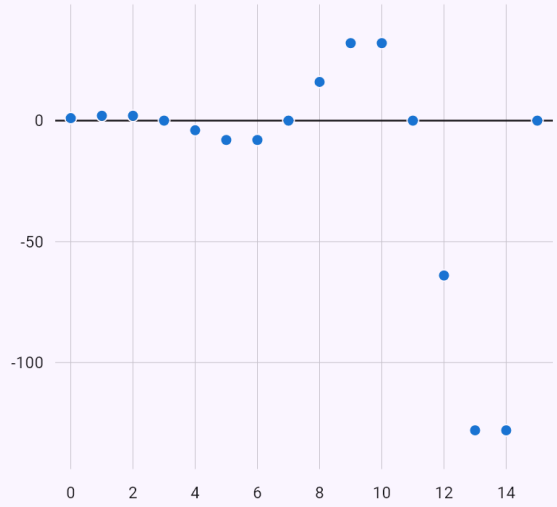
Result

$$a_n = \left(\sqrt{2}\right)^n \left(\cos\left(\frac{\pi n}{4}\right) + \sin\left(\frac{\pi n}{4}\right)\right)$$

Steps

- A linear recurrence of order 2 with constant coefficients: $a_{n+2} = 2a_{n+1} - 2a_n$
- We look for solutions $a_n = \lambda^n$; substituted and divided by λ^n, they give the characteristic equation: $\lambda^2 - 2\lambda + 2 = 0$
- The discriminant $D = (-2)^2 - 4 \cdot 2 = -4$; the roots: $\lambda_{1,2} = 1 \pm i$
- We write the complex roots $1 \pm i$ in polar form: $|\lambda| = \sqrt{2}$, $\varphi = \frac{\pi}{4}$, so $\lambda = \sqrt{2}\left(\cos\frac{\pi}{4} \pm i \sin\frac{\pi}{4}\right)$. By de Moivre's theorem $\lambda^n =$
  $$\left(\sqrt{2}\right)^n \left(\cos\left(\frac{\pi n}{4}\right) \pm i \sin\left(\frac{\pi n}{4}\right)\right)$$
- The general solution in the complex form (complex constants): $a_n = C_1(1+i)^n + C_2(1-i)^n$. Adding the conjugate terms, with new real constants we get the real form
  $$a_n = \left(\sqrt{2}\right)^n \left(C_1 \cos\left(\frac{\pi n}{4}\right) + C_2 \sin\left(\frac{\pi n}{4}\right)\right)$$
- We substitute the conditions into the general solution hence $C_1 = 1,\ C_2 = 1$
  $$C_1 = 1$$
  $$C_1 + C_2 = 2$$
- The solution satisfying the given conditions
  $$a_n = \left(\sqrt{2}\right)^n \left(\cos\left(\frac{\pi n}{4}\right) + \sin\left(\frac{\pi n}{4}\right)\right)$$
- Check: the recurrence gives $a_2 = 2,\ a_3 = 0,\ a_4 = -4$ – the same as the formula.

Graph



The points are the terms computed from the formula – the recurrence itself gives the same values.

The right side n, a double root 2.

Input: `a_{n+2} = 4a_{n+1} - 4a_n + n, a_0 = 0, a_1 = 0`

Result

$$a_n = \frac{n-4}{2} \cdot 2^n + n + 2$$

Steps

- A linear recurrence of order 2 with constant coefficients: $a_{n+2} = 4a_{n+1} - 4a_n + n$
- First we solve the homogeneous recurrence (the right side 0): $a_{n+2} - 4a_{n+1} + 4a_n = 0$
- We look for solutions $a_n = \lambda^n$; substituted and divided by λ^n, they give the characteristic equation: $\lambda^2 - 4\lambda + 4 = 0$
- The discriminant $D = (-4)^2 - 4 \cdot 4 = 0$; the roots: $\lambda_{1,2} = 2$
- The homogeneous recurrence's general solution: $a_n^{(h)} = (C_1 + C_2 n) \cdot 2^n$
- The right side n has a special form; the number 1 is not a root of the characteristic equation, so we look for a particular solution $a_n^{(p)} = An + B$
- Substituted into the recurrence, the terms collected: $An + B - 2A = n$
- We compare the coefficients of like terms on both sides hence $A = 1,\ B = 2$

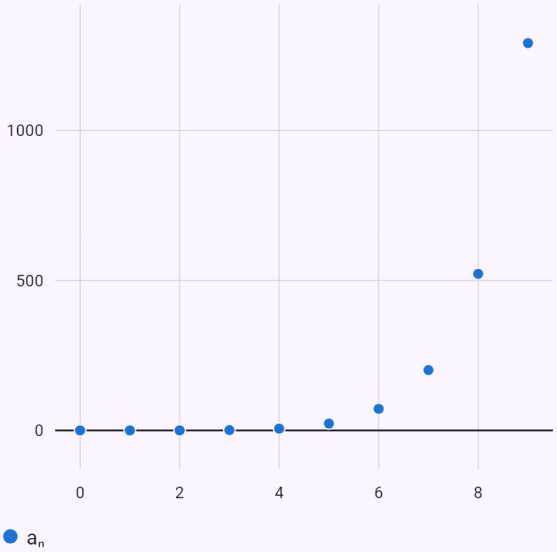
$$A = 1$$
$$B - 2A = 0$$

- A particular solution: $a_n^{(p)} = n + 2$
- The general solution: $a_n = a_n^{(h)} + a_n^{(p)} = (C_1 + C_2 n) \cdot 2^n + n + 2$
- We substitute the conditions into the general solution hence $C_1 = -2,\ C_2 = \frac{1}{2}$

$$C_1 + 2 = 0$$
$$2C_1 + 2C_2 + 3 = 0$$

- The solution satisfying the given conditions: $a_n = \frac{n-4}{2} \cdot 2^n + n + 2$
- Check: the recurrence gives $a_2 = 0,\ a_3 = 1,\ a_4 = 6$ – the same as the formula.

Graph



The points are the terms computed from the formula – the recurrence itself gives the same values.

The same recurrence by the generating function.

Input: `a_{n+2} = 5a_{n+1} - 6a_n, a_0 = 1, a_1 = 4` Generating function

Result

$$a_n = 2 \cdot 3^n - 2^n$$

Steps

- A linear recurrence of order 2 with constant coefficients: $a_{n+2} = 5a_{n+1} - 6a_n$
- We introduce the generating function $A(x) = \sum_{n=0}^{\infty} a_n x^n$
- The recurrence multiplied by x^{n+2} and summed over all n ≥ 0

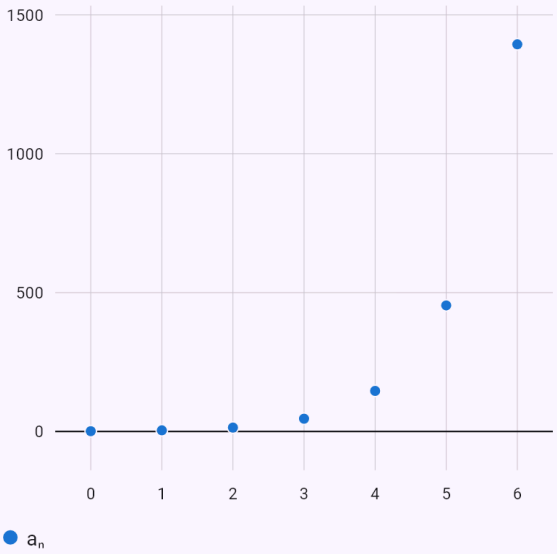
$$\sum_{n=0}^{\infty} a_{n+2} x^{n+2} = 5 \sum_{n=0}^{\infty} a_{n+1} x^{n+2} - 6 \sum_{n=0}^{\infty} a_n x^{n+2}$$

- The sums expressed by the generating function and the initial values:
  $A(x) - 1 - 4x = 5x(A(x) - 1) - 6x^2 A(x)$
- The generating function expressed, its denominator factored: $A(x) = \frac{1 - x}{(1 - 2x)(1 - 3x)}$
- Partial fractions: $A(x) = -\frac{1}{1 - 2x} + \frac{2}{1 - 3x}$
- The fractions as geometric series

$$\frac{1}{1 - 2x} = \sum_{n=0}^{\infty} 2^n x^n, \quad \frac{1}{1 - 3x} = \sum_{n=0}^{\infty} 3^n x^n$$

- The coefficient of x^n is the term: $a_n = 2 \cdot 3^n - 2^n$

Graph



The points are the terms computed from the formula – the recurrence itself gives the same values.

The same recurrence by matrix powers.

Input: `a_{n+2} = 5a_{n+1} - 6a_n, a_0 = 1, a_1 = 4` Matrix powers

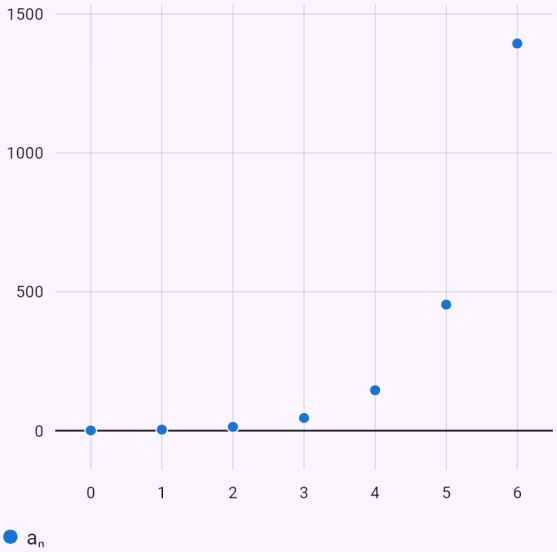
Result

$$a_n = 2 \cdot 3^n - 2^n$$

Steps

- A linear recurrence of order 2 with constant coefficients:
  $a_{n+2} = 5a_{n+1} - 6a_n$
- The recurrence in matrix form i.e. $v_{n+1} = M \cdot v_n$ with
  $$\begin{pmatrix} a_{n+2} \\ a_{n+1} \end{pmatrix} = \begin{pmatrix} 5 & -6 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix}$$
  $$M = \begin{pmatrix} 5 & -6 \\ 1 & 0 \end{pmatrix}$$
- Substituting again and again, $v_n = M^n \cdot v_0$
  $$\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = M^n \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$
- The matrix's eigenvalues:
  $\det(M - \lambda E) = \lambda^2 - 5\lambda + 6 = 0$, so $\lambda_1 = 2,\ \lambda_2 = 3$
- The eigenvectors – the solutions of $(M - \lambda E)u = 0$
  $$\mathbf{u}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$
- The matrix diagonalized, $M = P \cdot D \cdot P^{-1}$
  $$P = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$$
  $$D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$
  $$P^{-1} = \begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}$$
- The matrix power the coefficients so $v_n = P \cdot D^n \cdot (P^{-1} v_0)$.
  $$M^n = PD^nP^{-1}, \quad D^n = \begin{pmatrix} 2^n & 0 \\ 0 & 3^n \end{pmatrix}$$
  $$P^{-1}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$
- The solution satisfying the given conditions:
  $a_n = 2 \cdot 3^n - 2^n$
- Check: the recurrence gives
  $a_2 = 14,\ a_3 = 46,\ a_4 = 146$ – the same as the formula.

Graph



The points are the terms computed from the formula – the recurrence itself gives the same values.

A first-order recurrence by summing.

Input: `a_{n+1} = 2a_n + 1, a_0 = 0` Summing

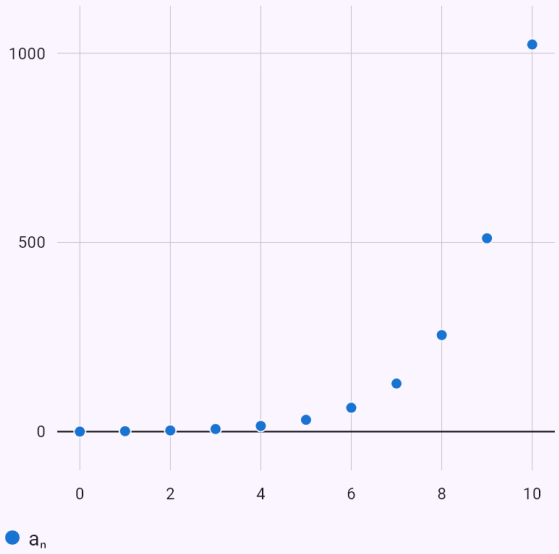
Result

$$a_n = 2^n - 1$$

Steps

- A linear recurrence of order 1 with constant coefficients: $a_{n+1} = 2a_n + 1$
- Substituting the recurrence into itself again and again: $a_n = 2a_{n-1} + 1 = 4a_{n-2} + \dots$, in general $a_n = a_0 \cdot 2^n + \sum_{j=0}^{n-1} 2^{n-1-j}$
- The sum evaluated: $\sum_{j=0}^{n-1} 2^{n-1-j} = 2^n - 1$
- The solution satisfying the given conditions: $a_n = 2^n - 1$
- Check: the recurrence gives $a_1 = 1,\ a_2 = 3,\ a_3 = 7$ – the same as the formula.

Graph



The points are the terms computed from the formula – the recurrence itself gives the same values.

A variable coefficient – the product gives a factorial.

Input: `a_{n+1} = (n+1)a_n, a_0 = 1`

Result

$$a_n = n!$$

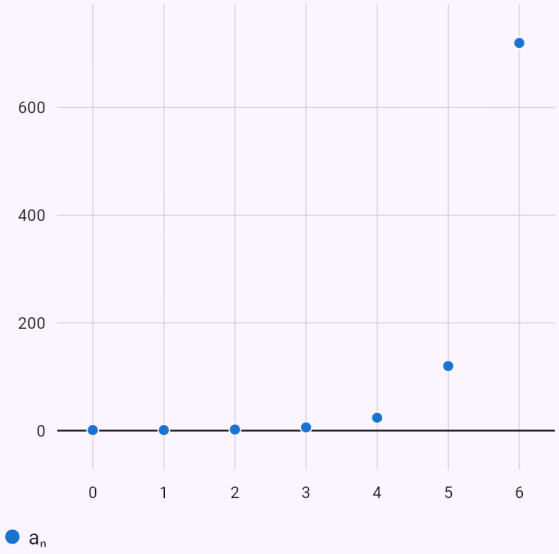
Steps

- A first-order linear recurrence with a variable coefficient: $a_{n+1} = a_n(n+1)$
- Substituting the recurrence into itself again and again – the term is a product

$$a_n = (n+1)a_{n-1} = n(n+1)a_{n-2} = \dots = a_0 \prod_{j=0}^{n-1} (j+1$$

- The product simplified (the factorials cancel): $\prod_{j=0}^{n-1} (j+1) = n!$
- The solution satisfying the given conditions: $a_n = n!$
- Check: the recurrence gives $a_1 = 1,\ a_2 = 2,\ a_3 = 6$ – the same as the formula.

Graph



The points are the terms computed from the formula – the recurrence itself gives the same values.

A nonlinear recurrence – the substitution $b_n = 1/a_n$.

Input: `a_{n+1} = a_n/(1 + a_n), a_0 = 1`

Result

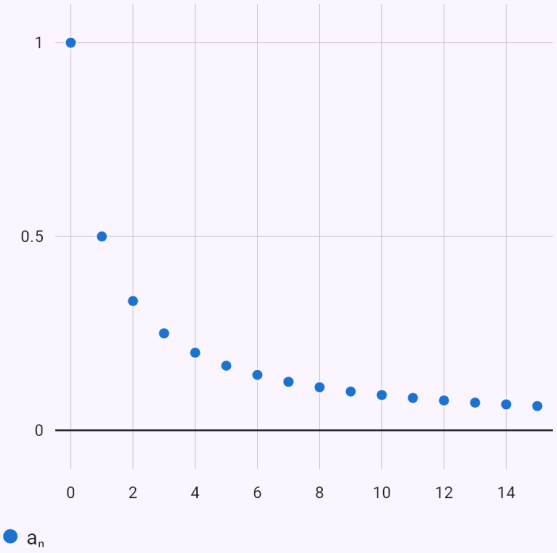
$$a_n = \frac{1}{n+1}$$

Steps

- A nonlinear recurrence – the reciprocals of the terms make it linear: $a_{n+1} = \frac{a_n}{a_n + 1}$
- Both sides reciprocated: $\frac{1}{a_{n+1}} = \frac{1}{a_n} + 1$; the substitution $b_n = \frac{1}{a_n}$ thus makes the recurrence linear: $b_{n+1} = b_n + 1$
- The initial conditions for the new sequence: $b_0 = 1$
- First we solve the homogeneous recurrence (the right side 0): $b_{n+1} - b_n = 0$
- We look for solutions $b_n = \lambda^n$; substituted and divided by λ^n, they give the characteristic equation: $\lambda - 1 = 0$
- The roots of the characteristic equation: $\lambda_1 = 1$
- The homogeneous recurrence's general solution: $b_n^{(h)} = C_1$
- The right side 1 has a special form; the number 1 is a root of the characteristic equation (multiplicity 1), so the form is multiplied by n: $b_n^{(p)} = An$
- Substituted into the recurrence, the terms collected: $A = 1$
- A particular solution: $b_n^{(p)} = n$
- The general solution: $b_n = b_n^{(h)} + b_n^{(p)} = C_1 + n$
- We substitute the conditions into the general solution hence $C_1 = 1$

  $C_1 = 1$
- The solution satisfying the given conditions: $b_n = 1 + n$
- Check: the recurrence gives $b_1 = 2,\ b_2 = 3,\ b_3 = 4$ – the same as the formula.
- Back to the original sequence: $a_n = \frac{1}{b_n} = \frac{1}{n+1}$

Graph



The points are the terms computed from the formula – the recurrence itself gives the same values.

A system of two recurrences by eliminating the sequence b_n.

Input: `a_{n+1} = a_n + 2b_n`
`b_{n+1} = 3a_n + b_n`

`a_0 = 1, b_0 = 0`

Result

$$a_n = \frac{\left(1-\sqrt{6}\right)^n}{2} + \frac{\left(1+\sqrt{6}\right)^n}{2}$$

$$b_n = \frac{\sqrt{6}}{4}\left(1+\sqrt{6}\right)^n - \frac{\sqrt{6}}{4}\left(1-\sqrt{6}\right)^n$$

Steps

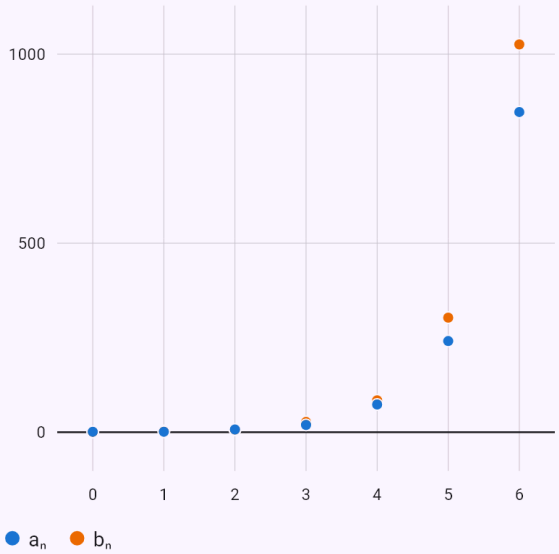
- A system of two linear recurrences
  $a_{n+1} = a_n + 2b_n$
  $b_{n+1} = 3a_n + b_n$
- From the first equation $b_n = \frac{a_{n+1} - a_n}{2}$, so $b_{n+1} = \frac{a_{n+2} - a_{n+1}}{2}$; put into the second one they give a recurrence for a single sequence:
  $a_{n+2} - 2a_{n+1} - 5a_n = 0$
- The second initial condition from the first equation:
  $a_1 = 1 + 0 = 1$
- We look for solutions $a_n = \lambda^n$; substituted and divided by λ^n, they give the characteristic equation:
  $\lambda^2 - 2\lambda - 5 = 0$
- The discriminant $D = (-2)^2 - 4 \cdot (-5) = 24$; the roots:
  $\lambda_1 = 1 - \sqrt{6},\ \lambda_2 = 1 + \sqrt{6}$
- The general solution:
  $a_n = C_1\left(1-\sqrt{6}\right)^n + C_2\left(1+\sqrt{6}\right)^n$
- We substitute the conditions into the general solution
  hence $C_1 = \frac{1}{2},\ C_2 = \frac{1}{2}$

$$C_1 + C_2 = 1$$

$$\left(1-\sqrt{6}\right)C_1 + \left(1+\sqrt{6}\right)C_2 = 1$$

- The solution satisfying the given conditions:
  $a_n = \frac{\left(1-\sqrt{6}\right)^n}{2} + \frac{\left(1+\sqrt{6}\right)^n}{2}$
- Check: the recurrence gives $a_2 = 7,\ a_3 = 19,\ a_4 = 73$ – the same as the formula.
- The other sequence from the expression
  $b_n = \frac{a_{n+1} - a_n}{2} = \frac{\sqrt{6}}{4}\left(1+\sqrt{6}\right)^n - \frac{\sqrt{6}}{4}\left(1-\sqrt{6}\right)^n$

Graph



The points are the terms computed from the formula – the recurrence itself gives the same values.

Divide and conquer (merge sort) – n = 2^k and the master theorem.

Input: `T(n) = 2T(n/2) + n, T(1) = 1`

Result

$$T(n) = n + n\log_2 n$$

Conditions: $n = 2^k,\ k \in \mathbb{N}_0$

Steps

- A divide-and-conquer recurrence: $T(n) = 2T\left(\frac{n}{2}\right) + n$
- For n a power of the base: the substitution $n = 2^k,\ S_k = T(2^k)$ gives a first-order linear recurrence: $S_{k+1} = 2S_k + 2^{k+1}$
- The initial conditions for the new sequence: $S_0 = 1$
- First we solve the homogeneous recurrence (the right side 0): $S_{k+1} - 2S_k = 0$
- We look for solutions $S_k = \lambda^k$; substituted and divided by λ^k, they give the characteristic equation: $\lambda - 2 = 0$
- The roots of the characteristic equation: $\lambda_1 = 2$
- The homogeneous recurrence's general solution: $S_k^{(h)} = C_1 \cdot 2^k$
- The right side 2^{k+1} has a special form; the number 2 is a root of the characteristic equation (multiplicity 1), so the form is multiplied by k: $S_k^{(p)} = Ak \cdot 2^k$
- Substituted into the recurrence, divided by 2^k, the terms collected: $2A = 2$
- Hence $A = 1$
- A particular solution: $S_k^{(p)} = k \cdot 2^k$
- The general solution: $S_k = S_k^{(h)} + S_k^{(p)} = C_1 \cdot 2^k + k \cdot 2^k$
- We substitute the conditions into the general solution hence $C_1 = 1$ $C_1 = 1$
- The solution satisfying the given conditions: $S_k = 2^k + k \cdot 2^k$
- Check: the recurrence gives $S_1 = 4,\ S_2 = 12,\ S_3 = 32$ – the same as the formula.
- Back to n ($k = \log_2 n$): $T(n) = n + n\log_2 n$
- The master theorem (a = 2, b = 2, f(n) = Θ(n)): $2 = 2^1$, so T(n) = Θ($n \log n$)

The generating function of the sequence $n^2 + 2^n$.

Input: `n^2 + 2^n` Generating function

Result

$$A(x) = \frac{1 - 2x + 2x^2 - 3x^3}{(1-x)^3(1-2x)}$$

Steps

- We introduce the generating function $A(x) = \sum_{n=0}^{\infty} (n^2 + 2^n)x^n$
- The series split by linearity: $A(x) = \sum_{n=0}^{\infty} 2^n x^n + \sum_{n=0}^{\infty} n^2 x^n$
- A known series: $\sum_{n=0}^{\infty} 2^n x^n = \frac{1}{1-2x}$
- A known series: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$
- Multiplying the terms by n is differentiating and multiplying by x $\sum_{n=0}^{\infty} nx^n = x \cdot \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{x}{(1-x)^2}$
- Multiplying the terms by n is differentiating and multiplying by x $\sum_{n=0}^{\infty} n^2 x^n = x \cdot \frac{d}{dx}\left(\frac{x}{(1-x)^2}\right) = \frac{x + x^2}{(1-x)^3}$
- The generating function: $A(x) = \frac{1 - 2x + 2x^2 - 3x^3}{(1-x)^3(1-2x)}$
- Check – the expansion starts with the sequence's terms: $A(x) = 1 + 3x + 8x^2 + 17x^3 + \ldots$

The formula for a_n from a generating function – partial fractions by irrational roots.

Input: `1/(1 - x - x^2)` Generating function

Result

$$a_n = \frac{5-\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n + \frac{5+\sqrt{5}}{10}\left($$

Steps

- The generating function expressed, its denominator factored

$$A(x) = \frac{1}{\left(1+\frac{-1+\sqrt{5}}{2}x\right)\left(1-\frac{1+\sqrt{5}}{2}x\right)}$$

- Partial fractions

$$A(x) = \frac{\frac{5-\sqrt{5}}{10}}{1+\frac{-1+\sqrt{5}}{2}x} + \frac{\frac{5+\sqrt{5}}{10}}{1-\frac{1+\sqrt{5}}{2}x}$$

- The fractions as geometric series

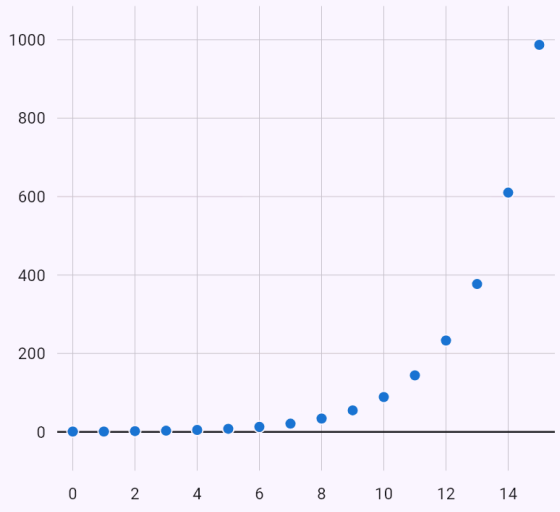
$$\frac{1}{1+\frac{-1+\sqrt{5}}{2}x} = \sum_{n=0}^{\infty}\left(\frac{1-\sqrt{5}}{2}\right)^n x^n, \quad \frac{1}{1-\frac{1+\sqrt{5}}{2}x} = \sum_{n=}^{\circ}$$

- The coefficient of x^n is the term

$$a_n = \frac{5-\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n + \frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n$$

- Check – the expansion starts with the sequence's terms: $A(x) = 1 + x + 2x^2 + 3x^3 + \dots$

Graph



The points are the terms computed from the formula – the recurrence itself gives the same values.

The number of solutions of $x_1 + x_2 + x_3 = 10$ with $x_1 \le 3$.

Input: `number of solutions x_1 + x_2 + x_3 = 10, x_1 <= 3`

Result

$$38$$

Steps

- The unknown x_1 $(0 \le x_1 \le 3)$ contributes the series $1 + x + x^2 + x^3 = \frac{1-x^4}{1-x}$
- The unknown x_2 $(x_2 \ge 0)$ contributes the series $1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$
- The unknown x_3 $(x_3 \ge 0)$ contributes the series $1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$
- The generating function is their product – the coefficient of x^N counts the solutions: $G(x) = \frac{1-x^4}{(1-x)^3}$
- The number of solutions is the coefficient of x^{10}: 38

A function's difference.

Input: `Δ(n^3)`

Result

$$\Delta f(n) = 3n^2 + 3n + 1$$

Steps

- The difference – the function's increment when n grows by 1

  $\Delta f(n) = f(n+1) - f(n) = (n+1)^3 - n^3 = 3n^2 + 3r$

A difference equation as a recurrence.

Input: `\Delta y_n = n`

Result

$$y_n = C_1 + \frac{n^2 - n}{2}, \quad C_1 \in \mathbb{R}$$

Steps

- A linear recurrence of order 1 with constant coefficients: $y_{n+1} - y_n = n$
- First we solve the homogeneous recurrence (the right side 0): $y_{n+1} - y_n = 0$
- We look for solutions $y_n = \lambda^n$; substituted and divided by λ^n, they give the characteristic equation: $\lambda - 1 = 0$
- The roots of the characteristic equation: $\lambda_1 = 1$
- The homogeneous recurrence's general solution: $y_n^{(h)} = C_1$
- The right side n has a special form; the number 1 is a root of the characteristic equation (multiplicity 1), so the form is multiplied by n: $y_n^{(p)} = n(An + B)$
- Substituted into the recurrence, the terms collected:

  $2An + A + B = n$
- We compare the coefficients of like terms on both sides hence $A = \frac{1}{2},\ B = -\frac{1}{2}$

  $2A = 1$

  $A + B = 0$
- A particular solution: $y_n^{(p)} = \frac{n^2 - n}{2}$
- The general solution: $y_n = y_n^{(h)} + y_n^{(p)} = C_1 + \frac{n^2 - n}{2}$

An arithmetic sequence from two terms and the sum of ten terms.

Input: `a_3 = 7, a_7 = 19, s_{10}`

Result

$$a_1 = 1,\ d = 3,\ s_{10} = 145$$

Steps

- An arithmetic sequence: $a_n = a_1 + (n-1)d$, $s_n = \frac{n}{2}(2a_1 + (n-1)d)$. The given values make a system
  $a_3 = a_1 + 2d = 7$
  $a_7 = a_1 + 6d = 19$
- The system's solution: $a_1 = 1,\ d = 3$
- The n-th term: $a_n = a_1 + (n-1)d = 3n - 2$
- The values asked for: $s_{10} = 145$

A geometric sequence from two terms.

Input: `geometric sequence a_2 = 6, a_5 = 162`

Result

$$a_1 = 2,\ q = 3$$

Steps

- A geometric sequence: $a_n = a_1 q^{n-1}$, $s_n = a_1 \frac{q^n - 1}{q - 1}$. The given values make a system
  $a_2 = a_1 q^1 = 6$
  $a_5 = a_1 q^4 = 162$
- The system's solution: $a_1 = 2,\ q = 3$
- The n-th term: $a_n = a_1 q^{n-1} = 2 \cdot 3^{n-1}$

The properties of a sequence given by a formula.

Input: `a_n = (n+1)/n`

Result

$$\lim_{n\to\infty} a_n = 1,\ 1 < a_n \leq 2$$

Steps

- The first five terms
  $a_1 = 2,\ a_2 = \frac{3}{2},\ a_3 = \frac{4}{3},\ a_4 = \frac{5}{4},\ a_5 = \frac{6}{5},\ \ldots$
- The difference of neighbouring terms
  $a_{n+1} - a_n = -\frac{1}{n^2+n}$ is negative for all $n \geq 1$ – the sequence is decreasing.
- The limit $\lim_{n\to\infty} a_n = 1$ – the sequence converges.
- The sequence is bounded: $1 < a_n \leq 2$

Known limitations

- Not yet: higher-order linear recurrences with variable coefficients, systems of three or more recurrences, divide and conquer with several terms (T(n/2) and T(n/3)) and nonlinear recurrences these substitutions don't make linear.
- A sequence from given terms needs exactly two values (terms, sums, d or q).

40. Numerical methods

Numerical methods are on the home screen in the “Applications” section, the field Numerical methods: roots of equations (bisection, Newton's, secant, regula falsi, fixed-point iteration), numerical integrals (the trapezoidal, Simpson's and Romberg's methods), numerical derivatives by difference quotients, differential equations (Euler's, Heun's and the Runge–Kutta methods), linear systems (Jacobi's and the Gauss–Seidel methods, the LU decomposition), interpolation (Lagrange's and Newton's polynomial, a cubic spline) and least squares. An integral and a derivative can be computed from a table of measured values too.

Differential equations also by multistep methods – Adams–Bashforth and the Adams–Bashforth–Moulton predictor-corrector (started by Runge–Kutta) –, the adaptive RK45 method (Dormand–Prince, a table of the steps accepted and rejected by the error estimate, the notation tol=…) and implicit methods for stiff equations: the implicit Euler method and the trapezoidal rule (Crank–Nicolson) with Newton's method in every step. For an equation with $\partial f/\partial y < 0$ the steps show the method's stability: $h\lambda$ and the amplification $|R(h\lambda)|$; if it is above 1, an explicit method diverges and the computation says so.

Boundary value problems $y'' = f(x, y, y')$ with the conditions $y(a)$, $y(b)$: by the shooting method (the initial slope found by the secant method, each shot solved by Runge–Kutta) and by finite differences (a linear equation as a tridiagonal system solved by the Thomas algorithm with tables of the forward sweep and the back substitution); where the ODE solver finds the exact solution, it is compared at the nodes. The equation of motion $x'' = a(x)$ is solved by the symplectic Verlet method beside RK4, showing how the total energy's error evolves for each.

The steps show the method's formula, the table of the iterations or of the values at the nodes, the stopping condition (the default tolerance $\varepsilon = 10^{-4}$, at most 50 iterations) and the error's estimate; where the exact value is known, the result is compared with it. The conditions of convergence are checked: a sign change on the interval, Fourier's condition for Newton's start, $|g'(x)| < 1$ for the fixed-point iteration, the diagonal dominance and the spectral radius for systems. The graph shows the tangents or secants, trapezoids or parabolas, a differential equation's numerical solution against the exact one, and the fitted polynomial with the points.

On an ordinary problem – an equation in one unknown, a definite integral, a differential equation with an initial condition – the app offers the chips “Solve by a numerical method” under the result; the form is prefilled with the problem. Where there's no exact way or it is needlessly complicated ($x = \cos x$, Cardano's formulas), the offer is highlighted, and an equation that can't be solved algebraically is solved by Newton's method at once. The applications that compute a number silently (the internal rate of return, the APR, a bond's yield, a polyprotic acid's pH, Runge–Kutta in the models) have a collapsed line “The method” under the result with a short table of the iterations.

How to enter it

`newton(x = cos x; x0=0.7; tol=1e-4)`	The method – its English name – and after semicolons the equation, the initial estimate x0, the tolerance tol, optionally max (iterations) and stop=fx.
`bisection(x^3 - x - 2; a=1; b=2)`	A function f(x) (f(x) = 0 is solved) and the interval a, b. Without an interval or an estimate the roots are isolated automatically.
`simpson(e^(-x^2); a=0; b=1; n=10)`	The integral of a function between a and b with n subintervals; also trapezoid and romberg (tol).
`trapezoid(x=0 1 2 3; y=0 2 3 5)`	The integral of measured values – x and y a space apart.
`finitediff(sin x; at=1; h=0.1)`	The derivative at the point at with the step h; from a table x=…; y=… (at optional).
`rk4(y' = x + y; y(0)=1; h=0.1; to=1)`	A first-order differential equation with its initial condition, the step h and the interval's end to; also euler and heun.
`gaussseidel(10x + y + z = 12; x + 10y + z = 12; x + y + 10z = 12)`	A system – the equations after semicolons or on lines; also jacobi and lu.
`lagrange(x=0 1 2 3; y=1 3 2 5; at=1.5)`	The points x, y and the point at; also newtonpoly and spline.
`leastsquares(x=0 1 2 3 4; y=1 1.8 3.3 4.5 6.3; degree=2)`	A least-squares polynomial of the degree degree.

Examples

An equation that can't be solved algebraically – the app solves it by Newton's method and offers the

others.

Input: `x = cos x`

Result

$$x \approx 0.7391$$

Solve by a numerical method:

Σ Bisection | Σ Secant | Σ Regula falsi | Σ Fixed point

Steps

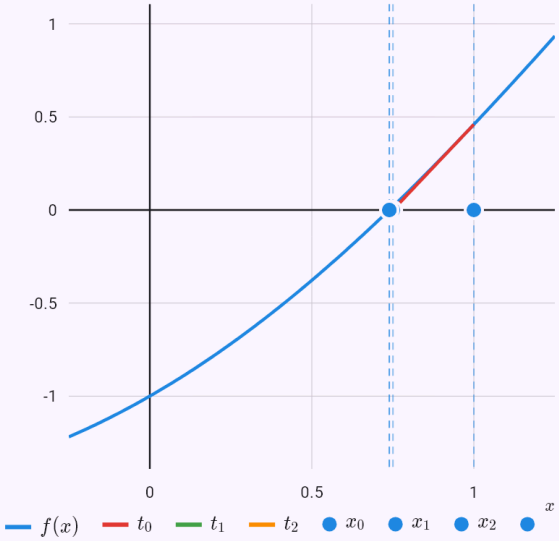
- We look for a root of the equation $x = \cos x$, i.e. a zero of the function $f(x) = x - \cos x$
- The roots isolated from a table of values: the function changes its sign in the intervals $[0,\ 1]$ ($f(0) \approx -1,\ f(1) \approx 0.4597$). A continuous function has a root in each such interval.
- Newton's method needs the derivative: $f'(x) = \sin x + 1$
- The initial estimate $x_0 = 1$ chosen by Fourier's condition – at the end where $f(1) \cdot f''(1) \approx 0.2484 > 0$ ($f''(x) = \cos x$); from it the tangents converge to the root monotonically.
- Newton's method: from the point x_n along the tangent to where it crosses the x-axis, $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\lvert x_n - x_{n-}$
0	1.000000	0.459698	1.841471	
1	0.750364	0.018923	1.681905	0.24963
2	0.739113	0.000046	1.673633	0.01125
3	0.739085	$2.84721 \cdot 10^{-10}$	1.673612	0.00002

- The stopping condition $\lvert x_n - x_{n-1} \rvert < 10^{-4}$ is met after 3 iterations.

- The error's estimate by the mean value theorem, $m_1 = \min \lvert f'(x) \rvert \approx 1$ on the interval $[0,\ 1]$: $\lvert \xi - x_n \rvert \le \dfrac{\lvert f(x_n) \rvert}{m_1} \approx 2.85 \cdot 10^{-10}$
- The method's result: $x \approx 0.7391$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

Newton's method with the root isolated automatically and the start by Fourier's condition.

Input: `newton(x^3 - x - 2)`

Result

$$x \approx 1.5214$$

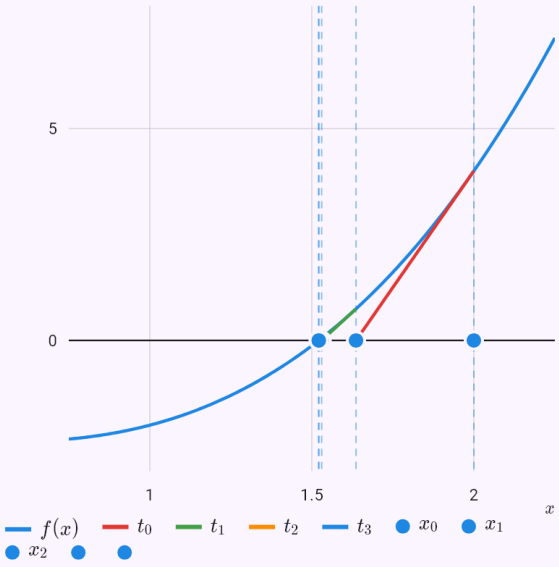
Steps

- We look for a root of the equation $x^3 - x - 2 = 0$, i.e. a zero of the function $f(x) = x^3 - x - 2$
- The roots isolated from a table of values: the function changes its sign in the intervals $[1,\ 2]$ ($f(1) \approx -2,\ f(2) \approx 4$). A continuous function has a root in each such interval.
- Newton's method needs the derivative: $f'(x) = 3x^2 - 1$
- The initial estimate $x_0 = 2$ chosen by Fourier's condition – at the end where $f(2) \cdot f''(2) \approx 48 > 0$ ($f''(x) = 6x$); from it the tangents converge to the root monotonically.
- Newton's method: from the point x_n along the tangent to where it crosses the x-axis, $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\lvert x_n - x_{n-1}\rvert$
0	2.000000	4.000000	11.000000	
1	1.636364	0.745304	7.033058	0.36363
2	1.530392	0.053939	6.026299	0.10597
3	1.521441	0.000367	5.944352	0.00895
4	1.521380	$1.74069 \cdot 10^{-8}$	5.943789	0.00006

- The stopping condition $|x_n - x_{n-1}| < 10^{-4}$ is met after 4 iterations.
- The error's estimate by the mean value theorem, $m_1 = \min |f'(x)| \approx 2$ on the interval $[1,\ 2]$: $|\xi - x_n| \le \dfrac{|f(x_n)|}{m_1} \approx 8.7 \cdot 10^{-9}$
- The method's result: $x \approx 1.5214$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

Bisection on a given interval with the error's estimate.

Input: `bisection(x^3 - x - 2; a=1; b=2)`

Result

$$x \approx 1.5214$$

Steps

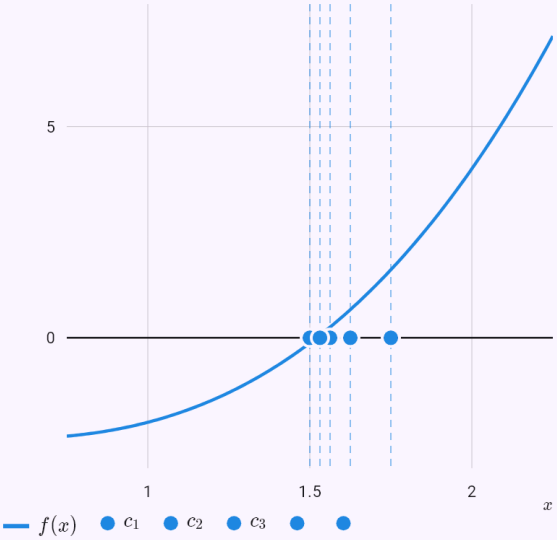
- We look for a root of the equation $x^3 - x - 2 = 0$, i.e. a zero of the function $f(x) = x^3 - x - 2$
- On the interval $[1,\ 2]$, $f(1) = -2.000000, \quad f(2) = 4.000000$ – opposite signs, the continuous function has a root there.
- Bisection: the interval's midpoint $c_n = \dfrac{a_n + b_n}{2}$, we keep the half on which f changes its sign.

n	a_n	b_n	c_n	$f(c_n)$	
1	1.000000	2.000000	1.500000	−0.125000	0
2	1.500000	2.000000	1.750000	1.609375	0
3	1.500000	1.750000	1.625000	0.666016	0
4	1.500000	1.625000	1.562500	0.252197	0
5	1.500000	1.562500	1.531250	0.059113	0
6	1.500000	1.531250	1.515625	−0.034054	0
7	1.515625	1.531250	1.523438	0.012250	0
8	1.515625	1.523438	1.519531	−0.010971	0
9	1.519531	1.523438	1.521484	0.000622	0
10	1.519531	1.521484	1.520508	−0.005179	0
11	1.520508	1.521484	1.520996	−0.002279	0
12	1.520996	1.521484	1.521240	−0.000829	0
13	1.521240	1.521484	1.521362	−0.000103	0
14	1.521362	1.521484	1.521423	0.000259	0

- The stopping condition $\dfrac{b_n - a_n}{2} < 10^{-4}$ is met after 14 iterations.
- The error's estimate: the root ξ lies in the last interval, $|\xi - c_n| \le \dfrac{b_n - a_n}{2} \approx 6.1 \cdot 10^{-5}$

- The method's result: $x \approx 1.5214$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

The secant method from two estimates.

Input: `secant(x^3 - 2x - 5; x0=2; x1=3)`

Result

$$x \approx 2.0946$$

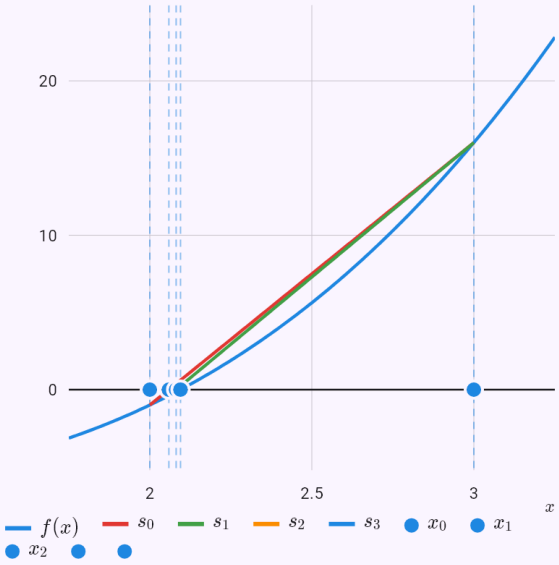
Steps

- We look for a root of the equation $x^3 - 2x - 5 = 0$, i.e. a zero of the function $f(x) = x^3 - 2x - 5$
- The secant method: the secant of the last two points replaces the tangent,
  $x_{n+1} = x_n - f(x_n)\,\dfrac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$.

n	x_n	$f(x_n)$	$\lvert x_n - x_{n-1}\rvert$
0	2.000000	-1.000000	
1	3.000000	16.000000	1.000000
2	2.058824	-0.390800	0.941176
3	2.081264	-0.147204	0.022440
4	2.094824	0.003044	0.013560
5	2.094549	-0.000023	0.000275
6	2.094551	$-3.51281 \cdot 10^{-9}$	0.000002

- The stopping condition $|x_n - x_{n-1}| < 10^{-4}$ is met after 6 iterations.
- The error's estimate by the mean value theorem, $m_1 = \min |f'(x)| \approx 10$ on the interval $[2,\ 3]$:
  $|\xi - x_n| \le \dfrac{|f(x_n)|}{m_1} \approx 3.51 \cdot 10^{-10}$
- The method's result: $x \approx 2.0946$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

The fixed-point iteration with the condition of convergence |g'(x)| < 1 checked.

Input: `fixedpoint(x = cos x; x0=0.7)`

Result

$$x \approx 0.7391$$

Steps

- The equation $x = \cos x$ written as x = g(x): $g(x) = \cos x$
- The condition of convergence: $g'(x) = -\sin x$, on the interval $[0,\ 1]$ $q = \max |g'(x)| \approx 0.8415 < 1$ – g is a contraction and the iteration converges.
- The fixed-point iteration: $x_{n+1} = g(x_n)$.

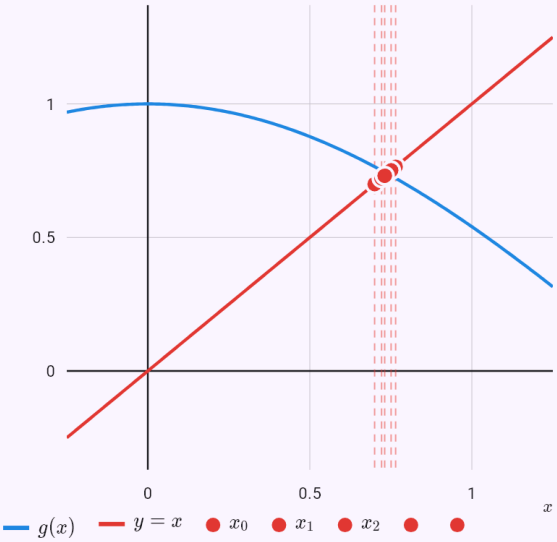
n	x_n	$\lvert x_n - x_{n-1}\rvert$
0	0.700000	
1	0.764842	0.064842
2	0.721492	0.043351
3	0.750821	0.029330
4	0.731129	0.019693
5	0.744421	0.013292
6	0.735480	0.008941
7	0.741509	0.006028
8	0.737450	0.004058
9	0.740185	0.002735
10	0.738344	0.001842
11	0.739584	0.001241
12	0.738749	0.000836
13	0.739312	0.000563
14	0.738932	0.000379
15	0.739188	0.000255
16	0.739016	0.000172
17	0.739132	0.000116
18	0.739054	0.000078

- The stopping condition $|x_n - x_{n-1}| < 10^{-4}$ is met after 18 iterations.
- The fixed-point iteration's error estimate

$$|\xi - x_n| \le \frac{q}{1-q}|x_n - x_{n-1}| \approx 4.14 \cdot 10^{-4}$$

- The method's result: $x \approx 0.7391$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

Simpson's rule, Runge's error estimate and the exact value for comparison.

Input: `simpson(e^(-x^2); a=0; b=1; n=10)`

Result

$$\int_0^1 e^{-x^2}\,dx \approx 0.746825$$

Steps

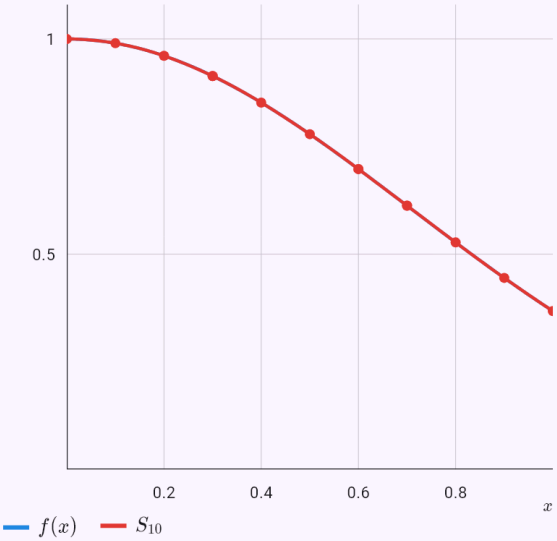
- The integral $\int_0^1 e^{-x^2}\,dx$ computed numerically.
- The step: $h = \frac{b-a}{n} = \frac{1-0}{10} = 0.1$
- The values at the nodes and their coefficients

i	x_i	$f(x_i)$	c_i	$c_i \cdot f(x_i)$
0	0	1.000000	1	1.000000
1	0.1	0.990050	4	3.960199
2	0.2	0.960789	2	1.921579
3	0.3	0.913931	4	3.655725
4	0.4	0.852144	2	1.704288
5	0.5	0.778801	4	3.115203
6	0.6	0.697676	2	1.395353
7	0.7	0.612626	4	2.450506
8	0.8	0.527292	2	1.054585
9	0.9	0.444858	4	1.779432
10	1	0.367879	1	0.367879
Σ				22.404748

- Simpson's rule
  $S_n = \frac{h}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + \cdots + 4f_{n-1} + f_n)$:
  $S_{10} = \frac{0.1}{3} \cdot 22.404748 \approx 0.746825$
- For comparison the exact value
  $I = \frac{\operatorname{erf}(1)\sqrt{\pi}}{2} \approx 0.74682413$, the method's error
  $|I - S_{10}| \approx 8.15 \cdot 10^{-7}$

- The method's result: $\int_0^1 e^{-x^2}\,dx \approx 0.746825$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

Romberg's table for an integral without an elementary antiderivative.

Input: `romberg(sin x / x; a=1; b=2)`

Result

$$\int_1^2 \frac{\sin x}{x}\,\mathrm{d}x \approx 0.659330$$

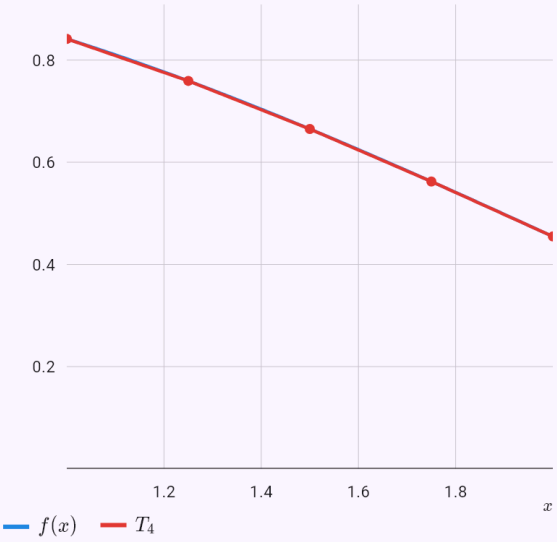
Steps

- The integral $\int_1^2 \frac{\sin x}{x}\,\mathrm{d}x$ computed numerically.
- Romberg's method: the first column are the trapezoids with the steps h, h/2, h/4, ..., the next columns refine them by Richardson's extrapolation,
  $R_{k,0} = T_{2^k}, \quad R_{k,j} = R_{k,j-1} + \frac{R_{k,j-1} - R_{k-1,j-1}}{4^j - 1}.$

k	n	$R_{k,0}$	$R_{k,1}$	$R_{k,2}$
0	1	0.648060		
1	2	0.656528	0.659351	
2	4	0.658630	0.659331	0.659330

- The stopping condition $|R_{k,k} - R_{k-1,k-1}| < 10^{-4}$ is met after 2 iterations.
- For comparison the exact value the method's error
  $|I - R_{2,2}| \approx 1.84 \cdot 10^{-8}$
  $I = -\operatorname{Si}(1) + \operatorname{Si}(2) \approx 0.65932991$
- The method's result: $\int_1^2 \frac{\sin x}{x}\,\mathrm{d}x \approx 0.659330$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

The distance from a measured speed by the trapezoidal rule.

Input: `trapezoid(x=0 10 20 30 40 50 60; y=0 4.2 7.5 9.8 11.1 11.6 11.8)`

Result

$$\int_{0}^{60} y \, \mathrm{d}x \approx 501.000000$$

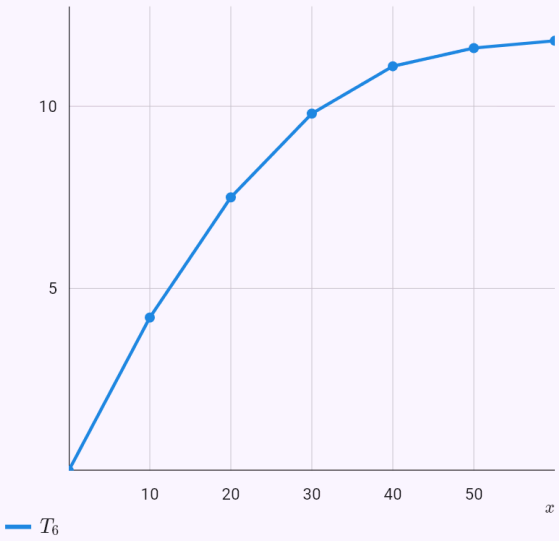
Steps

- We integrate a table of 6 subintervals on the interval $[0,\ 60]$
- The step: $h = 10$
- The values at the nodes and their coefficients

i	x_i	y_i	c_i	$c_i \cdot y_i$
0	0	0	1	0.000000
1	10	4.2	2	8.400000
2	20	7.5	2	15.000000
3	30	9.8	2	19.600000
4	40	11.1	2	22.200000
5	50	11.6	2	23.200000
6	60	11.8	1	11.800000
Σ				100.200000

- The trapezoidal rule $T_n = \frac{h}{2}(f_0 + 2f_1 + 2f_2 + \cdots + 2f_{n-1} + f_n)$:
  $T_6 = \frac{10}{2} \cdot 100.200000 \approx 501.000000$
- The error estimated by Runge's rule (the computation with the double step):
  $T_3 \approx 490.000000, \quad E \approx \frac{T_6 - T_3}{3} \approx 3.67$
- The method's result: $\int_{0}^{60} y \, \mathrm{d}x \approx 501.000000$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

The forward, backward and central differences for three steps.

Input: `finitediff(sin x; at=1; h=0.1)`

Result

$$f'(1) \approx 0.539402$$
$$f''(1) \approx -0.840770$$

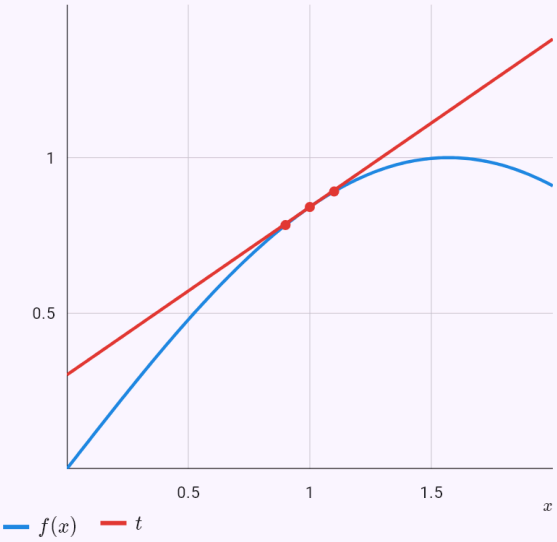
Steps

- The derivative of the function $f(x) = \sin x$ at $x_0 = 1$ estimated by difference quotients.
- The forward difference $f'(x_0) \approx D_h^+ = \frac{f(x_0+h) - f(x_0)}{h}$ has an error proportional to h, the backward one $f'(x_0) \approx D_h^- = \frac{f(x_0) - f(x_0-h)}{h}$ too; the central one $f'(x_0) \approx D_h^0 = \frac{f(x_0+h) - f(x_0-h)}{2h}$ only to h². The second derivative: $f''(x_0) \approx D_h^2 = \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2}$
-

h	D_h^+	D_h^-	D_h^0	D_h^2
0.1	0.497364	0.581441	0.539402	−0.840770
0.05	0.519045	0.561110	0.540077	−0.841296
0.025	0.529728	0.550764	0.540246	−0.841427

- Exactly: $f'(x) = \cos x, \quad f'(1) \approx 0.54030231$. The forward difference's error with the first step is 0.0429, the central one's only $9 \cdot 10^{-4}$
- The method's result: $f'(1) \approx 0.539402, \quad f''(1) \approx -0.840770$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

The Runge–Kutta method of the 4th order with the table of the slopes and the exact solution for

comparison.

Input: `rk4(y' = x + y; y(0)=1; h=0.1; to=1)`

Result

$$y(1) \approx 3.436559$$

Steps

- The initial value problem $y' = x + y, \quad y(0) = 1$ on the interval $[0,\ 1]$, the step $h = 0.1$ (10 steps).
- The Runge–Kutta method of the 4th order – a weighted average of four slopes

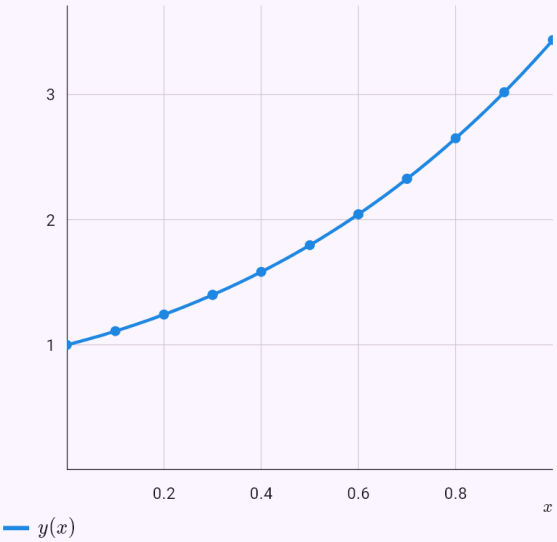
$$k_1 = f(x_n, y_n), \quad k_2 = f\left(x_n + \tfrac{h}{2}, y_n + \tfrac{h}{2}k_1\right)$$
$$k_3 = f\left(x_n + \tfrac{h}{2}, y_n + \tfrac{h}{2}k_2\right), \quad k_4 = f(x_n + h, y_n + hk$$
$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

- For comparison the exact solution: $y = -x + 2e^x - 1$
- Step by step

n	x_n	y_n	k_1	k_2	k_3
0	0	1.000000	1.000000	1.100000	1.105000
1	0.1	1.110342	1.210342	1.320859	1.326385
2	0.2	1.242805	1.442805	1.564945	1.571052
3	0.3	1.399717	1.699717	1.834703	1.841452
4	0.4	1.583648	1.983648	2.132831	2.140290
5	0.5	1.797441	2.297441	2.462313	2.470557
6	0.6	2.044236	2.644236	2.826448	2.835558
7	0.7	2.327503	3.027503	3.228878	3.238947
8	0.8	2.651079	3.451079	3.673633	3.684761
9	0.9	3.019203	3.919203	4.165163	4.177461
10	1	3.436559			

- At the interval's end exactly
$y(1) = 3.43656366, \quad |y(1) - y_{10}| \approx 4.17 \cdot 10^{-6}$
- The method's result: $y(1) \approx 3.436559$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

Euler's method for an equation without an elementary solution.

Input: `euler(y' = x^2 + y^2; y(0)=0; h=0.1; to=1)`

Result

$$y(1) \approx 0.292542$$

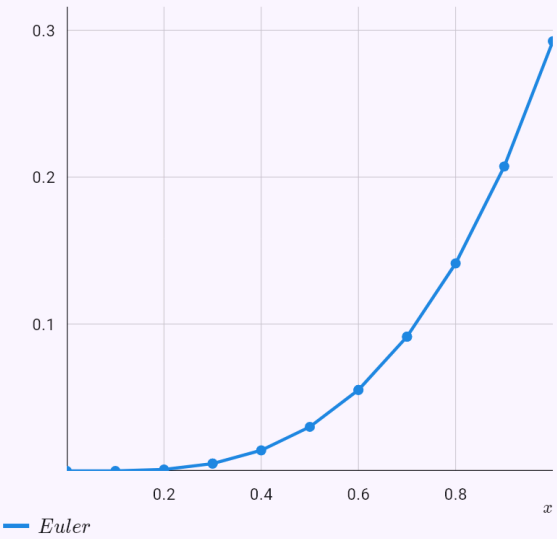
Steps

- The initial value problem $y' = x^2 + y^2, \quad y(0) = 0$ on the interval $[0,\ 1]$, the step $h = 0.1$ (10 steps).
- Euler's method – a step along the tangent: $y_{n+1} = y_n + h\, f(x_n, y_n)$
- Step by step

n	x_n	y_n	$f(x_n, y_n)$
0	0	0.000000	0.000000
1	0.1	0.000000	0.010000
2	0.2	0.001000	0.040001
3	0.3	0.005000	0.090025
4	0.4	0.014003	0.160196
5	0.5	0.030022	0.250901
6	0.6	0.055112	0.363037
7	0.7	0.091416	0.498357
8	0.8	0.141252	0.659952
9	0.9	0.207247	0.852951
10	1	0.292542	

- The method's result: $y(1) \approx 0.292542$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

The ABM predictor-corrector – more accurate than Adams–Bashforth alone.

Input: `abm(y' = x + y; y(0)=1; h=0.1; to=1)`

Result

$$y(1) \approx 3.436567$$

Steps

- The initial value problem $y' = x + y, \quad y(0) = 1$ on the interval $[0,\ 1]$, the step $h = 0.1$ (10 steps).
- A predictor-corrector: the Adams–Bashforth method estimates the new point, the Adams–Moulton method corrects it by the slope at the estimate. Started by the Runge–Kutta method

$$\tilde{y}_{n+1} = y_n + \frac{h}{24}(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3})$$
$$y_{n+1} = y_n + \frac{h}{24}\left(9\tilde{f}_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}\right)$$

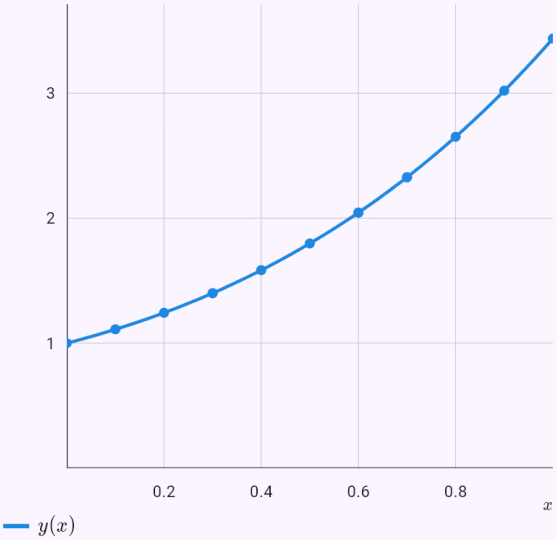
- For comparison the exact solution: $y = -x + 2e^x - 1$
- Step by step

n	x_n	y_n	f_n	$\tilde{y}_{n+1}$	
0	0	1.000000	1.000000		$RK4$
1	0.1	1.110342	1.210342		$RK4$
2	0.2	1.242805	1.442805		$RK4$
3	0.3	1.399717	1.699717	1.583640	$AB4 + A$
4	0.4	1.583649	1.983649	1.797433	$AB4 + A$
5	0.5	1.797443	2.297443	2.044227	$AB4 + A$
6	0.6	2.044238	2.644238	2.327495	$AB4 + A$
7	0.7	2.327507	3.027507	2.651070	$AB4 + A$
8	0.8	2.651084	3.451084	3.019194	$AB4 + A$
9	0.9	3.019209	3.919209	3.436551	$AB4 + A$
10	1	3.436567	4.436567		

- At the interval's end exactly $y(1) = 3.43656366, \quad |y(1) - y_{10}| \approx 3.58 \cdot 10^{-6}$

- The method's result: $y(1) \approx 3.436567$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

The adaptive RK45 – the first step is rejected and shortened, then the method controls its step itself.

Input: `rk45(y' = x + y; y(0)=1; h=0.5; to=2; tol=1e-6)`

Result

$$y(2) \approx 11.778114$$

Steps

- The initial value problem $y' = x + y, \quad y(0) = 1$ on the interval $[0,\ 2]$, the step $h = 0.5$ (4 steps).
- An adaptive step: in each step two methods (of the 5th and the 4th order) from the same slopes – their difference estimates the error. A step is accepted if the error is below the tolerance, and the next one grows or shrinks by it

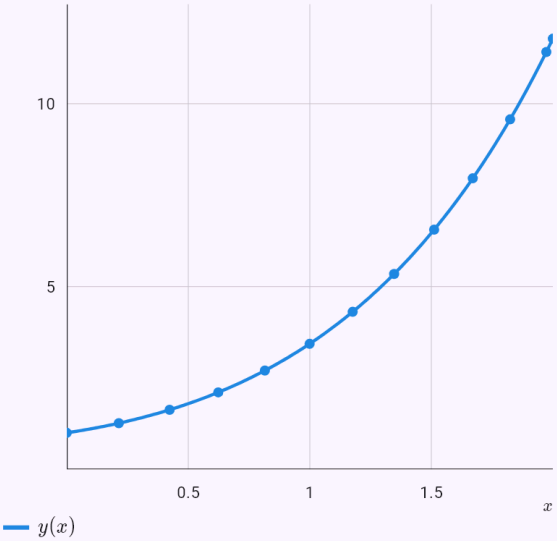
$$y_{k+1} = y_k + h\sum_i b_i k_i, \quad \varepsilon = |y_{k+1} - \hat{y}_{k+1}|, \quad h_{new} = 0$$

- For comparison the exact solution: $y = -x + 2e^x - 1$
- The steps with the tolerance $\varepsilon_{tol} = 10^{-6}$: ✓ an accepted step, × a rejected one (the step shrank and was repeated). Accepted 12, rejected 1.

k	x_k	h	y_{k+1}	ε	
1	0.0000	0.50000	1.797448	$4.1 \cdot 10^{-5}$	×
2	0.0000	0.21410	1.263397	$6.7 \cdot 10^{-7}$	✓
3	0.2141	0.20898	1.630247	$7.3 \cdot 10^{-7}$	✓
4	0.4231	0.20014	2.106643	$7.3 \cdot 10^{-7}$	✓
5	0.6232	0.19181	2.703462	$7.2 \cdot 10^{-7}$	✓
6	0.8150	0.18415	3.432946	$7.2 \cdot 10^{-7}$	✓
7	0.9992	0.17711	4.308380	$7.1 \cdot 10^{-7}$	✓
8	1.1763	0.17061	5.344079	$7.1 \cdot 10^{-7}$	✓
9	1.3469	0.16458	6.555405	$7 \cdot 10^{-7}$	✓
10	1.5115	0.15898	7.958790	$7 \cdot 10^{-7}$	✓
11	1.6705	0.15376	9.571752	$6.9 \cdot 10^{-7}$	✓
12	1.8242	0.14889	11.412926	$6.9 \cdot 10^{-7}$	✓
13	1.9731	0.02689	11.778114	$1.6 \cdot 10^{-10}$	✓

- At the interval's end exactly $y(2) = 11.77811220, \quad |y(2) - y_{12}| \approx 1.3 \cdot 10^{-6}$
- The method's result: $y(2) \approx 11.778114$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

A stiff equation – the explicit Euler method with the step 0.05 is unstable (|1 + hλ| = 1.5) and diverges.

Input: `euler(y' = -50(y - cos x); y(0)=0; h=0.05; to=1)`

Result

The numerical solution diverges with this step (see the steps).

Steps

- The initial value problem $y' = -50(y - \cos x), \quad y(0) = 0$ on the interval $[0,\ 1]$, the step $h = 0.05$ (20 steps).
- Euler's method – a step along the tangent: $y_{n+1} = y_n + h\, f(x_n, y_n)$
- For comparison the exact solution: $y = -\dfrac{2500e^{-50x}}{2501} + \dfrac{2500\cos x}{2501} + \dfrac{50\sin x}{2501}$
- Stability: the equation behaves like y' = λy with $\lambda = \dfrac{\partial f}{\partial y} \approx -50$. The method enlarges an error in every step: $h\lambda \approx -2.5, \quad |1 + h\lambda| \approx 1.5 > 1$ – with this step it is unstable, the values will oscillate and grow. A smaller step or an implicit method helps.
- Step by step

n	x_n	y_n	$f(x_n, y_n)$	$y(x_n)$
0	0	0.000000	50.000000	0.000000
1	0.05	2.500000	−75.062487	0.917298
2	0.1	−1.253124	112.406426	0.989867
3	0.15	4.367197	−168.921293	0.990810
4	0.2	−4.078868	252.946714	0.983601
5	0.25	8.568468	−379.977779	0.973467
6	0.3	−10.430421	569.287872	0.960862
7	0.35	18.033973	−854.729998	0.945852
8	0.4	−24.702527	1281.179410	0.928478
9	0.45	39.356443	−1922.799810	0.908783
10	0.5	−56.783547	2883.056488	0.886816
11	0.55	87.369277	−4325.837634	0.862633
12	0.6	−128.922605		0.836294

- The values grow beyond all bounds (at $x \approx 0.6000$) – the method is unstable with this step. Try a smaller step or an implicit method.

The same equation by the implicit Euler method – stable even with a large step.

Input: `beuler(y' = -50(y - cos x); y(0)=0; h=0.05; to=1)`

Result

$y(1) \approx 0.556616$

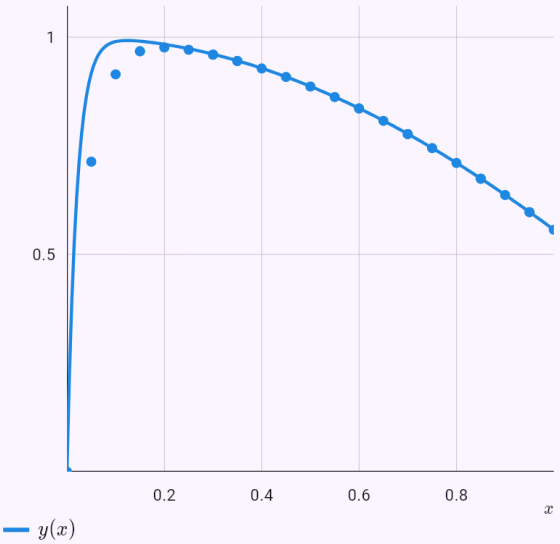
Steps

- The initial value problem $y' = -50(y - \cos x), \quad y(0) = 0$ on the interval $[0,\ 1]$, the step $h = 0.05$ (20 steps).
- The implicit Euler method – the slope at the end of the step, so y_{n+1} is on both sides of an equation. It is solved in each step by Newton's method: $y_{n+1} = y_n + h\, f(x_{n+1}, y_{n+1})$
- For comparison the exact solution: $y = -\dfrac{2500e^{-50x}}{2501} + \dfrac{2500\cos x}{2501} + \dfrac{50\sin x}{2501}$
- Stability: the equation behaves like y' = λy with $\lambda = \dfrac{\partial f}{\partial y} \approx -50$. The method shrinks an error in every step: $h\lambda \approx -2.5, \quad \left|\dfrac{1}{1 - h\lambda}\right| \approx 0.2857 \le 1$ – the computation is stable.
- Step by step

n	x_n	y_n	f_n	Newton	$y(x_n$
0	0	0.000000	50.000000	3	0.0000
1	0.05	0.713393	14.267861	3	0.9172
2	0.1	0.914544	4.023016	3	0.9898
3	0.15	0.967563	1.060389	3	0.9908
4	0.2	0.976494	0.178618	2	0.9836
5	0.25	0.971079	−0.108311	2	0.9734
6	0.3	0.959834	−0.224888	2	0.9608
7	0.35	0.945219	−0.292308	2	0.9458
8	0.4	0.927963	−0.345112	2	0.9284
9	0.45	0.908309	−0.393088	2	0.9087
10	0.5	0.886362	−0.438947	2	0.8868
11	0.55	0.862192	−0.483385	2	0.8626
12	0.6	0.835866	−0.526523	2	0.8362
13	0.65	0.807450	−0.568318	2	0.8078
14	0.7	0.777016	−0.608685	2	0.7774
15	0.75	0.744639	−0.647529	2	0.7450
16	0.8	0.710402	−0.684753	2	0.7107
17	0.85	0.674388	−0.720266	2	0.6747
18	0.9	0.636690	−0.753979	2	0.6370
19	0.95	0.597399	−0.785806	2	0.5977
20	1	0.556616	−0.815670		0.5569

- At the interval's end exactly $y(1) = 0.55690896, \quad |y(1) - y_{20}| \approx 2.93 \cdot 10^{-4}$
- The method's result: $y(1) \approx 0.556616$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

A nonlinear boundary value problem by the shooting method – five secant steps correct the slope.

Input: `shooting(y'' = -y^2; y(0)=1; y(1)=1)`

Result

$$y(0.5) \approx 1.160614$$

Steps

- The boundary value problem $y'' = -y^2, \quad y(0) = 1, \quad y(1) = 1$, the step $h = 0.05$ (20 subintervals).
- The shooting method: we pick an initial slope s = y'(a), solve the problem as an initial value problem (Runge–Kutta) and see how far it misses the right end. The slope is corrected by the secant method

$$F(s) = y(1;\, s) - 1, \quad s_{k+1} = s_k - F(s_k)\,\frac{s_k - s_{k-1}}{F(s_k) - F(s_k}$$

- The shots – each is one initial value problem; we correct until F(s) is below the tolerance.

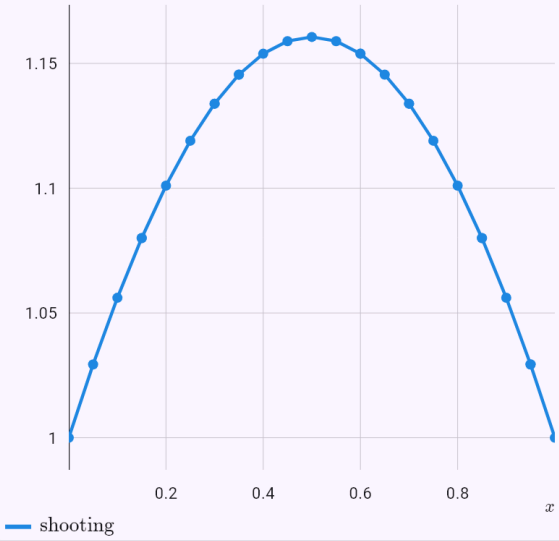
k	s_k	$y(1;\, s_k)$	$F(s_k)$
0	0.000000	0.571186	-0.429
1	1.000000	1.248638	0.249
2	0.632981	1.013343	0.0133
3	0.612168	0.999549	$-4.51 \cdot 10^{-4}$
4	0.612849	1.000001	$7.79 \cdot 10^{-7}$

- The initial slope with which the solution hits the boundary condition: $y'(0) \approx 0.612849$
- The solution at the nodes

i	x_i	y_i
0	0	1.000000
1	0.05	1.029367
2	0.1	1.056086
3	0.15	1.080018
4	0.2	1.101035
5	0.25	1.119022
6	0.3	1.133880
7	0.35	1.145525
8	0.4	1.153892
9	0.45	1.158931
10	0.5	1.160614
11	0.55	1.158931
12	0.6	1.153892
13	0.65	1.145526
14	0.7	1.133880
15	0.75	1.119022
16	0.8	1.101035
17	0.85	1.080019
18	0.9	1.056087
19	0.95	1.029368
20	1	1.000001

- The method's result: $y(0.5) \approx 1.160614$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

Finite differences and the Thomas algorithm – exact at the nodes for y = x³.

Input: `fdm(y'' = 6x; y(0)=0; y(1)=1; n=4)`

Result

$$y(0.5) \approx 0.125000$$

Steps

- The boundary value problem $y'' = 6x, \quad y(0) = 0, \quad y(1) = 1$, the step $h = 0.25$ (4 subintervals).
- Finite differences: at the nodes x_i, y" and y' are replaced by central differences

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = p_i \frac{y_{i+1} - y_{i-1}}{2h} + q_i y_i + r_i$$

- The tridiagonal system a_i y_{i−1} + b_i y_i + c_i y_{i+1} = d_i; the boundary values moved to the right side

i	x_i	a_i	b_i	c_i	d_i
1	0.25		−32.0000	16.0000	1.5000
2	0.5	16.0000	−32.0000	16.0000	3.0000
3	0.75	16.0000	−32.0000		−11.5000

- The Thomas algorithm – Gaussian elimination for a tridiagonal matrix: the forward sweep, then the back substitution from the end

$$c'_i = \frac{c_i}{b_i - a_i c'_{i-1}}, \quad d'_i = \frac{d_i - a_i d'_{i-1}}{b_i - a_i c'_{i-1}}, \quad y_i = d'_i - c'_i y_{i+1}$$

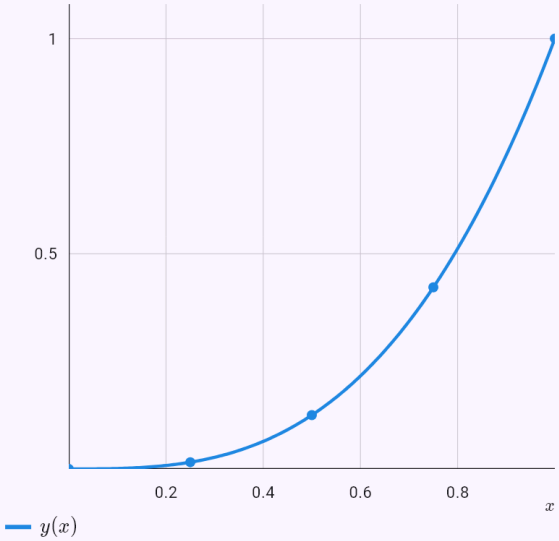
i	c'_i	d'_i
1	−0.500000	−0.046875
2	−0.666667	−0.156250
3		0.421875

- For comparison the exact solution: $y = x^3$
- The solution at the nodes

i	x_i	y_i	$y(x_i)$	$\lvert y(x_i) - y_i \rvert$
0	0	0.000000	0.000000	0
1	0.25	0.015625	0.015625	$2.78 \cdot 10^{-17}$
2	0.5	0.125000	0.125000	$5.55 \cdot 10^{-17}$
3	0.75	0.421875	0.421875	$5.55 \cdot 10^{-17}$
4	1	1.000000	1.000000	0

- The largest error at the nodes against the exact solution: $\max_i \lvert y(x_i) - y_i \rvert \approx 5.55 \cdot 10^{-17}$
- The method's result: $y(0.5) \approx 0.125000$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

A boundary value problem by finite elements – exact at the nodes.

Input: `fem(y'' = -x; y(0) = 0; y(1) = 0; n = 4)`

Result

$$y(0.5) \approx 0.062500$$

Steps

- The boundary value problem $y'' = -x, \quad y(0) = 0, \quad y(1) = 0$, the step $h = 0.25$ (4 subintervals).
- The finite element method: the equation is multiplied by a "hat" φ_i (a piecewise linear function, 1 at the node i and 0 at the others) and integrated by parts. The solution is sought as a combination of hats, the integrals over the elements by Gauss's rule

$$\int_0^1 (y'\varphi_i' + p\,y'\varphi_i + q\,y\varphi_i)\mathrm{d}x = -\int_0^1 r\,\varphi_i\,\mathrm{d}x, \quad y = \sum_j$$

- The assembled system – one row for each inner node (the stiffness matrix K is tridiagonal, the boundary values moved to the right side)

i	x_i	$K_{i,i-1}$	$K_{i,i}$	$K_{i,i+1}$	F_i
1	0.25		8.0000	−4.0000	0.0625
2	0.5	−4.0000	8.0000	−4.0000	0.1250
3	0.75	−4.0000	8.0000		0.1875

- The Thomas algorithm – Gaussian elimination for a tridiagonal matrix: the forward sweep, then the back substitution from the end

$$c_i' = \frac{c_i}{b_i - a_i c_{i-1}'}, \quad d_i' = \frac{d_i - a_i d_{i-1}'}{b_i - a_i c_{i-1}'}, \quad y_i = d_i' - c_i' y_{i+1}$$

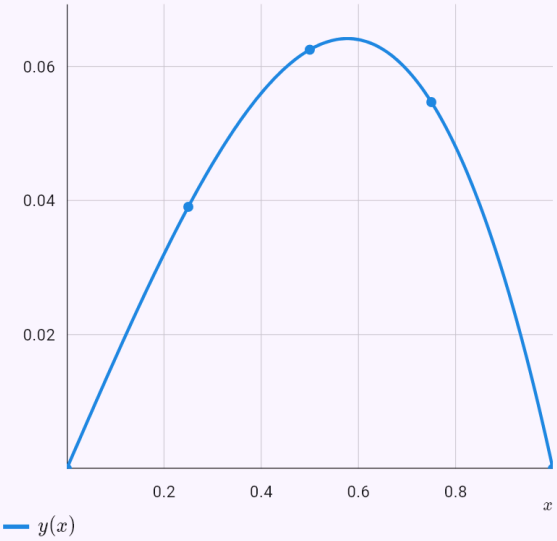
i	c_i'	d_i'
1	−0.500000	0.007812
2	−0.666667	0.026042
3		0.054687

- For comparison the exact solution: $y = -\frac{x^3}{6} + \frac{x}{6}$
- The solution at the nodes

i	x_i	y_i	$y(x_i)$	$\lvert y(x_i) - y_i \rvert$
0	0	0.000000	0.000000	0
1	0.25	0.039062	0.039063	$6.94 \cdot 10^{-18}$
2	0.5	0.062500	0.062500	$1.39 \cdot 10^{-17}$
3	0.75	0.054687	0.054688	$1.39 \cdot 10^{-17}$
4	1	0.000000	0.000000	0

- The largest error at the nodes against the exact solution: $\max_i |y(x_i) - y_i| \approx 1.39 \cdot 10^{-17}$
- The method's result: $y(0.5) \approx 0.062500$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

A pendulum by the Verlet method – its energy error stays, RK4's grows tenfold.

Input: `verlet(x'' = -sin(x); x(0)=1; x'(0)=0; h=0.5; to=500)`

Result

$$x(500) \approx -0.453415, \quad \max|\Delta E| \approx$$

Steps

- The equation of motion $x'' = -\sin x, \quad x(0) = 1, \quad x'(0) = 0$, the step $h = 0.5$ (1000 steps).
- The velocity Verlet method – the new position from the velocity and the acceleration, the velocity from the mean acceleration at the start and the end of the step. The total energy $E = \frac{v^2}{2} + V(x), \quad V(x) = -\cos x$ is computed for both methods

$$x_{n+1} = x_n + h\,v_n + \tfrac{h^2}{2}a(x_n)$$
$$v_{n+1} = v_n + \tfrac{h}{2}(a(x_n) + a(x_{n+1}))$$

- Verlet against RK4 with the same step (selected rows)

t	x_n	v_n	$E_n - E_0$	x_n^{RK4}	E_n^R
0	1.0000	0.0000	0	1.0000	
20	0.9969	−0.0714	$-8.7 \cdot 10^{-5}$	0.9924	–
40	0.9875	−0.1424	$-3.5 \cdot 10^{-4}$	0.9774	–
60	0.9719	−0.2128	$-8 \cdot 10^{-4}$	0.9574	–
80	0.9501	−0.2822	−0.0014	0.9340	-
100	0.9223	−0.3504	−0.0023	0.9091	-
120	0.8885	−0.4168	−0.0034	0.8839	-
140	0.8489	−0.4812	−0.0047	0.8597	-
160	0.8037	−0.5430	−0.0063	0.8375	-
180	0.7531	−0.6019	−0.0081	0.8181	-
200	0.6973	−0.6575	−0.01	0.8020	-
220	0.6366	−0.7091	−0.012	0.7897	
240	0.5715	−0.7564	−0.015	0.7815	-
260	0.5022	−0.7989	−0.017	0.7775	-
280	0.4293	−0.8361	−0.019	0.7776	-
300	0.3532	−0.8676	−0.022	0.7818	
320	0.2744	−0.8930	−0.024	0.7897	-
340	0.1936	−0.9121	−0.025	0.8009	-
360	0.1113	−0.9246	−0.026	0.8150	-
380	0.0281	−0.9303	−0.027	0.8311	
400	−0.0553	−0.9292	−0.026	0.8486	-
420	−0.1382	−0.9213	−0.026	0.8663	-
440	−0.2201	−0.9066	−0.025	0.8834	-
460	−0.3003	−0.8854	−0.023	0.8985	
480	−0.3783	−0.8580	−0.021	0.9102	-
500	−0.4534	−0.8245	−0.019	0.9171	-

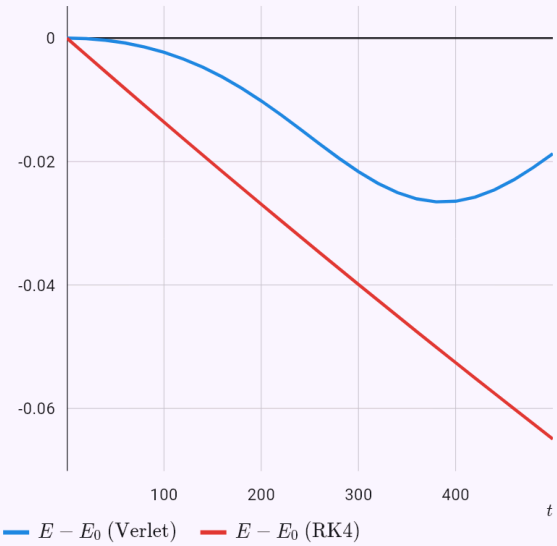
- Verlet's energy error stays the same, RK4's grows with time – over a long computation Verlet is more faithful. The largest error at the start and at the end, Verlet above, RK4 below

$$\max_{t \le t_{1/10}} |E - E_0| \approx 0.027, \quad \max_{t \ge t_{9/10}} |E - E_0| \approx 0.027$$

$$\max_{t \le t_{1/10}} |E - E_0| \approx 0.0069, \quad \max_{t \ge t_{9/10}} |E - E_0| \approx 0.065$$

- The method's result:
$x(500) \approx -0.453415, \quad \max|\Delta E| \approx 0.027$

Graph



The total energy's deviation from the initial one: Verlet oscillates around zero, RK4 slowly drifts away.

The Gauss–Seidel method – the equations reordered for the diagonal dominance.

Input: `gaussseidel(x + 3y = 7; 4x + y = 6)`

Result

$$x \approx 1.0000$$
$$y \approx 2.0000$$

Steps

- The system in the matrix form
$$\begin{pmatrix} 1 & 3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \end{pmatrix}$$
- The equations reordered to make the matrix diagonally dominant ($|4| > |1|, \quad |3| > |1|$)
$$\begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \end{pmatrix}$$
- The iteration matrix its spectral radius $\rho(T_{GS}) \approx 0.08333 < 1$ – the method converges.
$$T_{GS} = -(D+L)^{-1}U = \begin{pmatrix} 0 & -0.25 \\ 0 & 0.08333 \end{pmatrix}$$
- The Gauss–Seidel method – the new values used as soon as they are computed

$$x^{(k+1)} = \frac{6 - y^{(k)}}{4}$$
$$y^{(k+1)} = \frac{7 - x^{(k+1)}}{3}$$

- The iterations

k	$x^{(k)}$	$y^{(k)}$	$\left\|x^{(k)} - x^{(k-1)}\right\|_\infty$
0	0.000000	0.000000	
1	1.500000	1.833333	1.833333
2	1.041667	1.986111	0.458333
3	1.003472	1.998843	0.038194
4	1.000289	1.999904	0.003183
5	1.000024	1.999992	0.000265
6	1.000002	1.999999	0.000022

- The stopping condition $\left\|x^{(k)} - x^{(k-1)}\right\|_\infty < 10^{-4}$ is met after 6 iterations.
- The method's result: $x \approx 1.0000, \quad y \approx 2.0000$

The LU decomposition in exact fractions, the forward and back substitution.

Input: `lu(2x + y + z = 5; 4x + 3y + 3z = 13; 2x + 3y + 4z = 13)`

Result

$$x = 1$$
$$y = 1$$
$$z = 2$$

Steps

- The system in the matrix form
$$\begin{pmatrix} 2 & 1 & 1 \\ 4 & 3 & 3 \\ 2 & 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 13 \\ 13 \end{pmatrix}$$
- Gaussian elimination – the rows' multipliers make the matrix L: $l_{21} = \frac{4}{2} = 2, \quad l_{31} = \frac{2}{2} = 1, \quad l_{32} = \frac{2}{1} = 2$
- The LU decomposition A = LU
$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$
$$U = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

- The forward substitution Ly = b
$$y_1 = 5$$
$$y_2 = 13 - 2 \cdot 5 = 3$$
$$y_3 = 13 - 1 \cdot 5 - 2 \cdot 3 = 2$$

- The back substitution Ux = y
$$x = \frac{5 - 1 \cdot 1 - 1 \cdot 2}{2} = 1$$
$$y = \frac{3 - 1 \cdot 2}{1} = 1$$
$$z = \frac{2}{1} = 2$$

- The method's result: $x = 1, \quad y = 1, \quad z = 2$

Lagrange's interpolating polynomial and its value at a point.

Input: `lagrange(x=0 1 2 3; y=1 3 2 5; at=1.5)`

Result

$$L(x) = \frac{7}{6}x^3 - 5x^2 + \frac{35}{6}x + 1$$

$$L(1.5) = \frac{39}{16} \approx 2.437500$$

Steps

- The given points (4)

i	x_i	y_i
0	0	1
1	1	3
2	2	2
3	3	5

- Lagrange's basis polynomials $l_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$

$$l_0(x) = -\frac{(x-1)(x-2)(x-3)}{6}$$
$$l_1(x) = \frac{x(x-2)(x-3)}{2}$$
$$l_2(x) = -\frac{x(x-1)(x-3)}{2}$$
$$l_3(x) = \frac{x(x-1)(x-2)}{6}$$

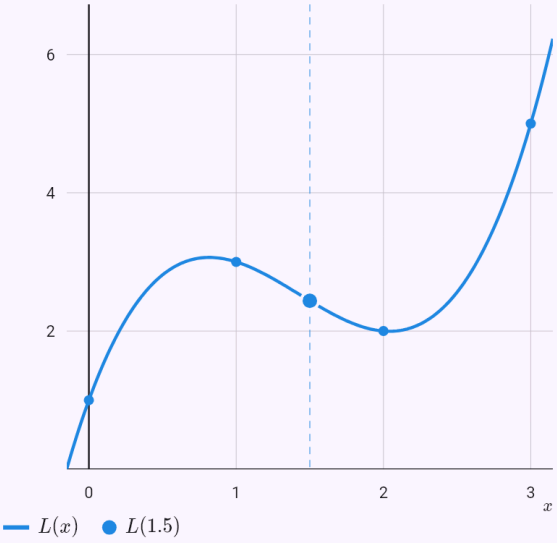
- Lagrange's interpolating polynomial

$$L(x) = 1 \cdot l_0(x) + 3 \cdot l_1(x) + 2 \cdot l_2(x) + 5 \cdot l_3(x) = \frac{7}{6}x^3$$

- The value at the given point: $L(1.5) = \frac{39}{16} \approx 2.437500$
- The method's result

$$L(x) = \frac{7}{6}x^3 - 5x^2 + \frac{35}{6}x + 1, \quad L(1.5) = \frac{39}{16} \approx 2.437$$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

Newton's polynomial from the table of divided differences.

Input: `newtonpoly(x=0 1 2 3; y=1 3 2 5)`

Result

$$N(x) = \frac{7}{6}x^3 - 5x^2 + \frac{35}{6}x + 1$$

Steps

- The given points (4)

i	x_i	y_i
0	0	1
1	1	3
2	2	2
3	3	5

- The table of divided differences

$$f[x_i, \ldots, x_{i+j}] = \frac{f[x_{i+1}, \ldots, x_{i+j}] - f[x_i, \ldots, x_{i+j-1}]}{x_{i+j} - x_i}$$

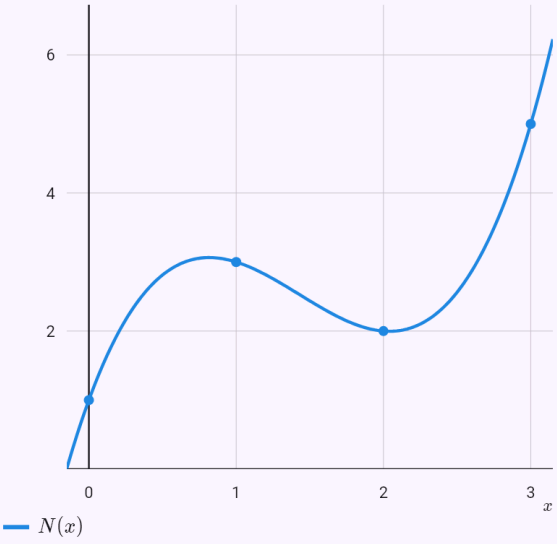
i	x_i	$f[x_i]$	$f[\cdot]_1$	$f[\cdot]_2$	$f[\cdot]_3$
0	0	1	2	$-\frac{3}{2}$	$\frac{7}{6}$
1	1	3	-1	2	
2	2	2	3		
3	3	5			

- Newton's interpolating polynomial (the coefficients from the table's top diagonal)

$$N(x) = 1 + 2x - \frac{3}{2}x(x-1) + \frac{7}{6}x(x-1)(x-2) = \frac{7}{6}$$

- The method's result: $N(x) = \frac{7}{6}x^3 - 5x^2 + \frac{35}{6}x + 1$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

A natural cubic spline – the system for the moments and the pieces.

Input: `spline(x=0 1 2 3; y=0 1 0 1; at=1.5)`

Result

$$S(1.5) \approx 0.500000$$

Steps

- The given points (4)

i	x_i	y_i
0	0	0
1	1	1
2	2	0
3	3	1

- The natural cubic spline: the second derivatives M⃗ at the nodes satisfy with $M_0 = M_3 = 0$

$$h_{i-1}M_{i-1} + 2(h_{i-1} + h_i)M_i + h_i M_{i+1} = 6\left(\frac{y_{i+1} - y_i}{h_i} -$$

$$4M_1 + 1M_2 = -12$$
$$1M_1 + 4M_2 = 12$$

- The system's solution:
$M_0 \approx 0, \quad M_1 \approx -4, \quad M_2 \approx 4, \quad M_3 \approx 0$

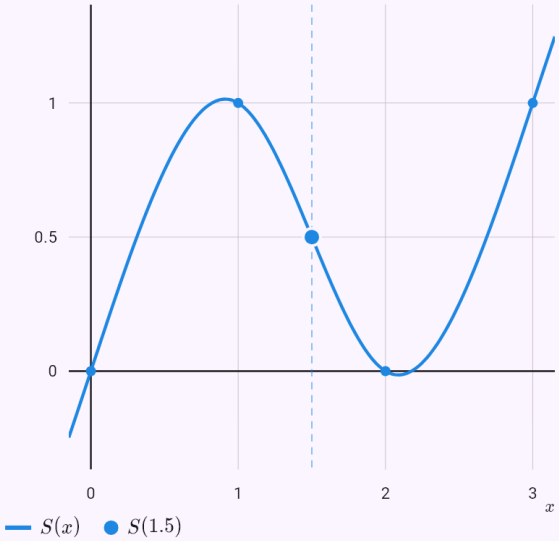
- On each subinterval

$$S_i(x) = y_i + b_i(x - x_i) + \frac{M_i}{2}(x - x_i)^2 + \frac{M_{i+1} - M_i}{6h_i}(x$$

$$S_0(x) = -0.666667x^3 + 1.66667x,$$
$$S_1(x) = 1.33333(x-1)^3 - 2(x-1)^2 - 0.333333(x -$$
$$S_2(x) = -0.666667(x-2)^3 + 2(x-2)^2 - 0.333333(x$$

- The value at the given point: $S(1.5) \approx 0.500000$
- The method's result: $S(1.5) \approx 0.500000$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

A quadratic approximation by least squares.

Input: `leastsquares(x=0 1 2 3 4 5; y=1.1 1.9 3.2 5.1 7.4 10.2; degree=2)`

Result

$$p(x) = 0.25x^2 + 0.575714x + 1.08571$$

Steps

- The given points (6)

i	x_i	y_i
0	0	$\frac{11}{10}$
1	1	$\frac{19}{10}$
2	2	$\frac{16}{5}$
3	3	$\frac{51}{10}$
4	4	$\frac{37}{5}$
5	5	$\frac{51}{5}$

- We look for the polynomial with $S = \sum_i (y_i - p(x_i))^2 \to \min$. The sums needed

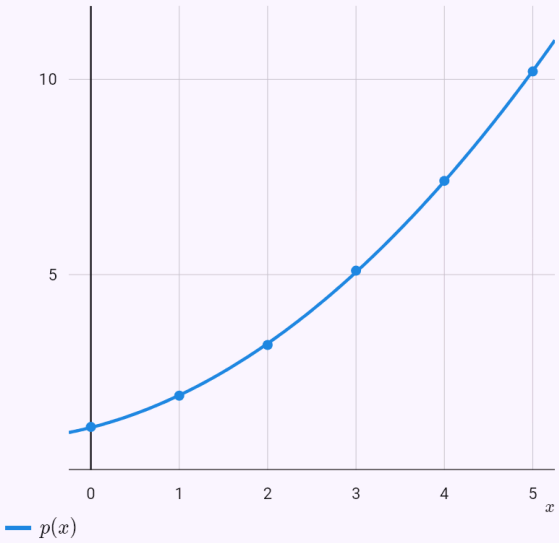
$$n = 6, \quad \sum x_i = 15, \quad \sum x_i^2 = 55, \quad \sum x_i^3 = 225,$$

- The normal equations for the coefficients $a_0 \ldots a_m$

$$\begin{aligned} 6a_0 + 15a_1 + 55a_2 &= 28.9 \\ 15a_0 + 55a_1 + 225a_2 &= 104.2 \\ 55a_0 + 225a_1 + 979a_2 &= 434 \end{aligned}$$

- The solution:
  $a_0 \approx 1.08571, \quad a_1 \approx 0.575714, \quad a_2 \approx 0.25$
- The sum of squared deviations:
  $S = \sum_i (y_i - p(x_i))^2 \approx 0.003429$
- The method's result:
  $p(x) = 0.25x^2 + 0.575714x + 1.08571$

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

The internal rate of return – the bisection's steps collapsed under the result.

Input: `model(npv; find=irr[one]; CF=-1000 300 400 500 200[one]; r=10[pct]; cur=CZK)`

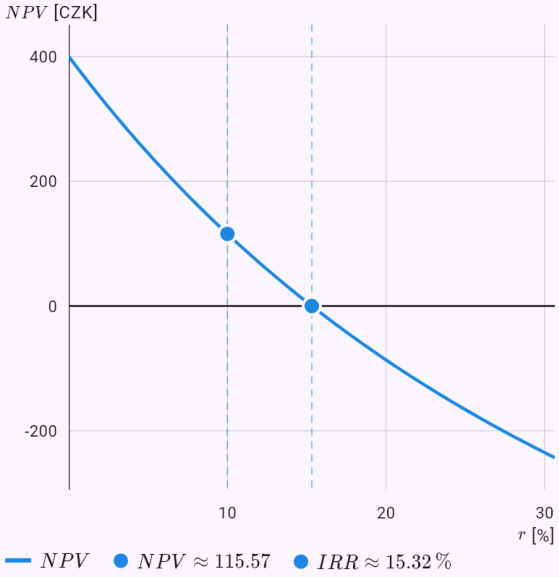
Result

$$IRR \approx 15.32\,\%$$

Steps

- Given:
  $CF_t = (-1{,}000;\ 300;\ 400;\ 500;\ 200), \quad r = 10\,\%$
- Converting the units: $r = 10\,\% = 0.1$
- The net present value NPV adds up a project's cash flows CF_0, CF_1, … discounted by the required return r: NPV = Σ CF_t/(1 + r)^t. A project with NPV > 0 earns more than the required return. The internal rate of return IRR is the rate at which NPV = 0 – a polynomial equation's root, found numerically. The payback period is when the (simple or discounted) running sum of the flows stops being negative. The model's equation:
  $$NPV = \sum_{t=0}^{n} \frac{CF_t}{(1+r)^t}$$
- Net present value
  $$NPV = -1{,}000 + \frac{300}{1.1^1} + \frac{400}{1.1^2} + \frac{500}{1.1^3} + \frac{200}{1.1^4} \approx 115.5$$
- NPV ≥ 0 – the project earns at least the required return.
- Solving the equation numerically (bisection):
  $NPV(IRR) = 0 \Rightarrow IRR \approx 15.32\,\%$
- › The method: bisection, 14 iterations

Graph



The graph computed from the model's solution; the dots are the values given or the answer.

Known limitations

- Roots: one equation in one unknown, real roots; at most three roots in ⟨−50; 50⟩ are isolated automatically.
- Differential equations: a single first-order equation y′ = f(x, y) only, at most 2,000 steps; systems and higher orders not yet.
- Systems: 2 to 6 equations; interpolation at most 12 points, a least-squares polynomial of degree 5 at most.
- Simpson's and Romberg's methods from a table need equal steps (Romberg 2, 4, 8, … subintervals).
- Boundary value problems only of the 2nd order with the values y(a), y(b) (not a derivative at an end); finite differences only for an equation linear in y and y′. The Verlet method only for motion x″ = a(x) in one coordinate – a planet's orbit in the plane not yet.